Problems Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
February 15, 2020

- **5565:** *Proposed by Kenneth Korbin, New York, NY*

  A trapezoid with integer length sides is inscribed in a circle with diameter $7^3 = 343$. Find the minimum and the maximum possible values of the perimeter.

- **5566:** *Proposed by Michael Brozinsky, Central Islip , NY,*

  Square ABCD (in clockwise order) with all sides equal to x has point E on AB at a distance $\alpha \cdot x$ from B where $0 < \alpha < 1$. The right triangle EBC is folded along segment EC so that what was previously corner B is now at point B'. A perpendicular from B' to AD intersects AD at H. If the ratio of the areas of trapezoids $AEB'H$ to $DCB'H$ is $\frac{7}{18}$ what is α?

- **5567:** *Proposed by D.M. Bătinetu-Giurgiu, National College "Matei Basarab" Bucharest, and Neulai Stanciu, "George Emil Palade" School, Buză, Romania*

  Let $[A_1A_2A_3A_4]$ be a tetrahedron with total area S, and with the area of S_k being the area of the face opposite the vertex $A_k,\ k = 1, 2, 3, 4$. Prove that

  $$\frac{20}{3} \leq \sum_{k=1}^{4} \frac{S + S_k}{S - S_k} < 8.$$

- **5568:** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Mehedinti, Romania*

  Given: $A \in M_5(\Re), \det(A^5 + I_5) \neq 0$, and $A^{20} - I^5 = A^5(A^5 + I_5)$. Prove that $\sqrt[4]{\det} A \in \Re$.

- **5569:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Compute:

  $$\lim_{x \to 0} \frac{\tan(x \cdot \cos(x)) - \tan(x) \cdot \cos(\tan(x))}{x^7}.$$

- **5570:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Calculate
$$\int_0^1 \frac{(\ln(1-x)+x)^2}{x^3}\mathrm{d}x.$$

Solutions

- **5547:** *Proposed by Kenneth Korbin, New York, NY*

  Given Heronian Triangle ABC with $\overline{AC} = 10201$ and $\overline{BC} = 10301$. Observe that the sum of the digits of $\overline{AC}$ is 4 and the sum of the digits of BC is 5. Find $\overline{AB}$ if the sum of its digits is 3.

  (An Heronian Triangle is one whose side lengths and area are integers.)

  Solutions 1 by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith, Angelo State University, San Angelo, TX

  We found two answers for $\overline{AB}$. To begin, many possibilities can be ruled out because of the triangle inequality which states that if x, y, z are the sides of a triangle and $x \leq y \leq z$, then $z < x + y$. This immediately eliminates the situations where $\overline{AB}$ has one, two, or six or more digits. Also, for the given values of $\overline{BC}$ and $\overline{AC}$, $\overline{AB}$ cannot have an odd units digit for this causes the semiperimeter s and the quantities $s - \overline{AB}$, $s - \overline{BC}$, and $s - \overline{AC}$ to be of the form $\frac{m}{2}$, where m is an odd integer. As a result, Heron's Formula for the area of $\triangle ABC$ will not yield an integer and the triangle will not be Heronian. Finally, the requirement that the sum of the digits of $\overline{AB}$ must be 3 restricts the digits of $\overline{AB}$ to satisfy one of the following scenarios:
  a) one 3 and two or more 0's; b) one 1, one 2, and one or more 0's; c) three 1's and perhaps some 0's. Of these, we found two solutions satisfying type b)

  Answer 1. Let $\overline{AB} = 102$. Then, the triangle inequality is satisfied because $\overline{BC} > \overline{AC} > \overline{AB}$ and $\overline{BC} < \overline{AB} + \overline{AC}$. Also, the semiperimeter s becomes
$$s = \frac{\overline{AB} + \overline{BC} + \overline{AC}}{2} = \frac{20604}{2} = 10302.$$

  Then,
$$\begin{aligned} s\left(s - \overline{AB}\right)\left(s - \overline{BC}\right)\left(s - \overline{AC}\right) &= (10302)(10200)(1)(101) \\ &= (10302)(102)(101)(100) \\ &= (10302)^2(10)^2 \end{aligned}$$

  and Heron's Formula yields
$$area\,(\triangle ABC) = (10302)(10) = 103020.$$

  Hence, this choice of $\overline{AB}$ produces a Heronian Triangle for which the sum of the digits of $\overline{AB}$ is 3.

Answer 2. Choose $\overline{AB} = 20100$. Then, $\overline{AB} > \overline{BC} > \overline{AC}$ with $\overline{AB} < \overline{BC} + \overline{AC}$ and the triangle inequality is satisfied. Further, in this case,

$$s = \frac{\overline{AB} + \overline{BC} + \overline{AC}}{2} = \frac{40602}{2} = 20301$$

and

$$\begin{aligned} s\left(s - \overline{AB}\right)\left(s - \overline{BC}\right)\left(s - \overline{AC}\right) &= (20301)\,(201)\,(10000)\,(10100) \\ &= (20301)\,(201)\,(101)\,(10)^6 \\ &= (20301)^2\,(10)^6\,. \end{aligned}$$

Then, Heron's Formula gives

$$area\,(\triangle ABC) = (20301)\,(10)^3 = 20301000.$$

Once again, this choice of $\overline{AB}$ determines a Heronian Triangle for which the sum of the digits of $\overline{AB}$ is 3.

Solution 2 by Brian D. Beasley, Presbyterian College, Clinton, SC

We let $x = \overline{AB}$ and note that x must be an integer with $100 < x < 20502$. Next, we let Δ be the area of triangle ABC. Using Heron's formula with $s = (x + 10201 + 10301)/2 = (1/2)x + 10251$, we have

$$\Delta^2 = s(s-x)(s-10201)(s-10301) = (-x^2/4 + 10251^2)(x^2/4 - 50^2).$$

Finally, we check via computer search the 29 values of x between 100 and 20502 with digit sum equal to 3, and only two produce an integer area for ABC:

When $x = 102$, we have $\Delta = 103020$.

When $x = 20100$, we have $\Delta = 20301000$.

Addenda. (i) It is interesting to note that in each solution, the sum of the digits of Δ is 6.

(ii) The 29 integers between 100 and 20502 with digit sum equal to 3 are:

102, 111, 120, 201, 210, 300

1002, 1011, 1020, 1101, 1110, 1200, 2001, 2010, 2100, 3000

10002, 10011, 10020, 10101, 10110, 10200, 11001, 11010, 11100, 12000, 20001, 20010, 20100

Solution 3 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

If a, b, and c are the lengths of the sides of a triangle and A, B, and C are the opposite angles, then we have: $b = 10201$ and $a = 10301$. So, by the cosine law we have:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{210171002 - c^2}{2210161002}$$

Since $-1 \leq \cos C \leq 1$, then $100 \leq c \leq 20502$. Furthermore, by Heron's formula for the area A of the triangle ABC, we have-

$$4A = \sqrt{(a+b+c)(a+b-c)(b-c-a)(c+a-b))},$$

or:

$$4A = \sqrt{(20502+c)(20502-c)(c-100)(c+100)} = \sqrt{-c^4 + 420342004c^2 - 4203320040000}$$

So, the side c must satisfy the following properties:

•(1) The side c must be an even number since the perimeter of a Heronian triangle is always an even number. Thus, every Heronian triangle has an odd number of sides of even length (Buchholz and MacDougall, 2008, p. 19).

•(2) For the side c we have: $100 \le c \le 20502$.

•(3) The sum of the digits of the side c is 3.

•(4) The side c satisfies the Diophantine equation:

$$16A^2 = -c^4 + 420342004c^2 - 4203320040000. \quad (a)$$

So, the values of side c come from the permutations of the digits of the sets:

$\{0,0,0,1,2\}, \{0,0,1,1,1\}, \{0,0,0,3\}$.

So, by $(1), (2)$, and (3), we have the possible values for side c:

$102, 120, 210, 1002, 1020, 1200, 2100, 2010, 10002, 10020,$

$10200, 12000, 20100, 20010, 1110, 11100, 11010, 10110, 300, 3000$. We can check by equation (a) that $c = 102$ or $c = 20100$.

[1] Buchholz, Ralph H. and Mac Dougall, James A. (2008). Cyclic polygons with rational sides and area. *Journal of Number Theory,* **128**:17-48.

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5548:** *Proposed by Michel Bataille, Rouen, France*

Given nonzero real numbers p and q, solve the system

$$\begin{cases} 2p^2x^3 - 2pqxy^2 - (2p-1)x = y \\ 2q^2y^3 - 2pqx^2y + (2q+1)y = x \end{cases}$$

Solution 1 by Anthony J. Bevelacqua, North Dakota State University, Grand Forks, ND

If we multiply the first equation by qy and the second by px we have

$$\begin{cases} 2p^2qx^3y - 2pq^2xy^3 - (2p-1)qxy = qy^2 \\ 2pq^2xy^3 - 2p^2qx^3y + (2q+1)pxy = px^2. \end{cases}$$

Now collect terms to find

$$\begin{cases} 2pqxy(px^2 - qy^2) - (2p-1)qxy = qy^2 \\ -2pqxy(px^2 - qy^2) + (2q+1)pxy = px^2 \end{cases}$$

and sum these equations to yield $(p+q)xy = px^2 + qy^2$. Thus

$$\begin{aligned}(px-qy)(x-y) &= px^2-(p+q)xy+qy^2 \\ &= 0.\end{aligned}$$

Thus in any solution to our original system we have $x = y$ or $px = qy$.
Note that if (x, y) is a solution to our system we have $x = 0$ if and only if $y = 0$. Since $(0, 0)$ is a solution to our system for any p and q, we suppose hereafter that (x, y) is a solution with $xy \neq 0$.
When $x = y = t$ with $t \neq 0$ our system becomes

$$\begin{cases} 2p^2t^3 - 2pqt^3 - (2p-1)t = t \\ 2q^2t^3 - 2pqt^3 + (2q+1)t = t \end{cases}$$

and we collect terms to find

$$\begin{cases} 2p(p-q)t^3 - (2p-1)t = t \\ 2q(q-p)t^3 + (2q+1)t = t. \end{cases}$$

Each of these equations reduces to $(p-q)t^3 = t$. Thus $p > q$ and $t = \dfrac{\pm 1}{\sqrt{p-q}}$.
When $x = t/p$ and $y = t/q$ with $t \neq 0$ our system becomes

$$\begin{cases} \dfrac{2}{p}t^3 - \dfrac{2}{q}t^3 - \dfrac{2p-1}{p}t = \dfrac{1}{q}t \\ \dfrac{2}{q}t^3 - \dfrac{2}{p}t^3 + \dfrac{2q+1}{q}t = \dfrac{1}{p}t. \end{cases}$$

If we multiply both equations by pq we have

$$\begin{cases} 2(q-p)t^3 - (2p-1)qt = pt \\ 2(p-q)t^3 + (2q+1)pt = qt \end{cases}$$

and each of these reduces to $2(q-p)t^3 = (2pq - (q-p))t$. Thus $\dfrac{pq}{q-p} \geq \dfrac{1}{2}$ and $t = \pm\sqrt{\dfrac{pq}{p-q} - \dfrac{1}{2}}$. (Note that since $t \neq 0$, $p = q$ would give $2pqt = 0$, a contradiction with p and q nonzero.)
Therefore the *real* solutions (x, y) to our original system are

1. $(0, 0)$ for any p and q,
2. $\left(\dfrac{\pm 1}{\sqrt{p-q}}, \dfrac{\pm 1}{\sqrt{p-q}}\right)$ when $p > q$, and
3. $\left(\dfrac{\pm 1}{p}\sqrt{\dfrac{pq}{q-p} - \dfrac{1}{2}}, \dfrac{\pm 1}{q}\sqrt{\dfrac{pq}{q-p} - \dfrac{1}{2}}\right)$ when $\dfrac{pq}{q-p} \geq \dfrac{1}{2}$.

If we want *all* solutions, we just ignore the restrictions on p and q.
Some real solutions:

- When $p = 1$, $q = -2$ we have both cases (2) and (3) above so the solutions to our system are
$$(0,0), \quad \left(\pm\sqrt{\frac{1}{3}}, \pm\sqrt{\frac{1}{3}}\right), \quad \left(\pm\sqrt{\frac{1}{6}}, \mp\frac{1}{2}\sqrt{\frac{1}{6}}\right)$$
- When $p = 2$, $q = 1$ we have case (2) but not (3), and our solutions are
$$(0,0), \quad (\pm 1, \pm 1)$$
- When $p = -3$, $q = -2$ we have case (3) but not (2). Our solutions are
$$(0,0), \quad \left(\pm\frac{1}{3}\sqrt{\frac{11}{2}}, \pm\frac{1}{2}\sqrt{\frac{11}{2}}\right)$$
- When $p = -1$, $q = 1$ we have neither case (2) nor (3) so the only solution is $(0,0)$.
- Finally, we note that case (3) can give the trivial solution $(0,0)$ as in $p = 1$, $q = -1$ or $p = 1/3$, $q = 1$.

Solution 2 by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC

Clearly, $(0,0)$ is a solution of the system. Now, if we set $x = 0$, then from the first equation we get $y = 0$, also if we set $y = 0$ in the second equation, we get $x = 0$, from these we conclude that any nontrivial solution (x,y) of the system should have both $x \neq 0$, and $y \neq 0$.

To find the nontrivial solutions, we add both sides of the equations and rearrange them, to obtain:
$$2px^2(px - qy) - 2qy^2(px - qy) - 2(px - qy) + x + y = x + y,$$
from this, we have
$$(px - qy)(px^2 - qy^2 - 1) = 0,$$
then
$$\begin{cases} px - qy = 0, \text{or} \\ px^2 - qy^2 - 1 = 0 \qquad (1) \end{cases}$$
From $px - qy = 0$, with $q \neq 0$, we have $y = \dfrac{px}{q}$, and substituting this into the equation $2p^2x^3 - 2pqxy^2 - (2p-1)x = y$ with $x \neq 0$, we obtain
$$x^2 = \frac{-p - 2pq + q}{2p^2(p-q).}$$
Since we want the value of x to be real, we must have $\dfrac{-p - 2pq + q}{p - q} > 0$.

For this, we consider two cases. Case 1: $\begin{cases} p - q > 0, \text{ and} \\ -q - p - 2pq + q > 0. \end{cases}$
From these we conclude that $-2pq > p - q > 0$, and $-2pq > 0$, or $pq < 0$. ,

That is, $\dfrac{-p-2pq+q}{2p^2(p-q)} > 0$, if and only if $\begin{cases} pq < 0 \\ p-q > 0' \end{cases}$ and then $x = \pm\dfrac{1}{p} \cdot \sqrt{\dfrac{-p-2pq+q}{2(p-q)}}$, and by $px = qy$, we obtain $y = \pm\dfrac{1}{q} \cdot \sqrt{\dfrac{-p-2pq+q}{2(p-q)}}$.

Case 2: To have $\dfrac{-p-2pq+q}{p-q} > 0$, we consider $\begin{cases} p-q = 0, \\ -p-2pq+q < 0 \end{cases}$

From these we can write $-2pq < p-q < 0$, or $-2pq < 0$ and $pq > 0$. That is, $\dfrac{-p-2pq+q}{2p^2(p-q)} > 0$, if and only if $\begin{cases} pq > 0, \\ p-q < 0 \end{cases}$ and then $x = \pm\dfrac{1}{p} \cdot \sqrt{\dfrac{-p-2pq+q}{2(p-q)}}$, and by $px = qy$, we obtain $y = \pm\dfrac{1}{q} \cdot \sqrt{\dfrac{-p-2pq+q}{2(p-q)}}$.

It is worth nothing that $-p-2pq+q = 0$ is possible and by looking at the graph of $q = \dfrac{p}{1-2p}$, we see that for nonzero p, q we have $p - q \neq 0$ and we get the solution $x = y = 0$.

Now, we look at $px^2 - qy^2 - 1 = 0$, of (1). We rearrange the first equation and use $px^2 - qy^2 = 1$, then $y = 2p^2x^2 - 2pqxy^2 - (2p-1)x = 2px(px^2 - qy^2) - 2px + x = 2px - 2px + x = y$, or $x = y$.

Using $x = y$ and $px^2 - qy^2 = 1$, gives us the solutions $x = y = \pm\dfrac{1}{\sqrt{p-q}}$, if $p - q > 0$.

Also solved by Pat Costello, Eastern Kentucky University, Richmond, KY; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5549:** *Proposed by Arkady Alt, San Jose, CA; Albert Stadler, Herrliberg, Switzerland,*

Let P be an arbitrary point in $\triangle\ ABC$ that has side lengths a, b, and c .

a) Find minimal value of

$$F(P) := \frac{a^2}{d_a(P)} + \frac{b^2}{d_b(P)} + \frac{c^2}{d_c(P)};$$

b) Prove the inequality $\dfrac{a^2}{d_a(P)} + \dfrac{b^2}{d_b(P)} + \dfrac{c^2}{d_c(P)} \geq 36r$, where r is the inradius.

Solution 1 by Michael N. Fried, Ben-Gurion University of the Negev, Beer-Sheva, Israel

a) Define $x = d_a a,\ y = d_b b,\ z = d_c c$. These are the twice the areas of the triangles determined by P and the sides of $\triangle ABC$, and, thus, if 2Δ is the area of $\triangle ABC$, we have:

$$g(x, y, z) = x + y + z = 2\Delta$$

In terms of these variables, the function we need to minimize is:

$$F(x, y, z) = \frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z}$$

Using Lagrange multipliers, we have for the point minimizing $F(x, y, z)$ constrained to $g(x, y, z) = 2\Delta$ (it is easy to see that no maximum exists since any one of x, y, or z can be made a small as one pleases):

$$grad(F) = \lambda\ grad(g)$$

or,

$$\left(-\frac{a^3}{x^2}, -\frac{b^3}{y^2}, -\frac{c^3}{z^2}\right) = \lambda\,(1,1,1)$$

From this, we have:

$$\frac{a^3}{x^2} = \frac{b^3}{y^2} = \frac{c^3}{z^2}$$

so that (keeping in mind that $x, y, z \geq 0$), we have:

$$y = \left(\frac{b}{a}\right)^{3/2} x$$

$$z = \left(\frac{c}{a}\right)^{3/2} x$$

Combined with the condition, $g(x, y, z) = 2\Delta$, we have:

$$x = \frac{2\Delta a^{3/2}}{a^{3/2} + b^{3/2} + c^{3/2}}$$

Similarly,

$$y = \frac{2\Delta b^{3/2}}{a^{3/2} + b^{3/2} + c^{3/2}}$$

$$z = \frac{2\Delta c^{3/2}}{a^{3/2} + b^{3/2} + c^{3/2}}$$

Substituting these values into F we obtain for the minimum value:

$$F_{min} = \frac{\left(a^{3/2} + b^{3/2} + c^{3/2}\right)^2}{2\Delta}$$

b) Since $a^{2/3}, b^{2/3}, c^{2/3}$ are all positive values, we know by, for example, the AG inequality that F_{min} will itself be minimized when these terms are equal, that is, when $a = b = c$, and the minimum will be, accordingly, $\frac{\left(3a^{3/2}\right)^2}{2\Delta} = \frac{9a^3}{2\Delta}$. Now, since in this case $\triangle ABC$ is equilateral and has inradius r, it follows that $a = 2r\sqrt{3}$ and $2\Delta = 6r^2\sqrt{3}$. Hence, we have:

$$F_{min} = \frac{\left(a^{3/2} + b^{3/2} + c^{3/2}\right)^2}{2\Delta} \geq \frac{9a^3}{2\Delta} = 36r$$

Solution 2 by Moti Levy, Rehovot, Israel

a) Let $d_a(P)$ denotes the distance from a point P to side a of the triangle.
Let $x := d_a(P)$, $y := d_b(P)$ and $z := d_a(P)$. Then our problem can be reformulated as:

$$\text{Minimize } \frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \text{ subject to the constraint } ax + by + cz = 2S,$$

where S is the area of the triangle.
The Lagrangian function is

$$L(x, y, z, \lambda) := \frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} - \lambda\,(ax + by + cz - 2S)\,.$$

$$\begin{aligned}\frac{\partial f}{\partial x} &= -\frac{a^2}{x^2} - \lambda a x = 0 \qquad (1)\\ \frac{\partial f}{\partial y} &= -\frac{b^2}{y^2} - \lambda b y = 0\\ \frac{\partial f}{\partial z} &= -\frac{c^2}{z^2} - \lambda c z = 0\\ \frac{\partial f}{\partial \lambda} &= -ax - by - cz + 2S = 0\end{aligned}$$

The real solution of (1) is

$$x = -\left(\frac{a}{\lambda}\right)^{\frac{1}{3}}, y = -\left(\frac{b}{\lambda}\right)^{\frac{1}{3}}, z = -\left(\frac{c}{\lambda}\right)^{\frac{1}{3}},\ \lambda^{\frac{1}{3}} = -\frac{a^{\frac{4}{3}} + b^{\frac{4}{3}} + c^{\frac{4}{3}}}{2S},$$

Therefore, the point $\left(\frac{2Sa^{\frac{1}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}, \frac{2Sb^{\frac{1}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}, \frac{2Sc^{\frac{1}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}\right)$ is critical point.
To verify that it is local minimum, we compute the bordered Hessian

$$H_4 = \begin{bmatrix} 0 & -a & -b & -c \\ -a & 2\frac{a^2}{x^3} & 0 & 0 \\ -b & 0 & 2\frac{b^2}{y^3} & 0 \\ -c & 0 & 0 & 2\frac{c^2}{z^3} \end{bmatrix}$$

$$-\det(H_4) = 4\frac{a^2b^2c^2x^3 + a^2b^2c^2y^3 + a^2b^2c^2z^3}{x^3y^3z^3}.$$

$$H_3 = \begin{bmatrix} 0 & -a & -b \\ -a & 2\frac{a^2}{x^3} & 0 \\ -b & 0 & 2\frac{b^2}{y^3} \end{bmatrix}$$

$$-\det(H_3) = 2\frac{a^2b^2x^3 + a^2b^2y^3}{x^3y^3}.$$

Since $-\det(H_4) > 0$ and $-\det(H_3) > 0$ at the critical point, then the point $\left(\frac{2Sa^{\frac{1}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}, \frac{2Sb^{\frac{1}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}, \frac{2Sc^{\frac{1}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}\right)$ is indeed local minimum.
By evaluation of $F(P)$ at the local minimum, we conclude that the minimal value of $F(P)$ is

$$\frac{1}{2S}\left(a^{\frac{5}{3}} + b^{\frac{5}{3}} + c^{\frac{5}{3}}\right)\left(a^{\frac{4}{3}} + b^{\frac{4}{3}} + c^{\frac{4}{3}}\right).$$

b) By the mean power inequality

$$\left(\frac{a^{\frac{5}{3}} + b^{\frac{5}{3}} + c^{\frac{5}{3}}}{3}\right)^{\frac{3}{5}} \geq \frac{a+b+c}{3},$$

and

$$\left(\frac{a^{\frac{4}{3}} + b^{\frac{4}{3}} + c^{\frac{4}{3}}}{3}\right)^{\frac{3}{4}} \geq \frac{a+b+c}{3}.$$

Hence

$$\frac{1}{2S}\left(a^{\frac{5}{3}}+b^{\frac{5}{3}}+c^{\frac{5}{3}}\right)\left(a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}\right) \geq 3^{-\frac{2}{3}}(a+b+c)^{\frac{5}{3}} * 3^{-\frac{1}{3}}(a+b+c)^{\frac{4}{3}}$$
$$= \frac{1}{2S} * \frac{1}{3}(a+b+c)^3.$$

The following two facts are well known

$$S = \frac{1}{2}r(a+b+c)$$

and

$$(a+b+c)^2 \geq 108r^2,$$

(Bottema, "*Geometric inequalities*," page 52, inequality No. 5.11).
Therefore,

$$F(P) \geq \frac{1}{2S} * \frac{1}{3}(a+b+c)^3 \geq \frac{1}{2S} * \frac{1}{3} * 108r^2 \frac{2S}{r} = 36r.$$

Solution 3 by Albert Stadler, Herrliberg, Switzerland

(a) Clearly $ad_a + bd_b + cd_c = 2\Delta$, where Δ is the area of the triangle. $F(P)$ is the minimum of $\frac{a^2}{d_a}+\frac{b^2}{d_b}+\frac{c^2}{d_c}$ under the constraint $ad_a + bd_b + cd_c = 2\Delta$. To find $F(P)$ we use Lagrange multipliers. Let

$$L(d_a, d_b, d_c, \lambda) = \frac{a^2}{d_a}+\frac{b^2}{d_b}+\frac{c^2}{d_c} + \lambda(ad_a + bd_b + cd_c).$$

Then $\frac{\partial}{\partial d_a}L = \frac{\partial}{\partial d_b}L = \frac{\partial}{\partial d_c}L = 0$ and thus

$$-\frac{a^2}{d_a^2} + \lambda a = -\frac{b^2}{d_b^2} + \lambda b = -\frac{c^2}{d_c^2} + \lambda c = 0, \ ad_a + bd_b + cd_c = 2\delta.$$

We conclude that

$$d_a = \sqrt{\frac{a}{\lambda}}, \ d_b = \sqrt{\frac{b}{\lambda}}, \ d_c = \sqrt{\frac{c}{\lambda}},$$

$$2\Delta = ad_a + b_d + c_c = \frac{a^{3/2}+b^{3/2}+c^{3/2}}{\sqrt{\lambda},}$$

$$d_a = \frac{2\Delta\sqrt{a}}{a^{3/2}+b^{3/2}+cd_a^{3/2}}, \ d_b = \frac{2\Delta\sqrt{b}}{a^{3/2}+b^{3/2}+cd_a^{3/2}}, \ d_c = \frac{2\Delta\sqrt{c}}{a^{3/2}+b^{3/2}+cd_a^{3/2}},$$

$$F(P) = \frac{\left(a^{3/2}+b^{3/2}+c^{3/2}\right)^2}{2\Delta}$$

(b) We prove the stronger inequality $F(P) \geq 18R$, where R is the circumradius of the triangle. (The inequality is stronger, since $R \geq 2r$ by Euler's inequality).

It is known that $R = \frac{abc}{4\Delta}$. Therefore the inequality $F(P) \geq 18R$ is equivalent to

$$\left(a^{3/2}+b^{3/2}+c^{3/2}\right)^2 \geq 9abc$$

which is obviously true by the AM-GM inequality.

Solution 4 by Michel Bataille, Rouen, France

Note. Part b was Junior Problem 58 in Mathproblems (proposed by the same author). Two solutions appear in Vol. 6 Issue 1 (2016) (http://www.mathproblems-ks.org). The solution below borrows from these two solutions.

a) Let S_a, S_b, S_c denote the areas of $\Delta BPC, \Delta CPA, \Delta APB$, respectively, and let $S = S_a + S_b + S_c$ be the area of ΔABC. Since $2S_x = x \cdot d_x(P)$ for $x = a, b, c$, we have $F(P) = \frac{1}{2}\left(\frac{a^3}{S_a} + \frac{b^3}{S_b} + \frac{c^3}{S_c}\right)$. From the Cauchy-Schwarz inequality, we deduce

$$\left(\frac{a^3}{S_a} + \frac{b^3}{S_b} + \frac{c^3}{S_c}\right)(S_a + S_b + S_c) \geq (a^{3/2} + b^{3/2} + c^{3/2})^2$$

and so

$$F(P) \geq \frac{(a^{3/2} + b^{3/2} + c^{3/2})^2}{2S}. \qquad (1)$$

For the point P_0 with trilinear coordinates $(\sqrt{a} : \sqrt{b} : \sqrt{c})$, that is, with barycentric coordinates $(a\sqrt{a} : b\sqrt{b} : c\sqrt{c})$, we have $2S_a = \lambda a\sqrt{a}, 2S_b = \lambda b\sqrt{b}, 2S_c = \lambda c\sqrt{c}$ for some λ. By addition, $2S = \lambda(a^{3/2} + b^{3/2} + c^{3/2})$, hence

$$F(P_0) = \frac{a^3}{\lambda a^{3/2}} + \frac{b^3}{\lambda b^{3/2}} + \frac{c^3}{\lambda c^{3/2}} = \frac{a^{3/2} + b^{3/2} + c^{3/2}}{\lambda} = \frac{(a^{3/2} + b^{3/2} + c^{3/2})^2}{2S}. \qquad (2)$$

From (1) and (2), and with $s = \frac{a+b+c}{2}$, the minimal value of $F(P)$ is

$$\frac{(a^{3/2} + b^{3/2} + c^{3/2})^2}{2S} = \frac{(a^{3/2} + b^{3/2} + c^{3/2})^2}{2\sqrt{s(s-a)(s-b)(s-c)}}$$

b) An inequality of means yields $\left(\frac{a^{3/2}+b^{3/2}+c^{3/2}}{3}\right)^{2/3} \geq \frac{a+b+c}{3}$, hence $(a^{3/2} + b^{3/2} + c^{3/2})^2 \geq \frac{(a+b+c)^3}{3}$. Since $2S = 2rs = r(a+b+c)$, we see that the minimal value of $F(P)$ found above is greater than or equal to $\frac{(a+b+c)^2}{3r} = \frac{4s^2}{3r}$.
But, from the geometric mean-arithmetic mean, we have

$$r^2 s = \frac{r^2 s^2}{s} = \frac{S^2}{s} = (s-a)(s-b)(s-c) \leq \left(\frac{s-a+s-b+s-c}{3}\right)^3 = \frac{s^3}{27}$$

so that $s^2 \geq 27r^2$. Thus, $\frac{4s^2}{3r} \geq 36r$ and the required result follows.

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece, and the proposer.

- **5550:** *Proposed by Ángel Plaza, University of the Las Palmas de Gran Canaria, Spain*

Prove that

$$\sum_{n=4}^{\infty} \sum_{k=2}^{n-2} \frac{1}{k\binom{n}{k}} = \frac{1}{2}.$$

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA

For $k \geq 1$,

$$k \binom{n}{k} = n \binom{n-1}{k-1}.$$

Moreover,

$$\binom{n}{k}^{-1} = (n+1) \int_0^1 t^k (1-t)^{n-k} \, dt.$$

Thus,

$$\begin{aligned}
\sum_{n=4}^{\infty} \sum_{k=2}^{n-2} \frac{1}{k \binom{n}{k}} &= \sum_{n=4}^{\infty} \sum_{k=2}^{n-2} \frac{1}{n \binom{n-1}{k-1}} \\
&= \sum_{n=4}^{\infty} \sum_{k=2}^{n-2} \int_0^1 t^{k-1} (1-t)^{n-k} \, dt \\
&= \sum_{k=2}^{\infty} \sum_{n=k+2}^{\infty} \int_0^1 t^{k-1} (1-t)^{n-k} \, dt \\
&= \sum_{k=2}^{\infty} \int_0^1 \frac{t^{k-1}}{(1-t)^k} \sum_{n=k+2}^{\infty} (1-t)^n \, dt \\
&= \sum_{k=2}^{\infty} \int_0^1 \frac{t^{k-1}}{(1-t)^k} \cdot \frac{(1-t)^{k+2}}{t} \, dt \\
&= \sum_{k=2}^{\infty} \int_0^1 t^{k-2} (1-t)^2 \, dt \\
&= \int_0^1 (1-t)^2 \sum_{k=2}^{\infty} t^{k-2} \, dt \\
&= \int_0^1 (1-t) \, dt = \frac{1}{2}.
\end{aligned}$$

Solution 2 by Ulrich Abel, Technische Hochschule Mittelhessen, Germany and Manfred Hauben, Pfizer Inc and NYU Langone Health, USA

We show that

$$s := \sum_{n=4}^{\infty} \sum_{k=2}^{n-2} \frac{1}{k \binom{n}{k}} = \frac{1}{2}.$$

Proof: Observing that

$$\frac{1}{k \binom{n}{k}} = \frac{\Gamma(k) \, \Gamma(n-k+1)}{\Gamma(n+1)} = B(k, n+1-k) = \int_0^1 t^{k-1} (1-t)^{n-k} \, dt,$$

where B denotes the beta function, we obtain

$$s = \sum_{n=4}^{\infty} \sum_{k=0}^{n-4} \int_0^1 t^{k+1} (1-t)^{n-2-k} \, dt.$$

Application of the formula for geometric series yields

$$s = \sum_{n=4}^{\infty} \int_0^1 t (1-t)^2 \, \frac{t^{n-3} - (1-t)^{n-3}}{t - (1-t)} dt.$$

Inserting

$$(1-t)\sum_{n=4}^{\infty} t^{n-3} = t, \qquad t\sum_{n=4}^{\infty}(1-t)^{n-3} = 1-t,$$

leads to

$$s = \int_0^1 (1-t)\,\frac{t^2-(1-t)^2}{2t-1}dt = \int_0^1 (1-t)\,dt = \frac{1}{2}.$$

Solution 3 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

We have

$$\begin{aligned}
\sum_{n=4}^{+\infty}\sum_{k=2}^{n-2}\frac{1}{k\binom{n}{k}} &= \sum_{n=4}^{+\infty}\sum_{k=2}^{n-2}\frac{(n-k)(k-1)!}{n!} = \sum_{n=4}^{+\infty}\sum_{k=2}^{n-2}\frac{\Gamma(n-k+1)\Gamma(k)}{\Gamma(n+1)} \\
&= \sum_{n=4}^{+\infty}\sum_{k=2}^{n-2} B(n-k+1,k) = \sum_{n=4}^{+\infty}\sum_{k=2}^{n-2}\int_0^1 x^{n-k}(1-x)^{k-1}dx \\
&= \int_0^1 \sum_{n=4}^{+\infty}\sum_{k=2}^{n-2} x^n(1-x)^{k-1}x^{-k}dx = \int_0^1 \sum_{k\geq 2}(1-x)^{k-1}x^{-k}\sum_{n\geq k+2}x^n dx \\
&= \int_0^1 \sum_{k\geq 2}(1-x)^{k-1}x^{-k}\frac{x^{k+2}}{1-x}dx = \int_0^1 x^2\sum_{k\geq 2}(1-x)^{k-2}dx = \int_0^1 x dx = \frac{1}{2}.
\end{aligned}$$

Also solved by Michel Bataille, Rouen, France; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Carl Libis, Columbia Southern University, Orange Beach, AL; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5551:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $n \geq 2$ be positive real numbers. Prove that the following inequality holds:

$$1 + \frac{1}{n^2}\sum_{1\leq i<j\leq n}\frac{\left(\sqrt{\alpha_i\alpha_{j+1}}-\sqrt{\alpha_j\alpha_{i+1}}\right)^2}{\alpha_i\alpha_j} \leq \left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{\alpha_{k+1}}{\alpha_k}\right)^2\right)^{1/2}$$

(Here the subscripts are taken modulo n.)

Solution 1 by Kee-Wai Lau, Hong Kong, China

Let $S_n = \sum_{1\leq i<j\leq n}\frac{\left(\sqrt{\alpha_i\alpha_{j+1}}-\sqrt{\alpha_j\alpha_{i+1}}\right)^2}{\alpha_i\alpha_j}$, which equals

$\sum_{1\leq j<i\leq n}\frac{\left(\sqrt{\alpha_i\alpha_{j+1}}-\sqrt{\alpha_j\alpha_{i+1}}\right)^2}{\alpha_i\alpha_j}$ by symmetry. Hence, S_n equals

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\left(\sqrt{\alpha_i\alpha_{j+1}}-\sqrt{\alpha_j\alpha_{i+1}}\right)^2}{\alpha_i\alpha_j}$$

$$= \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{a_{j+1}}{a_j}+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{a_{i+1}}{a_i}\right)-\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{\frac{a_{i+1}}{a_i}}\sqrt{\frac{a_{j+1}}{a_j}}$$

$$= n\sum_{i=1}^{n}\frac{a_{i+1}}{a_i}-\left(\sum_{i=1}^{n}\sqrt{\frac{a_{i+1}}{a_i}}\right)^2.$$

Now by the Cauchy-Schwatz inequality, we have $\sum_{i=1}^{n}\frac{a_{i+1}}{a_i} \le \left(n\sum_{i=1}^{n}\left(\frac{a_{i+1}}{a_i}\right)^2\right)^{\frac{1}{2}}$,

and by the AM-GM inequality, we have $\sum_{i=1}^{n}\sqrt{\frac{a_{i+1}}{a_i}} \ge n\sqrt[2n]{\prod_{i=1}^{n}\frac{a_{i+1}}{a_i}} = n.$

The inequality of the problem follows easily.

Solution 2 by Brian Bradie, Christopher Newport University, Newport News, VA

By the arithmetic mean - quadratic mean inequality,

$$\frac{1}{n}\sum_{k=1}^{n}\frac{\alpha_{k+1}}{\alpha_k} \le \left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{\alpha_{k+1}}{\alpha_k}\right)^2\right)^{1/2},$$

so it suffices to show that

$$1+\frac{1}{n^2}\sum_{1\le i<j\le n}\frac{(\sqrt{\alpha_i\alpha_{j+1}}-\sqrt{\alpha_j\alpha_{i+1}})^2}{\alpha_i\alpha_j} \le \frac{1}{n}\sum_{k=1}^{n}\frac{\alpha_{k+1}}{\alpha_k}.$$

Now,

$$\begin{aligned}\frac{(\sqrt{\alpha_i\alpha_{j+1}}-\sqrt{\alpha_j\alpha_{i+1}})^2}{\alpha_i\alpha_j} &= \left(\frac{\sqrt{\alpha_i\alpha_{j+1}}-\sqrt{\alpha_j\alpha_{i+1}}}{\sqrt{\alpha_i\alpha_j}}\right)^2\\ &= \left(\sqrt{\frac{\alpha_{j+1}}{\alpha_j}}-\sqrt{\frac{\alpha_{i+1}}{\alpha_i}}\right)^2\\ &= \frac{\alpha_{j+1}}{\alpha_j}-2\sqrt{\frac{\alpha_{j+1}\alpha_{i+1}}{\alpha_j\alpha_i}}+\frac{\alpha_{i+1}}{\alpha_i},\end{aligned}$$

so,

$$\begin{aligned}\sum_{1\le i<j\le n}\frac{(\sqrt{\alpha_i\alpha_{j+1}}-\sqrt{\alpha_j\alpha_{i+1}})^2}{\alpha_i\alpha_j} &= \sum_{1\le i<j\le n}\left(\frac{\alpha_{j+1}}{\alpha_j}-2\sqrt{\frac{\alpha_{j+1}\alpha_{i+1}}{\alpha_j\alpha_i}}+\frac{\alpha_{i+1}}{\alpha_i}\right)\\ &= (n-1)\sum_{j=1}^{n}\frac{\alpha_{j+1}}{\alpha_j}-2\sum_{1\le i<j\le n}\sqrt{\frac{\alpha_{j+1}\alpha_{i+1}}{\alpha_j\alpha_i}}.\end{aligned}$$

By the arithmetic mean - geometric mean inequality,

$$\sum_{j=1}^{n}\frac{\alpha_{j+1}}{\alpha_j} \ge n\sqrt[n]{\prod_{j=1}^{n}\frac{\alpha_{j+1}}{\alpha_j}} = n,$$

and
$$\sum_{1\le i<j\le n}\sqrt{\frac{\alpha_{j+1}\alpha_{i+1}}{\alpha_j\alpha_i}} \ge \frac{n(n-1)}{2}\sqrt[n(n-1)/4]{\prod_{1\le i<j\le n}\frac{\alpha_{j+1}\alpha_{i+1}}{\alpha_j\alpha_i}} = \frac{n(n-1)}{2}.$$

Therefore,

$$\begin{aligned}
&1+\frac{1}{n^2}\sum_{1\le i<j\le n}\frac{(\sqrt{\alpha_i\alpha_{j+1}}-\sqrt{\alpha_j\alpha_{i+1}})^2}{\alpha_i\alpha_j}\\
&\quad = 1+\frac{1}{n}\sum_{j=1}^{n}\frac{\alpha_{j+1}}{\alpha_j}-\frac{1}{n^2}\left(\sum_{j=1}^{n}\frac{\alpha_{j+1}}{\alpha_j}+2\sum_{1\le i<j\le n}\sqrt{\frac{\alpha_{j+1}\alpha_{i+1}}{\alpha_j\alpha_i}}\right)\\
&\quad \le 1+\frac{1}{n}\sum_{j=1}^{n}\frac{\alpha_{j+1}}{\alpha_j}-\frac{1}{n^2}\left(n+2\cdot\frac{n(n-1)}{2}\right)\\
&\quad = 1+\frac{1}{n}\sum_{j=1}^{n}\frac{\alpha_{j+1}}{\alpha_j}-\frac{1}{n^2}\cdot n^2\\
&\quad = \frac{1}{n}\sum_{j=1}^{n}\frac{\alpha_{j+1}}{\alpha_j}.
\end{aligned}$$

Also solved by Michel Bataille, Rouen, France; ; Carl Libis, Columbia Southern University, Orange Beach, AL; Moti Levy, Rehovot, Israel; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5552:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Find all differentiable functions $f:\Re\to\Re$ such that $f'(x)-f(-x)=e^x, \forall x\in\Re$, with $f(0)=0$.

Solution 1 by Anthony J. Bevelacqua, University of North Dakota, Great Falls, ND

Suppose $f(x)$ is a solution to

$$f'(x)-f(-x)=e^x,\quad \forall x\in\Re. \tag{1}$$

Since $f'(x)=e^x+f(-x)$ we see that $f'(x)$ is differentiable. Now

$$\begin{aligned}
f''(x) &= e^x-f'(-x)\\
&= e^x-(e^{-x}+f(x))\\
&= 2\sinh x-f(x).
\end{aligned}$$

Thus $f(x)$ is a solution to the linear differential equation

$$y''+y=2\sinh x \tag{2}$$

Therefore $f(x)=\sinh x+C\sin x+D\cos x$ for some constants C and D. (Why? It's clear that $\sinh x$ is a particular solution to (2) and that the general solution to the associated homogeneous equation $y''+y=0$ is $C\sin x+D\cos x$ for constants C and D.)

Since $f(0) = 0$ we must have $D = 0$. So $f(x) = \sinh x + C \sin x$ for some constant C. Since

$$\begin{aligned} f'(x) - f(-x) &= \cosh x + C\cos x - (-\sinh x - C\sin x) \\ &= e^x + C(\sin x + \cos x) \end{aligned}$$

and $f(x)$ is a solution of (1) we must have

$$C(\sin x + \cos x) = 0, \quad \forall x \in \Re.$$

If we set $x = 0$ we find $C = 0$. Therefore $f(x) = \sinh x$.
Since $\sinh x$ is, in fact, a solution to (1), we see that the unique solution to our original equation is $f(x) = \sinh x$.

Solution 2 by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith, Angelo State University, San Angelo, TX

We note first that since

$$f'(x) = f(-x) + e^x \tag{1}$$

for all $x \in R$ and $f(0) = 0$, we have

$$f'(0) = e^0 = 1. \tag{2}$$

Also, because equation (1) is true for all $x \in R$, it follows that $f''(x)$ exists for all $x \in R$ and

$$f''(x) = -f'(-x) + e^x.$$

Then, (1) implies that

$$f''(x) = -\left[f(x) + e^{-x}\right] + e^x$$

or

$$f''(x) + f(x) = e^x - e^{-x} \tag{3}$$

for all $x \in R$.
We can solve (3) by the usual procedures taught in introductory
differential equations courses. The first step is to solve the equation

$$f_h''(x) + f_h(x) = 0. \tag{4}$$

Since $\sin x$ and $\cos x$ are two independent solutions of (4), the general solution for (4) is $f_h(x) = c_1 \cos x + c_2 \sin x$.
Then to find a particular solution $f_p(x)$ for (3), we try $f_p(x) = ae^x + be^{-x}$ for some constants a and b. When this is substituted in (3) and the result is simplified, we get

$$2ae^x + 2be^{-x} = e^x - e^{-x}$$

for all $x \in R$. This yields $a = \frac{1}{2}$ and $b = -\frac{1}{2}$ and hence,

$$f_p(x) = \frac{1}{2}e^x - \frac{1}{2}e^{-x} = \sinh x.$$

This presents

$$\begin{aligned} f(x) &= f_h(x) + f_p(x) \\ &= c_1 \cos x + c_2 \sin x + \sinh x \end{aligned}$$

as the general solution of (3). Since $f(0) = 0$, we get

$$0 = c_1$$

and thus,

$$f(x) = c_2 \sin x + \sinh x.$$

If we follow this with condition (2), the result is

$$\begin{aligned} 1 &= f'(0) \\ &= c_2 \cos 0 + \cosh 0 \\ &= c_2 + 1 \end{aligned}$$

and hence, $c_2 = 0$ also. As a result, the only feasible solution for this problem is $f(x) = \sinh x$. To wrap things up, if $f(x) = \sinh x$, then $f'(x) = \cosh x = \dfrac{e^x + e^{-x}}{2}$ and we have

$$\begin{aligned} f'(x) - f(-x) &= \cosh x - \sinh(-x) \\ &= \frac{e^x + e^{-x}}{2} - \frac{e^{-x} - e^x}{2} \\ &= e^x \end{aligned}$$

and $f(0) = 0$. This completes our solution.

Solution 3 by Rob Downes, Newark Academy, Livingston, NJ

Assume $f(x)$ has the series solution:

$$\begin{aligned} f(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots \text{. Then,} \\ f'(x) &= c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots \text{. And,} \\ f(-x) &= c_0 - c_1 x + c_2 x^2 - c_3 x^3 + c_4 x^4 + \cdots . \end{aligned}$$

Using the fact that $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + + \dfrac{x^4}{4!} + \cdots$, and the series for $f'(x)$ and $f(-x)$ above, we substitute into the given differential equation:

$$\left(c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots\right) - \left(c_0 - c_1 x + c_2 x^2 - c_3 x^3 + c_4 x^4 + \cdots\right) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Equating coefficients, yields the system:

$$\begin{aligned} c_1 - c_0 &= 1 \\ 2c_2 + c_1 &= 1 \\ 3c_3 - c_2 &= \frac{1}{2!} \\ 4x_4 + c_3 &= \frac{1}{3!} \\ &\vdots \end{aligned}$$

Using the initial condition $f(0) = 0$ gives $c_0 = 0$. It is easy to see that

$$c_k = \begin{cases} 0 & \text{for } k \text{ even} \\ \\ \dfrac{1}{k!} & \text{for } k \text{ odd} \end{cases}$$

Substituting these coefficients back into the series for $f(x)$ gives:
$f(x) = \dfrac{x}{1!} + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \cdots$, which is the well-known series expansion for $\sinh x$.
Lastly, we note that $f(x) = \sinh(x)$ satisfies the statement of the problem.

Solution 4 by Moti Levy, Rehovot, Israel

It is straightforward to check that $f(x) = \sinh(x)$ satisfies the differential equation and initial condition.

Now it remains to check if there are more functions which satisfy the differential equation and initial condition.

Suppose $\sinh(x) + r(x)$ is such a function, then

$$\begin{aligned} (\sinh(x) + r(x))^{'-(\sinh(-x)+r(-x))} &= \cosh(x) + r^{'}(x) + \sinh(x) - r(-x) \\ &= r^{'}(x) - r(-x) + e^x. \end{aligned}$$

It follows that the function $r(x)$ must satisfy

$$r'(x) - r(-x) = 0, \quad r(0) = 0, \tag{1}$$

$$\frac{d}{dr}r'(x) = r''(x), \quad \frac{d}{dx}r(-x) = -r'(-x)$$

$$r''(x) + r'(-x) = 0, \tag{2}$$

By setting $-x$ in (1),

$$r'(-x) - (x) = 0, \tag{3}$$

Subtracting (3) from (2) results in the following differential equation for $r(x)$.

$$r^{''}(x) + r(x) = 0, \quad r(0) = 0. \tag{4}$$

Solution of (4) is

$$r(x) = \alpha \sin(x).$$

Substitution of (5) into (1) gives

$$r^{'}(x) - r(-x) = \alpha \cos(x) + \alpha \sin(x) = 0 \tag{1}$$

Hence $\alpha = 0$, and we conclude that only $f(x) = \sinh(x)$ satisfies the differential equation and initial condition.

Also solved by Ulrich Abel,Technische Hochschule Mittelhessen, Friedberg, Germany; Yagub N. Aliyev, ADA University, Baku,Azerbaijan; Hatef I. Arshagi,

Guilford Technical Community College, Jamestown NC; Brian Bradie, Christopher Newport University, Newport News, VA; Bruno Salgueiro Fanego, Viveiro, Spain; Michael N. Fried, Ben-Gurion University of the Negev, Beer-Sheva, Israel; Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News, VA; Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposer.

$$Mea - Culpa$$

Albert Stadler of Herrliberg, Switzerland noted a mistake in the featured solution to problem **5544.** The author of the solution "argues that $f(0) = 0$ and $f'(x) > 0$ for every real x. However $f(x)$ is discontinuous at $x = (2k+1)\dfrac{\pi}{2}$. So the argument is invalid."

Two solutions to the problem were received, the featured solution and one by Ed Gray of Highland Beach, FL. I shared Ed's solution with Albert and he responded as follows:
"Thanks for sharing with me the e-mail of Ed Gray. As Ed correctly points out there are infinitely many solutions to the given system of equations, contrary to what (the author of the featured solution wrote, namely that $x = y = z = 0$ is the only real solution.) Ed specifically highlights that there are infinitely many solutions with $x = y = z$. It turns out that there are many more solutions. I have devised an algorithm to find all solutions (see attachment). Initially I was not intrigued by this problem. However when I was working on it it I realized that this problem has more in it than what I originally thought. What do the French say? L'appétit vient en mangeant."

Albert's solution now follows:

- **5544:** *Proposed by Seyran Brahimov, Baku State University, Masalli, Azerbaijan*

Solve in $\Re$:

$$\begin{cases} \tan^{-1} x = & \tan y + \tan z \\ \tan^{-1} y = & \tan x + \tan z \\ \tan^{-1} z = & \tan x + \tan y \end{cases}$$

Solution by Albert Stadler, Herrliberg, Switzerland

The given system of equations is equivalent to

$$\arctan x + \tan x = \arctan y + \tan y = \arctan z + \tan z,$$

$$2(\tan x + \tan y + \tan z) = \arctan x + \arctan y + \arctan z.$$

Before we set out to solve this system of equations we make a few preliminary remarks: (i) If (x, y, z) is a solution then $(-x, -y, -z)$ is as well a solution.
(ii) If $x = y = z$ the system of equations collapses to $2 \tan x = \arctan x$. Given an integer k there is a unique real root t_k in the interval $I_k := (k\pi - \pi/2, k\pi + \pi/2)$, because

$$\lim_{x \to \pi - \pi/2} 2 \tan x - \arctan x = \infty,$$

$$\lim_{x\to\pi-\pi/2} 2\tan x - \arctan x = \infty,$$

$$\frac{d}{dx}(2\tan x - \arctan x) = \frac{2}{\cos^2 x} - \frac{1}{1+x^2} > 0,$$

so that the function $2\tan x - \arctan x$ is monotonically increasing in I_k. Therefore $(x,y,z) = (t_k, t_k, t_k)$ is a solution of the system of equations for every integer k.

(iii) If $x = 0$ then $\arctan y + \tan y = \arctan z + \tan z = 0$ & $2(\tan y + \tan z) = \arctan y + \arctan z$. Given an integer k the equation $\arctan y + \tan y = 0$ has a unique root s_k in the interval $(k\pi - \pi/2, k\pi + \pi/2)$. (Use the same argument as above.) Clearly $s_k = -s_k$. Furthermore

$$3(\tan y + \tan z) = \tan y + \arctan y + \tan z + \arctan z = 0$$

from which we deduce that $+\tan z = 0$. Then $\arctan y = -\arctan z$, and $(0, s_k, s_k), (s_k, 0, -s_k), (s_k, -s_k, 0)$ are solutions of the given system of equations for every integer k.

(iv) If (x,y,z) is a solution of the system of equations there is a real number t such that $\arctan x + \tan x = \arctan y + \tan y = \arctan z + \tan z = t$.

By remark (i) and (iii) we may assume that $t > 0$. Given $t > 0$ there is a unique real root $u_{k,t}$ of the equation $\arctan u + \tan u = t$ in the interval $(k\pi - \pi/2, k\pi + \pi/2)$. So $x = u_{a,t}, y = u_{b,t}, z = u_{c,t}$ for some integers a, b, c. The equation

$$2(\tan x + \tan y + \tan z) = \arctan x + \arctan y + \arctan z$$

implies

$3(\tan x + \tan y + \tan z) = \arctan x + \arctan y + \arctan z + \tan x + \tan y + \tan z = 3t$, or equivalently $\tan x + \tan y + \tan z = t$, as well as

$$\arctan x + \arctan y + \arctan z = 2t.$$

Therefore $\tan u_{a,t} + \tan u_{b,t} + \tan u_{ct} = t$ and $|t| < 3\pi/4$. Thus if

$$\arctan u + \tan u = t$$

for $u = u_{k,t} \in (k\pi - \pi/2, k\pi + \pi/2)$ then $|u_{k,t} - k_\pi| < 1.4$, , i.e. $u_{k,t}$ is bounded away from $k\pi \pm \pi/2$.

Let L_k be the circle centered at $k\pi$ and radius 1.4 that is run through once in the positive direction. By Cauchys integral theorem the equation $\tan u_{a,t}, + \tan u_{b,t} + \tan u_{c,t} = t$ translates to

$$t = \frac{\frac{1}{2\pi i}\int_{L_{at}} \tan z \frac{\frac{1}{1+z^2} + \frac{1}{\cos^2 z}}{\arctan z + \tan z - t} dz}{\tan u_{a,t}} + \frac{\frac{1}{2\pi i}\int_{L_{bt}} \tan z \frac{\frac{1}{1+z^2} + \frac{1}{\cos^2 z}}{\arctan z + \tan z - t} dz}{\tan u_{b,t}}$$

$$+ \frac{\frac{1}{2\pi i}\int_{L_{ct}} \tan z \frac{\frac{1}{1+z^2} + \frac{1}{\cos^2 z}}{\arctan z + \tan z - t} dz}{\tan u_{c,t}} =$$

$$\frac{1}{2\pi i}\int_{|z|=1.4} \tan z \left(\frac{\frac{1}{1+(z+\pi a)^2} + \frac{1}{\cos^2 z}}{\arctan(z+\pi a) + \tan z - t} + \frac{\frac{1}{1+(z+\pi b)^2} + \frac{1}{\cos^2 z}}{\arctan(z+\pi b) + \tan z - t} + \right.$$

$$+\frac{\frac{1}{1+(z+\pi c)^2}+\frac{1}{\cos^2 z}}{\arctan(z+\pi c)+\tan z-t}dz$$

So given a, b, c this is a transcendental equation for t that we may solve using Newton's method (https://en.wikipedia.org/wiki/Newton27s method). Once we have t we can calculate $x = u_{a,t}, y = u_{b,t}, z = u_{c,t}$ using the formula

$$u_{kt} = \frac{1}{2\pi i}\int_{|z|=1.4}(z+nk)\frac{\frac{1}{1+(z+\pi k)^2}+\frac{1}{\cos^2 z}}{\arctan(z+\pi k)+\tan z-t}dz,\ \ k \in \{a, b, c\}.$$

Let

$$f(k,t) = \frac{1}{2\pi i}int_{|z|=1.4}\tan z\frac{\frac{1}{1+(z+\pi k)^2}+\frac{1}{\cos^2 z}}{\arctan(z+\pi k)+\tan z-t}dz,$$

$$(k,t) = \frac{\partial}{\partial t}f(k,t) = \frac{1}{2\pi i}\int_{|z|=1.4}\tan z\frac{\frac{1}{1+(z+\pi k)^2}+\frac{1}{\cos^2 z}}{(\arctan(z+\pi k)+\tan z-t)^2}dz$$

Then Newton's method implies the recursion

$$\begin{aligned} t_{n+1} &= t_n - \frac{t_n - f(a,t_n) - f(b,t_n) - f(c,t_n)}{1 - g(a,t_n) + g_(b,t_n) + g(ct_n)} \\ &= \frac{f(a,t_n) - t_n g(a,t_n) + f(b,t_n) - t_n g(b,t_n) + f(c,t_n) - t_n g(c,t_n)}{1 - g(a,t_n) - g(b,t_n) - g(c,t_n)} \qquad (1) \end{aligned}$$

Let's consider as an example the case a=1, b=2, c=3. Starting with $t_1 = 1$, we find (with the help of Mathematica using above iteration):

$t_2 = 2.1209406577190077$,
$t_3 = 2.1058473431512574$,
$t_4 = 2.105843425064334$,
$t_5 = 2.105843425064067$,

We then find with $t = 2.105843425064$:
$x = u_{1,t} = 3.811219014066947$,
$y = u_{2,t} = 6.879942228618484$,
$z = u_{3,t} = 9.99040120284607$,
and we verify that these values indeed satisfy the original system of equations.

To summarize:

- We obtain all solutions of the given system of equations by starting with a triple (a,b,c) of integers.
- We next calculate t from (1) via Newton's method.
- We calculate

$$\frac{1}{2\pi i}\int_{|z|=1.4}(z+\pi k)\frac{\frac{1}{1+(z+\pi k)^2}+\frac{1}{\cos^2 z}}{\arctan(z+\pi k)+\tan z-t}dz$$

for $k = a, k = b$, and $k = c$ to obtain the solution (x, y, z) with $x \in (a\pi - \pi/2, a\pi + \pi/2), y \in (b\pi - \pi/2, b\pi + \pi/2), z \in (c\pi - \pi/2, c\pi + \pi/2)$.