Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
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- **5571:** *Proposed by Kenneth Korbin, New York, NY*

  Solve the equation; $\sqrt[3]{x^2+x} = \sqrt[3]{x} + \sqrt[3]{x^2-x}$, with $x > 0$.

- **5572:** *Proposed by Titu Zvonaru, Comănesti, Romania*

  Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$\frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b} + \frac{64}{a+b+c} \geq 34.$$

- **5573:** *Proposed by D.M. Bătinetu-Giurgiu, National College "Matei Basarab," Bucharest, Anastasios Kotronis, Athens, Greece, and Neulai Stanciu, "George Emil Palade" School, Buzău, Romania*

  Let $f : \Re_+ \to \Re_+$ be a continuous function such that $\lim\limits_{x\to\infty} \dfrac{f(x)}{x^2} = a \in \Re_+$. ($\Re_+$ stands for the positive real numbers.) Calculate:

$$\lim_{x\to\infty} \left(\sqrt[n+1]{\prod_{k=1}^{n+1} \frac{f(k)}{k}} - \sqrt[n]{\prod_{k=1}^{n} \frac{f(k)}{k}} \right).$$

- **5574:** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Mehedinti, Romania*

  Prove: If $0 < a \leq b \leq c$ then:

$$\frac{1}{1+e^{a-b+c}} + \frac{1}{1+e^{b}} \leq \frac{1}{1+e^{a}} + \frac{1}{1+e^{c}}.$$

- **5575:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Compute:
$$\int_1^{\infty} \frac{dt}{[t]^3 + 9[t]^2 + 26[t] + 24,}$$
  where $[x]$ represents the integer part of x.

- **5576:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  (a) Calculate
$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{2}{n+1} + \frac{2}{n+2} - \frac{2}{n+3} + \cdots \right).$$
  (b) Find the domain of convergence and the sum of the power series
$$\sum_{n=1}^{\infty} x^n \left(\frac{1}{n} - \frac{2}{n+1} + \frac{2}{n+2} - \frac{2}{n+3} + \cdots \right).$$

Solutions

5553: *Proposed by Kenneth Korbin, New York, NY*

A triangle with sides $(x, x, 57)$ has the same area as a triangle with sides $(x+1, x+1, 55)$. Find x.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

From Heron's Formula the area of the isosceles triangle with sides $(x, x, 57)$ is:
$$\sqrt{\frac{x+x+57}{2}\left(\frac{x+x+57}{2} - x\right)\left(\frac{x+x+57}{2} - x\right)\left(\frac{x+x+57}{2} - 57\right)} =$$
$$\sqrt{\left(x + \frac{57}{2}\right)\left(\frac{57}{2}\right)^2\left(x - \frac{57}{2}\right)},$$

and the area of the isosceles triangle with sides $(x+1, x+1, 55)$ that has the same perimeter as the one above is, after simplification, $\sqrt{\left(x + \frac{57}{2}\right)\left(\frac{55}{2}\right)^2\left(x - \frac{53}{2}\right)}$.
Hence, both triangles have the same area if, and only if,
$$\left(x + \frac{57}{2}\right)\left(\frac{57}{2}\right)^2\left(x - \frac{57}{2}\right) = \left(x + \frac{57}{2}\right)\left(\frac{55}{2}\right)^2\left(x - \frac{53}{2}\right), \text{ which implies}$$
$$x = \frac{6217}{112}.$$

Solution 2 by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Karl Havlak, Angelo State University, San Angelo, TX

We begin by considering a slightly more general problem. For positive integers n and k, let $\triangle_1$ be the triangle with sides $(x, x, n+2k)$ and $\triangle_2$ be the triangle with sides $(x+k, x+k, n)$. The object is to find the value of x (in terms of n and k) which makes $area\,(\triangle_1) = area\,(\triangle_2)$. Note that both triangles have the same semiperimeter

$$s = \frac{2x+n+2k}{2}.$$

By Heron's Formula,

$$\begin{aligned} area\,(\triangle_1) &= \sqrt{s\,(s-x)\,(s-x)\,(s-n-2k)} \\ &= \sqrt{\left(\frac{2x+n+2k}{2}\right)\left(\frac{n+2k}{2}\right)^2\left(\frac{2x-n-2k}{2}\right)} \\ &= \frac{n+2k}{4}\sqrt{(2x+n+2k)\,(2x-n-2k)} \end{aligned} \tag{1}$$

and

$$\begin{aligned} area\,(\triangle_2) &= \sqrt{s\,(s-x-k)\,(s-x-k)\,(s-n)} \\ &= \sqrt{\left(\frac{2x+n+2k}{2}\right)\left(\frac{n}{2}\right)^2\left(\frac{2x+2k-n}{2}\right)} \\ &= \frac{n}{4}\sqrt{(2x+2k+n)\,(2x+2k-n)}. \end{aligned} \tag{2}$$

Then, $area\,(\triangle_1) = area\,(\triangle_2)$ if and only if

$$(n+2k)^2\,(2x-n-2k) = n^2\,(2x-n+2k)\,.$$

If we solve for x, the result is

$$\begin{aligned} x &= \frac{2n^2+3nk+2k^2}{2\,(n+k)} \\ &= \frac{n^2}{2\,(n+k)} + \frac{(n+k)\,(n+2k)}{2\,(n+k)} \\ &= \frac{n^2}{2\,(n+k)} + \frac{n+2k}{2}. \end{aligned} \tag{3}$$

For the given problem, we have $n = 55$ and $k = 1$ and hence (3) gives

$$x = \frac{55^2}{2\,(56)} + \frac{57}{2} = \frac{6217}{112}.$$

With some perseverance, it follows that (1) and (2) yield

$$area\,(\triangle_1) = \frac{(55)\,(57)\,(97)}{224} = area\,(\triangle_2)\,.$$

Thus, when $n = 55$ and $k = 1$, both x and the equal areas of our triangles are rational numbers.

It is natural to ask whether it's possible to find integral values of n and k for which (1), (2), and (3) provide positive integral values for x and the areas of $\triangle_1$ and $\triangle_2$. This would present Heronian Triangles $\triangle_1$ and $\triangle_2$ of the types described above for which

$area\,(\triangle_1) = area\,(\triangle_2)$. After much experimentation and trial and error, we found an infinite family of examples where this is true.

If we restrict our attention to triangles $\triangle_1$ and $\triangle_2$ where $n = 3k$, then by (3),

$$x = \frac{9k^2}{8k} + \frac{5k}{2} = \frac{29k}{8}.$$

Next, set $k = 8i$ for some positive integer i, and the results are $n = 24i$ and $x = 29i$. Then, $\triangle_1$ and $\triangle_2$ have the common semiperimeter

$$s = \frac{2x + n + 2k}{2} = \frac{98i}{2} = 49i$$

and (1) and (2) provide

$$\begin{aligned} area\,(\triangle_1) &= \frac{40i}{4}\sqrt{(98i)\,(18i)} \\ &= (10i)\,(42i) \\ &= 420i^2 \end{aligned}$$

and

$$\begin{aligned} area\,(\triangle_2) &= \frac{24i}{4}\sqrt{(98i)\,(50i)} \\ &= (6i)\,(70i) \\ &= 420i^2. \end{aligned}$$

Hence, for $i \geq 1$, the triangles $\triangle_1$ with sides $(29i, 29i, 40i)$ and $\triangle_2$ with sides $(37i, 37i, 24i)$ are Heronian Triangles of our prescribed type with common area $420i^2$.

Editor: **David Stone and John Hawkins of Georgia Southern University** also placed the problem into a more general setting. First they showed that there is a unique solution; the one given above. Then they claimed that in the statement of the problem we are given two triangles with identical areas and identical perimeters. So instead of using 57 as the base of the initial triangle, let its side lengths be $(x, x, 2b)$ and let the second triangle have side lengths of $(x+1, x+1, 2(b-1))$, where $b > 1$. They then showed that in such a triangle $x = \dfrac{4b^2 - 5b + 2}{2b - 1}$. Returning to the problem with base 57, gave them in their generalization $b = 57/2$, and thus $x = \dfrac{6217}{112} = 55 + 57/112 = 55.5089$.

They commented on this answer as follows:

The result is surprising in two ways.

There is a unique solution for any size base.

That unique triangle satisfying the given transformation is almost equilateral.

To see this, let's rewrite our result in terms of $B = 2b$, the base of the original triangle: $x = \dfrac{2B^2 - 5B + 4}{2(B-1)} = \left(B - \dfrac{3}{2}\right) + \dfrac{1}{2B - 2}$.

That is, the original triangle is (approximately) $\left(B - \dfrac{3}{2}, B - \dfrac{3}{2}, B\right)$, while the transformed triangle is (approximately) $\left(B - \dfrac{1}{2}, B - \dfrac{1}{2}, B - 2\right)$.

Very close to equilateral! In fact, the base is the longest side of the original triangle, but simply shifting 2 units to the sides forces the base to become the shortest side.

With the data from the given problem, the triangle $(55.51, 55.51, 57)$ is transformed to the triangle $(56.51, 56.51, 55)$, which has the same area and perimeter.

The result is even more striking with a larger base, say $B = 1000$ (so $x = 998.5$). Then the triangle $(998.5, 998.5, 1000)$ is transformed to the triangle $(999.5, 999.5, 998)$.

Also solved by Ulrich Abel, Technische Hochschule Mittelhessen, Germany and Manfred Hauben, Pzer Inc and NYU Langone Health, USA; Michel Bataille, Rouen, France; Brian D. Beasley, Presbyterian College, Clinton, SC; Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND; Brian Bradie, Christopher Newport University, Newport News, VA; Michael C. Faleski, Delta College, University Center, Midland, MI; Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; David A. Huckaby, Angelo State University, San Angelo,TX; Kee-Wai Lau, Hong Kong, China; Carl Libis, Columbia Southern University Orange Beach, AL; David E. Manes, Oneonta, NY; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Seán M. Stewart, Bomaderry, NSW, Australia; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Southern Georgia University, Statesboro, GA; Daniel Văcaru, Pitesti, Romania, and the proposer.

5554: *Proposed by Michel Bataille, Rouen, France*

Find all pairs of complex numbers (a, b) such that the polynomial $x^5 + x^2 + ax + b$ has two roots of multiplicity 2.

Solution 1 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

Let p, q, and r be complex numbers. We calculate $(x^2 + px + q)^2(x + r)$ to find

$$x^5 + (2p + r)x^4 + (p^2 + 2q + 2pr)x^3 + (2pq + p^2r + 2qr)x^2 + (q^2 + 2pqr)x + q^2r.$$

So $x^5 + x^2 + ax + b = (x^2 + px + q)^2(x + r)$ if and only if

1. $0 = 2p + r$
2. $0 = p^2 + 2q + 2pr$
3. $1 = 2pq + p^2r + 2qr$
4. $a = q^2 + 2pqr$
5. $b = q^2r$

Now suppose $f(x) := x^5 + x^2 + ax + b$ has two roots of multiplicity two. Then it will factor as in the last paragraph. From (1) we see $r = -2p$. Substituting this in (2) we find $q = \dfrac{3}{2}p^2$. Substituting these expressions for q and r into (3) yields $p^3 = -\dfrac{1}{5}$. Now (4) and (5) give $a = \dfrac{3}{4}p$ and $b = \dfrac{9}{10}p^2$.

Conversely if p is a root of $p^3 = -\dfrac{1}{5}$ and we set $q := \dfrac{3}{2}p^2$, $r := -2p$, $a := \dfrac{3}{4}p$, and $b := \dfrac{9}{10}p^2$ then the first paragraph shows that $f(x) = (x^2 + px + q)^2(x + r)$. The roots of

$x^2 + px + q$ are $\dfrac{-p \pm p\sqrt{5}i}{2}$. These are distinct and neither is $-r$. Thus $f(x)$ has three distinct roots of multiplicities two, two, and one.

Thus $x^5 + x^2 + ax + b$ has two roots of multiplicity two if and only if

$$a = \frac{3}{4}p \text{ and } b = \frac{9}{10}p^2$$

where p is one of the three roots of $p^3 = -\dfrac{1}{5}$.

Solution 2 by Brian Bradie, Christopher Newport University, Newport News, VA

Let $p(x) = x^5 + x^2 + ax + b$, and let α and β denote the two roots of multiplicity 2 of p. Because the coefficient of x^4 in p is zero, the sum of the roots of p is zero. This means the five roots of p are

$$r_1 = \alpha, \quad r_2 = \alpha, \quad r_3 = \beta, \quad r_4 = \beta, \quad \text{and} \quad r_5 = -2\alpha - 2\beta.$$

By Vieta's formulas,

$$\sum_{cyclic} r_i r_j = -3\alpha^2 - 4\alpha\beta - 3\beta^2 = -3(\alpha+\beta)^2 + 2\alpha\beta = 0 \tag{1}$$

and

$$\begin{aligned} \sum_{cyclic} r_i r_j r_k &= -2\alpha^3 - 8\alpha^2\beta - 8\alpha\beta^2 - 2\beta^3 \\ &= -2(\alpha+\beta)\left[(\alpha+\beta)^2 + \alpha\beta\right] = -1. \end{aligned} \tag{2}$$

From (1),

$$\alpha\beta = \frac{3}{2}(\alpha+\beta)^2.$$

Substitution into (2) yields

$$(\alpha+\beta)^3 = \frac{1}{5},$$

or

$$\alpha + \beta = \sqrt[3]{\frac{1}{5}}\left(\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right),$$

for $k = 0, 1, 2$. Once again using Vieta's formulas,

$$\begin{aligned} a &= \sum_{cyclic} r_i r_j r_k r_\ell = -4\alpha^3\beta - 7\alpha^2\beta^2 - 4\alpha\beta^2 \\ &= -\alpha\beta\left[4(\alpha+\beta)^2 - \alpha\beta\right] = -\frac{15}{4}(\alpha+\beta)^4, \end{aligned}$$

and

$$b = -r_1 r_2 r_3 r_4 r_5 = 2(\alpha+\beta)\alpha^2\beta^2 = \frac{9}{2}(\alpha+\beta)^5.$$

Thus, there are three pairs of complex numbers (a, b) such that the polynomial $x^5 + x^2 + ax + b$ has two roots of multiplicity 2:

$$\begin{aligned}
(a,b) &= \left(-\frac{3}{4\sqrt[3]{5}}, \frac{9}{10\sqrt[3]{25}}\right) \\
(a,b) &= \left(-\frac{3}{4\sqrt[3]{5}}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right), \frac{9}{10\sqrt[3]{25}}\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\right) \\
(a,b) &= \left(-\frac{3}{4\sqrt[3]{5}}\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right), \frac{9}{10\sqrt[3]{25}}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right).
\end{aligned}$$

Solution 3 by Albert Stadler, Herrliberg, Switzerland

By assumption there are complex numbers u, v, w such that

$$x^5 + x^2 + ax + b = (x-u)^2(x-u)^2(x-w).$$

Comparing coefficients we find that

$$\begin{gathered}
2u + 2v + w = 0, \\
(u+v)^2 + 2uv + 2vw + 2wu = 0, \\
2u^2v + 2uv^2 + u^2w + 24uvw + v^2w = -1, \\
u^2v^2 + 2u^2vw + 2uv^2w = a, \\
u^2v^2w = -b.
\end{gathered}$$

Let $x = u + v,\ y = uv$. Then above equations translate to

$$\begin{gathered}
w = -2x, \\
3x^2 = 2y, \\
2xy - 2x(x^2 + 2y) = -1 \\
y^2 - 4x^2y = a, \\
2xy^2 = b.
\end{gathered}$$

Combining the second and third equation we find that $3x^3 - 2x(4x^2) = -1$. So

$$(x,y) \in \left\{\left(\frac{1}{\sqrt[3]{5}}e^{\frac{2\pi ik}{3}}, \frac{3}{2\sqrt[3]{25}}e^{-\frac{2\pi ik}{3}}\right\}.$$

Given x and y the variables u and v can be recovered from the equations $x = u + v,\ y = uv$:

$$\{u, v\} = \left\{\frac{x - \sqrt{x^2 - 4y}}{2}, \frac{x + \sqrt{x^2 - 4y}}{2}\right\}.$$

The case $k = 0$ yields $a = y^2 - 4x^2y = -\frac{3}{4\sqrt[3]{5}},\ b = 2xy^2 = \frac{9}{10\sqrt[3]{25}}$ and

$$x^5 + x^2 - \frac{3}{4\sqrt[3]{5}}x + \frac{9}{10\sqrt[3]{25}} = \left(x - \frac{1}{2\sqrt[3]{5}}(1+i\sqrt{5})\right)^2\left(x - \frac{1}{2\sqrt[3]{5}}(1-i\sqrt{5})\right)^2\left(x + \frac{2}{\sqrt[3]{5}}\right).$$

The case $k=1$ yields $a=y^2-4x^2y=-\dfrac{3}{4\sqrt[3]{5}}e^{\frac{2\pi i}{3}}$, $b=2xy^2=\dfrac{9}{10\sqrt[3]{25}}e^{\frac{-2\pi i}{3}}$ and

$$x^5+x^2-\frac{3e^{\frac{2\pi i}{3}}}{4\sqrt[3]{5}}x+\frac{9e^{\frac{-2\pi i}{3}}}{10\sqrt[3]{25}}=\left(x-\frac{e^{\frac{2\pi i}{3}}}{2\sqrt[3]{5}}(1+i\sqrt{5})\right)^2\left(x-\frac{e^{\frac{2\pi i}{3}}}{2\sqrt[3]{5}}(1-i\sqrt{5})\right)^2\left(x+\frac{2e^{\frac{2\pi i}{3}}}{\sqrt[3]{5}}\right).$$

The case $k=2$ yields $a=y^2-4x^2y=-\dfrac{3}{4\sqrt[3]{5}}e^{-\frac{2\pi i}{3}}$, $b=2xy^2=\dfrac{9}{10\sqrt[3]{25}}e^{\frac{2\pi i}{3}}$ and

$$x^5+x^2-\frac{3e^{\frac{-2\pi i}{3}}}{4\sqrt[3]{5}}x+\frac{9e^{\frac{2\pi i}{3}}}{10\sqrt[3]{25}}=\left(x-\frac{e^{-\frac{2\pi i}{3}}}{2\sqrt[3]{5}}(1+i\sqrt{5})\right)^2\left(x-\frac{e^{-\frac{2\pi i}{3}}}{2\sqrt[3]{5}}(1-i\sqrt{5})\right)^2\left(x+\frac{2e^{-\frac{2\pi i}{3}}}{\sqrt[3]{5}}\right).$$

Solution 4 by Kee-Wai Lau, Hong Kong, China

We show that the polynomial x^5+x^2+ax+b has two roots of multiplicity 2 if and only if

$$(a,b)=\left(\frac{-3}{4\sqrt[3]{5}},\frac{9}{10\sqrt[3]{25}}\right),\ \left(\frac{3\left(1-\sqrt{3}i\right)}{8\sqrt[3]{5}},\frac{-9\left(1+\sqrt{3}i\right)}{20\sqrt[3]{25}}\right),\ \left(\frac{3\left(1+\sqrt{3}i\right)}{8\sqrt[3]{5}},\frac{9\left(-1+\sqrt{3}i\right)}{20\sqrt[3]{25}}\right),$$

where $i=\sqrt{-1}$.

Suppose that $x^5+x^2+ax+b\equiv(x-p)^2(x-q)^2(x-r)$, where p,q,r are complex numbers. By equating coefficients of x^4,x^3,x^2,x and the constants, we obtain

$$r=-2(p+q) \tag{1}$$

$$p^2+4pq+q^2+2r(p+q)=0 \tag{2}$$

$$2pq(p+q)+r(p^2+4pq+q^2)=-1 \tag{3}$$

$$1=pq(pq+2r(p+q)), \tag{4}$$

and

$$b=-(pq)^2r, \tag{5}$$

By substituting r of (1) into (2), we obtain $3p^2+4pq+3q^2=0$, or

$$pq=\frac{3(p+q)^2}{2} \tag{6}$$

By substituting r of (1) into (3), we obtain $2(p+q)((p+q)^2+pq)=1$, and so by (6), we have

$$(p+q)^3=\frac{1}{5} \tag{7}$$

It follows that $p+q=\dfrac{1}{\sqrt[3]{5}},\ \dfrac{-1+\sqrt{3}i}{2\sqrt[3]{5}},\ \dfrac{-(1+\sqrt{3}i)}{2\sqrt[3]{5}}$. By (1), (4), (6) and (7) we have

$a=\dfrac{-15(p+q)^4}{4}=\dfrac{-3(p+q)}{4}$. By (1), (5), (6) and (7), we have

$b=\dfrac{(p+q)^5}{2}=\dfrac{9(p+q)^2}{10}$, and we obtain (a,b) as stated above. It can be checked readily that in fact

$$x^5+x^2-\frac{3}{4\sqrt[3]{5}}x+\frac{9}{10\sqrt[3]{25}} \equiv \left(x-\frac{\sqrt[3]{25}-\sqrt[6]{78125}i}{10}\right)^2\left(x-\frac{\sqrt[3]{25}+\sqrt[6]{78125}i}{10}\right)^2\left(x+\frac{2}{\sqrt[3]{5}}\right),$$

$$x^5+x^2+\frac{3(1-\sqrt{3}i)}{8\sqrt[3]{5}}x-\frac{9(1+\sqrt{3}i)}{20\sqrt[3]{25}} \equiv$$
$$\left(x+\frac{1+\sqrt{15}+(\sqrt{5}-\sqrt{3})i}{4(\sqrt[3]{5})}\right)^2\left(x-\frac{\sqrt{15}-1+(\sqrt{5}+\sqrt{(3)})i}{4(\sqrt[3]{5})}\right)^2\left(x+\frac{-1+\sqrt{3}i}{\sqrt[3]{5}}\right),$$

and

$$x^5+x^2+\frac{3(1+\sqrt{3}i)}{8\sqrt[3]{5}}x+\frac{9(-1+\sqrt{3}i)}{20\sqrt[3]{25}} \equiv$$
$$\left(x+\frac{1+\sqrt{15}-(\sqrt{5}-\sqrt{3})i}{4(\sqrt[3]{5})}\right)^2\left(x-\frac{\sqrt{15}-1-(\sqrt{5}+\sqrt{3})i}{4(\sqrt[3]{5})}\right)^2\left(x-\frac{-1+\sqrt{3}i}{\sqrt[3]{5}}\right).$$

Solution 5 by Albert Natian, Los Angeles Valley College, Valley Glen, CA

First Solution:

Set

$$f(x) := x^5+x^2+ax+b$$

and let u and v be the two roots each of multiplicity 2, and r another root, so that

$$f(x) = (x-u)^2(x-v)^2(x-r)$$

which can be re-expressed as

$$f(x) = x^5-(2s+r)x^4+\left(2sr+s^2+2p\right)x^3-\left(r\left[s^2+2p\right]+2sp\right)x^2+\left(2spr+p^2\right)x-rp^2$$

where

$$s = u+v \qquad \text{and} \qquad p = uv.$$

Comparing the two expressions of $f(x)$, we see that

$$2s+r=0, \qquad 2sr+s^2+2p=0, \qquad r\left[s^2+2p\right]+2sp=-1,$$
$$a=2spr+p^2, \qquad b=-rp^2,$$

which imply

$$s^3=\frac{1}{5}, \qquad a=-\frac{3}{4}s, \qquad b=\frac{9}{50}s^{-1}.$$

Since there are three solutions

$$s_k = \frac{1}{\sqrt[3]{5}}e^{ik\frac{2\pi}{3}} \quad \text{with} \quad k\in\{0,1,2\}$$

to the first of the latter three equalities, then the sought-after pairs of complex numbers (a,b) are:

$$(a_k,b_k) = \left(-\frac{3}{4\sqrt[3]{5}}\,e^{ik\frac{2\pi}{3}},\ \frac{9\sqrt[3]{5}}{50}\,e^{-ik\frac{2\pi}{3}}\right) \quad \text{with} \quad k\in\{0,1,2\}.$$

Second Solution:

It's a fact that a polynomial $P(x)$ and its derivative $P'(x)$ have a common root ρ if and only if ρ is a root of $P(x)$ of multiplicity greater than 1.
Set

$$f(x) := x^5 + x^2 + ax + b$$

and let u and v be the two distinct roots each of multiplicity 2, and set $s := u + v$ and $p := uv$. Then

$$f'(x) = 5x^4 + 2x + a$$

and

$$3x^2 + 4ax + 5b = 5f(x) - xf'(x)$$

which implies the polynomial $3x^2 + 4ax + 5b$ – which is a greatest common divisor of $f(x)$ and $f'(s)$ – has distinct roots u and v, common to $f(x)$ and $f'(x)$, and so

$$-\frac{4a}{3} = s \quad \text{and} \quad \frac{5b}{3} = p;$$

$$4a = -3s \quad \text{and} \quad 5b = 3p,$$

$$a = \frac{-3}{4}\, s \quad \text{and} \quad b = \frac{3}{5}\, p,$$

Since

$$f'(u) = f'(v) = 0,$$

then

$$f'(u) - f'(v) = 5\left(u^4 - v^4\right) + 2\,(u - v) = 0,$$

$$5\left((u+v)^2 - 2uv\right)(u+v) + 2 = 0,$$

$$5\left(s^2 - 2p\right)s + 2 = 0,$$

$$10ps = 5s^3 + 2,$$

$$p = \frac{5s^3 + 2}{10s},$$

Also since

$$5u^4 = -2u - a \quad \text{and} \quad 2v + a = -5v^4,$$

then

$$\left(5u^4\right)(2v + a) = (-2u - a)\left(-5v^4\right),$$

$$2u^4v + au^4 = 2v^4u + av^4,$$

$$2u^4v - 2v^4u + +au^4 - av^4 = 0,$$

$$2uv\left(u^3 - v^3\right) + a\left(u^4 - v^4\right) = 0,$$

$$2uv\left(u^2 + uv + v^2\right) + a\left(u^2 + v^2\right)(u+v) = 0,$$

$$2p\left(s^2 - p\right) + \left(-\frac{3s}{4}\right)\left(s^2 - 2p\right)s = 0,$$

$$14ps^2 - 8p^2 - 3s^4 = 0,$$

$$35s^3\,(10ps) - 2\,(10ps)^2 - 75s^6 = 0,$$

$$35s^3\left(5s^3 + 2\right) - 2\left(5s^3 + 2\right)^2 - 75s^6 = 0,$$

$$25s^6 + 15s^3 - 4 = 0,$$

$$s^3 = \frac{1}{5} \qquad \text{or} \qquad s^3 = -\frac{4}{5}.$$

We cannot have $s^3 = -4/5$ for if it were so, then for the real solution

$$s = \left(-\frac{4}{5}\right)^{\frac{1}{3}} = -2^{\frac{2}{3}}\ 5^{-\frac{1}{3}},$$

the above quadratic polynomial, that must have two distinct roots u and v, can be written as

$$3x^2 + 4ax + 5b = 3x^2 + 6\left(10^{-\frac{1}{3}}\right)x + 3\left(10^{-\frac{1}{3}}\right)^2 = 3\left(x + 10^{-\frac{1}{3}}\right)^2$$

whose single root of multiplicity 2 is $-10^{-\frac{1}{3}}$, which poses a contradiction! Same goes with the other solutions of $s^3 = -4/5$.
So, we can only have $s^3 = \frac{1}{5}$, which has three solutions

$$s_k = \frac{1}{\sqrt[3]{5}} e^{ik\frac{2\pi}{3}} \quad \text{with} \quad k \in \{0, 1, 2\}$$

leading to three sought-after pairs of complex numbers (a, b):

$$(a_k, b_k) = \left(-\frac{3}{4\sqrt[3]{5}}\ e^{ik\frac{2\pi}{3}},\ \frac{9\sqrt[3]{5}}{50}\ e^{-ik\frac{2\pi}{3}}\right) \quad \text{with} \quad k \in \{0, 1, 2\}.$$

Solution 6 by Seán M. Stewart, Bomaderry, NSW, Australia

We will show there are three pairs of complex numbers (a, b) such that the polynomial $p(x) = x^5 + x^2 + ax + b$ has two roots of multiplicity 2.
Denote the five roots of the polynomial $p(x)$ by: $\alpha, \alpha, \beta, \beta, z_1$. As the polynomial has two roots of multiplicity 2, the two roots α and β are also roots of the polynomial

$$p'(x) = 5x^4 + x + a.$$

The two roots with multiplicities of 2 therefore satisfy both of the following equations

$$x^5 + x^2 + ax + b = 0, \tag{1}$$

and

$$5x^4 + x + a = 0. \tag{2}$$

Multiplying (1) by a factor of 5 and (2) by x before taking their difference yields

$$3x^2 + 4ax + 5b = 0, \tag{3}$$

with the two roots to (3) being α and β. Applying Viète's formulae to the quadratic appearing in (3) gives

$$\sum_{i=1}^{2} r_i = \alpha + \beta = -\frac{4a}{3}, \tag{4}$$

and

$$\sum_{\substack{i,j=1\\ i\neq j}}^{2} r_i r_j = \alpha\beta = \frac{5b}{3}. \tag{5}$$

Here $r_1 = \alpha$ and $r_2 = \beta$ are the 2 roots of (3).
Next, applying Viète's formulae to $p(x) = 0$ appearing in (1) gives

$$\sum_{i=1}^{5} r_i = 2\alpha + 2\beta + z_1 = 0, \tag{6}$$

$$\sum_{\substack{i,j=1\\ i<j}}^{5} r_i r_j = \alpha^2 + \beta^2 + 4\alpha\beta + 2z_1(\alpha+\beta) = (\alpha+\beta)^2 + 2\alpha\beta + 2z_1(\alpha+\beta) = 0, \tag{7}$$

and

$$\sum_{\substack{i,j,k,l,m\\ i<j<k<l<m}}^{5} r_i r_j r_k r_l r_m = z_1(\alpha\beta)^2 = -b. \tag{8}$$

Here $r_1 = r_2 = \alpha, r_3 = r_4 = \beta$, and $r_5 = z_1$ are the 5 roots of (1). Combining (4) with (6) gives

$$z_1 = -2(\alpha+\beta) = \frac{8a}{3}. \tag{9}$$

Combining (7) with (4), (5), and (9) gives

$$b = \frac{8a^2}{5}. \tag{10}$$

Finally, combining (8) with (5) and (9) gives

$$b\left(1 + \frac{200}{27}ab\right) = 0. \tag{11}$$

If $b = 0$ then from (10), $a = 0$ and it follows the polynomial $p(x)$ would reduce to

$$p(x) = x^5 + x^2 = x^2(x^3+1) = x^2(x+1)(x^2-x+1),$$

a polynomial with only one root of multiplicity 2. Thus $a, b \neq 0$. Returning to (11), we therefore have

$$ab = -\frac{27}{200}. \tag{12}$$

Eliminating b from (10) and (12) leads to the following cubic equation in a

$$a^3 + \frac{27}{320} = 0.$$

Solving for a, the three solutions give rise to three pairs of complex numbers (a, b) that result in the polynomial $p(x)$ having two roots of multiplicity 2. They are:

1. $(a,b) = \left(-\dfrac{3}{4\sqrt[3]{5}}, \dfrac{9\sqrt[3]{5}}{50}\right)$,

2. $(a,b) = \left(-\dfrac{3}{8\sqrt[3]{5}}(-1+i\sqrt{3}), -\dfrac{9\sqrt[3]{5}}{100}(1+i\sqrt{3})\right)$, and

3. $(a,b) = \left(\dfrac{3}{8\sqrt[3]{5}}(1+i\sqrt{3}), \dfrac{9\sqrt[3]{5}}{100}(-1+i\sqrt{3})\right)$.

Remark:
When $a, b \in R$ the 5 roots to the polynomial $p(x)$ have a particularly nice closed form. They are:

$$\begin{aligned} z_1 &= -\frac{2}{\sqrt[3]{5}} \\ \alpha &= \frac{1}{2}\left(\frac{1}{\sqrt[3]{5}} + i\sqrt[6]{5}\right) \\ \alpha &= \frac{1}{2}\left(\frac{1}{\sqrt[3]{5}} + i\sqrt[6]{5}\right) \\ \beta &= \frac{1}{2}\left(\frac{1}{\sqrt[3]{5}} - i\sqrt[6]{5}\right) \\ \beta &= \frac{1}{2}\left(\frac{1}{\sqrt[3]{5}} - i\sqrt[6]{5}\right) \end{aligned}$$

Editor : **David Stone and John Hawkins of Georgia Southern University** accompanied their solution with a general comment about the multiplicity of roots which was used less explicitly in some of the above solutions. They stated: "Let $p(x) = x^5 + x^2 + ax + b$, so $p'(x) = 5x^4 + 2x + a$. A complex number r is a double root of multiplicity greater than 1 (of $p(x)$) if and only if r is also a root of the derivative. In this case, the factor $x - r$ is a factor of the GCD (p, p'). So in order to have two double roots of $p(x)$, the GCD must be a quadratic with two roots. This GCD can be computed by the Euclidean Algorithm. Stop the repeated division process so the GCD is a quadratic involving a and b; the next would-be remainder is a linear term which also involves a and b in its coefficients. Then solve for a and b which force the linear term to become zero and the quadratic to have distinct roots."

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown NC; Brian D. Beasley, Presbyterian College, Clinton, SC; Daniel Văcaru Pitesti, Romania; David Stone and John Hawkins of Georgia Southern University, Statesboro,GA, and the proposer.

5555: *Proposed by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy*

Show that $x^x - 1 \leq x^{1-x^2} e^{x-1} (x-1)$ for $0 < x \leq 1$.

Solution 1 by Michel Bataille, Rouen, France

Equality holds when $x = 1$. Let x be such that $0 < x < 1$. Since $0 < x^x < 1$, the required inequality rewrites as

$$\frac{1 - x^x}{1 - x} \geq x^{1-x^2} e^{x-1}.$$

It is known that $x^x \leq x^2 - x + 1$ (four proofs can be found in *Crux Mathematicorum*, 40(2), February 2014, p. 82-3), hence $\frac{1-x^x}{1-x} \geq x$. As a result, it is enough to show that $x \geq x^{1-x^2} e^{x-1}$.

Taking logarithms, the latter is equivalent to $x^2 \ln x + 1 - x \geq 0$, which itself is equivalent to $f(1/x) \geq 0$ where $f(u) = u^2 - u - \ln u$.
Now, $f'(u) = \frac{(u-1)(2u+1)}{u}$ is nonnegative on the interval $[1, \infty)$, hence f is increasing on $[1, \infty)$. Thus $f(1/x) \geq f(1) = 0$, as desired.

Solution 2 by Moti Levy, Rehovot, Israel

We substitute $x = 1 - y$ in the original inequality. The equivalent inequality is

$$(1-y)^{1-y} + (1-y)^{y(2-y)}\, e^{-y} y \leq 1. \tag{1}$$

The binomial theorem:

$$(1-y)^z = 1 - zy + \frac{1}{2!} z(z-1) y^2 - \frac{1}{3!} z(z-1)(z-2) y^3 + \cdots \tag{2}$$

$$= 1 - zy + \frac{1}{2!} z(z-1) y^2 + \sum_{n=3}^{\infty} (-1)^n \frac{(z)_n}{n!} \tag{3}$$

If $0 \leq y < 1$ and $0 \leq z < 1$ then the tail is negative,

$$\sum_{n=3}^{\infty} (-1)^n \frac{(z)_n}{n!} \leq 0. \tag{4}$$

This fact follows from

$$\frac{|(z)_n|}{n!} \geq \frac{|(z)_{n+1}|}{(n+1)!}, \quad n \geq 3$$

which is true since

$$\frac{|(z)_n|\,(n+1)!}{n!\,|(z)_{n+1}|} = (n+1)\,|z - n + 2| \geq 1, \quad n \geq 3.$$

It follows from (4) that

$$(1-y)^z \leq 1 - zy + \frac{1}{2!} z(z-1) y^2. \tag{5}$$

Using (5), we obtain two inequalities.
Set $z = 1 - y$ in (5) to obtain

$$(1-y)^{1-y} \leq 1 - y(1-y) + \frac{1}{2} y^2 (1-y)(-y) \tag{6}$$

$$= \frac{1}{2} y^4 - \frac{1}{2} y^3 + y^2 - y + 1, \quad 0 \leq y < 1,$$

and set $z = y(2-y)$ to obtain

$$(1-y)^{y(2-y)} \leq 1 - y(2-y)y + \frac{1}{2} y(2-y)(y(2-y) - 1) y^2 \tag{7}$$

$$= 1 - 2y^2 + \frac{5}{2} y^4 - 2y^5 + \frac{1}{2} y^6 \qquad 0 \leq y < 1.$$

It is also known that

$$e^{-y} \leq 1 - y + \frac{1}{2} y^2 \qquad 0 \leq y < 1. \tag{8}$$

Now by plugging (6), (7) and (8) instead of the respective terms in the left hand side of (1), we see that it is enough to prove that for $0 \le y < 1$

$$\left(1 - y + y^2 - \frac{1}{2}y^3 + \frac{1}{}^4\right) + \left(1 - 2y^2 + \frac{5}{2}y^4 - 2y^5 + \frac{1}{2}y^6\right)\left(1 - y + \frac{1}{2}y^2\right)y \le 1 \quad (9)$$

Simplification of the left hand side of (9) gives

$$\frac{1}{4}y^9 - \frac{3}{2}y^8 + \frac{15}{4}y^7 - \frac{9}{2}y^6 + \frac{3}{2}y^5 + \frac{5}{2}y^4 - 2y^3 + 1 \le 1, \qquad 0 \le y < 1.$$

or

$$\frac{1}{4}y^9 - \frac{3}{2}y^8 + \frac{15}{4}y^7 - \frac{9}{2}y^6 + \frac{3}{2}y^5 + \frac{5}{2}y^4 - 2y^3 \le 0 \quad 0 \le y < 1. \qquad (10)$$

Factoring the left hand side of (10) gives

$$\frac{1}{4}y^3\,(y-1)\left(8 - 2y - 8y^2 + 10y^3 - 5y^4 + y^5\right) \le 0, \qquad 0 \le y < 1.$$

We end up with showing that the polynomial $P(y) := 8 - 2y - 8y^2 + 10y^3 - 5y^4 + y^5$ is positive in the interval $[0, 1)$.

This task can be achieved by Sturm's theorem. (Nice reference is the "Sturm's theorem" entry in Wikipedia).

We start by calculating the Sturm chain:

$$\begin{aligned}
p_0(y) &= P(y) = 8 - 2y - 8y^2 + 10y^3 - 5y^4 + y^5. \\
p_1(y) &= -2 - 16y + 30y^2 - 20y^3 + 5y^4. \\
p_2(y) &= -rem\left(8 - 2y - 8y^2 + 10y^3 - 5y^4 + y^5, -2 - 16y + 30y^2 - 20y^3 + 5y^4\right) \\
&= \frac{6}{5}y^2 - \frac{24}{5}y + \frac{38}{5}. \\
p_3(y) &= -rem\left(-2 - 16y + 30y^2 - 20y^3 + 5y^4, \frac{6}{5}y^2 - \frac{24}{5}y + \frac{38}{5}\right) = \frac{68}{3}y - \frac{77}{9}. \\
p_4(y) &= -rem\left(\frac{6}{5}y^2 - \frac{24}{5}y + \frac{38}{5}, \frac{68}{3}y - \frac{77}{9}\right) = -\frac{41\,333}{6936}.
\end{aligned}$$

$$p_0(0) = 8,\ p_1(0) = -2,\ p_2(0) = \frac{38}{5},\ p_3(0) = -\frac{77}{9},\ p_4(0) = -\frac{41\,333}{6936}.$$

The number of sign changes at $y = 0$ is $V(0) = 3$.

$$p_0(1) = 4,\ p_1(1) = -3,\ p_2(1) = 4,\ p_3(1) = \frac{127}{9},\ p_4(1) = -\frac{41\,333}{6936}.$$

The number of sign changes at $y = 1$ is $V(1) = 3$. Therefore, by Sturm's theorem, the number of real roots of $P(y)$ is $V(0) - V(1) = 3 - 3 = 0$ in the interval $[0, 1)$. We conclude that polynomial $P(y)$ is positive in the interval $[0, 1)$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

For $0 < x \le 1$, we have

$$1 - x^x \;=\; 1 - e^{x\ln(1-(1-x))} \ge 1 - e^{-x\left(1-x+\frac{(1-x)^2}{2}\right)} = 1 - e^{\frac{-x(1-x)(3-x)}{2}}$$

$$\geq \quad \frac{x(1-x)(3-x)}{2} - \frac{(x(1-x)(3-x))^2}{8} = \frac{\big(x(1-x)(3-x)(4-3x+4x^2-x^3)}{8}.$$

Hence to prove the inequality of the problem, it suffices to show that for $0 < x \leq 1$,

$x^{(x^2)} \geq \dfrac{8e^{x-1}}{(3-x)(4-3x+4x^2-x^3)}$ or

$$x^2 \ln x + \ln(3-x) + \ln(4-3x+4x^2-x^3) - x + 1 - \ln 8 \geq 0, \tag{1}$$

Denote the left side of (1) by $f(x)$. Sine $f(1) = 0$, (1) will follow from

$$f'(x) < 0 \tag{2}$$

for $0 < x < 1$. We have $\dfrac{f'(x)}{x} = 12 \ln x + \dfrac{x^4 - 11x^3 + 36x^2 - 43x + 25}{x(3-x)(x^3 - 4x^2 + 3x - 4)}$ so that

$f'(1) = 0$. Thus, to prove (2), it suffices to show that $\left(\dfrac{f'(x)}{x}\right)' > 0$, for $0 < x < 1$. It can be checked readily that $\left(\dfrac{f'(x)}{x}\right)' = \dfrac{g(x)}{x^2(3-x)^2(x^3-4x^2+3x-4)^2}$, where

$$g(x) = 300 - 362x + 628x^2 - 668x^3 + 273x^4 + 212x^5 - 302x^6 + 136x^7 - 27x^8 + 2x^9.$$

By writing $g(x)$ as

$$\frac{(300-181x)^2 + 239x^2}{300} \quad + \quad \frac{2x^2(259-167x)^2 + x^4(14929-14763x)}{259} +$$

$$+ \quad \frac{x^5(269-151x)^2 + x^7(13783-7263x)}{269} + 2x^9,$$

we see that $g(x) > 0$ so that $\left(\dfrac{f'(x)}{x}\right)' > 0$. This proves (2) and completes the solution.

Solution 4 by Daniel Văcaru, Pitesti, Romania

First we prove that $x^x - 1 \leq x^2 - x,\ x \in (0,1) \qquad (1)$.

Let $a \neq \dfrac{1}{x}$. Then (1) is equivalent to $\left(\dfrac{1}{a}\right)^{\frac{1}{n}} \leq \dfrac{1}{a^2} - \dfrac{1}{a} + 1$. After rearranging and raising both sides to the power of a, this relationship becomes $\dfrac{1}{a} \leq \left(1 + \dfrac{1-a}{a^2}\right)^a$. We now use Bernoulli's inequality in the form $1 + bx \leq (1+x)b$ for $x \geq -1$ and $b \geq 1$ and

$$\frac{1}{a} = 1 + \frac{a \cdot 1 - a}{a^2} \leq \left(1 + \frac{1-a}{a^2}\right)^a,$$

which concludes this part of the solution.

Now we prove that

$$x^2 - x \leq x^{1-x^2} e^{x-1}(x-1).$$

Since $x-1 \le 0, \forall 0 < x \le 1$, we obtain

$$x \ge x^{1-x^2} e^{x-1} \implies x^{x^2} \ge e^{x-1} \implies e^{\log x \cdot x^2} \ge e^{x-1}.$$

Equivalently, we have to prove that

$$\log x \cdot x^2 \ge x-1,\ \forall 0 < x \le 1\ (^*).$$

Consider $f:(0,1] \implies \Re,\ f(x) = x^2 \cdot \log x - x + 1$. Then
$f'(x) = 2x \ln x \le 0,\ \forall 0 < x \le 1$. So $f(x) \ge f(1) = 0$, and this proves (*).

Also solved by Ed Gray, Highland Beach, FL; G. C. Greuber, Newport News, VA; Albert Natian, Los Angeles Valley College, Valley Glen, CA; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5556: *Proposed by Pedro Jesús Rodríguez de Rivera (student) and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

Let $\alpha_k = \dfrac{k+\sqrt{k^2+4}}{2}$. Evaluate $\lim\limits_{k\to\infty} \dfrac{\prod\limits_{n=1}^{\infty}\left(1+\dfrac{(k-1)\alpha_k+1}{\alpha_k^n+\alpha_k}\right)}{\alpha_k}$.

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA

Let $\alpha_k = (k+\sqrt{k^2+4})/2$. Then

$$\alpha_k^2 - k\alpha_k - 1 = \frac{k^2+2k\sqrt{k^2+4}+k^2+4}{4} - \frac{k^2+k\sqrt{k^2+4}}{2} - 1 = 0,$$

and

$$k\alpha_k + 1 = \alpha_k^2.$$

It follows that

$$\begin{aligned} 1+\frac{(k-1)\alpha_k+1}{\alpha_k+\alpha_k} &= \frac{\alpha_k+k\alpha_k+1}{2\alpha_k} = \frac{\alpha_k+\alpha_k^2}{2\alpha_k} = \frac{1+\alpha_k}{2}, \\ 1+\frac{(k-1)\alpha_k+1}{\alpha_k^2+\alpha_k} &= \frac{\alpha_k^2+k\alpha_k+1}{\alpha_k^2+\alpha_k} = \frac{2\alpha_k^2}{\alpha_k^2+\alpha_k} = \frac{2\alpha_k}{\alpha_k+1}, \end{aligned}$$

and for $n \ge 3$,

$$1+\frac{(k-1)\alpha_k+1}{\alpha_k^n+\alpha_k} = \frac{\alpha_k^n+k\alpha_k+1}{\alpha_k^n+\alpha_k} = \frac{\alpha_k^n+\alpha_k^2}{\alpha_k^n+\alpha_k} = \frac{\alpha_k(\alpha_k^{n-2}+1)}{\alpha_k^{n-1}+1}.$$

Thus, for $N \ge 3$,

$$\begin{aligned} \prod_{n=1}^{N}\left(1+\frac{(k-1)\alpha_k+1}{\alpha_k^n+\alpha_k}\right) &= \frac{\alpha_k+1}{2}\cdot\frac{2\alpha_k}{\alpha_k+1}\cdot\frac{\alpha_k(\alpha_k+1)}{\alpha_k^2+1}\cdot\dots\cdot\frac{\alpha_k(\alpha_k^{N-2}+1)}{\alpha_k^{N-1}+1} \\ &= \frac{\alpha_k^{N-1}(\alpha_k+1)}{\alpha_k^{N-1}+1}, \end{aligned}$$

and

$$\prod_{n=1}^{\infty}\left(1+\frac{(k-1)\alpha_k+1}{\alpha_k^n+\alpha_k}\right)=\lim_{N\to\infty}\frac{\alpha_k^{N-1}(\alpha_k+1)}{\alpha_k^{N-1}+1}=\alpha_k+1.$$

Finally,

$$\lim_{k\to\infty}\frac{\prod_{n=1}^{\infty}\left(1+\frac{(k-1)\alpha_k+1}{\alpha_k^n+\alpha_k}\right)}{\alpha_k}=\lim_{k\to\infty}\frac{\alpha_k+1}{\alpha_k}=1.$$

Remark: This problem is similar to Elementary Problem B-1237 that appeared in the November 2018 issue of the Fibonacci Quarterly and to Problem 12110 that appeared in the April 2019 issue of the American Mathematical Monthly.

Solution 2 by Moti Levy, Rehovot, Israel

Let $L=\lim_{k\to\infty}\frac{\prod_{n=1}^{\infty}\left(1+\frac{(k-1)\alpha_k+1}{\alpha_k^n+\alpha_k}\right)}{\alpha_k}$. We rewrite the product,

$$\frac{\prod_{n=1}^{\infty}\left(1+\frac{(k-1)\alpha_k+1}{\alpha_k^n+\alpha_k}\right)}{\alpha_k}=\left(\frac{1+\frac{(k-1)\alpha_k+1}{\alpha_k+\alpha_k}}{\alpha_k}\right)\left(1+\frac{(k-1)\,\alpha_k+1}{\alpha_k^2+\alpha_k}\right)\prod_{n=1}^{\infty}\left(1+\frac{(k-1)\,\alpha_k+1}{\alpha_k^{n+2}+\alpha_k}\right).$$

It is easy to show that

$$\begin{aligned}\lim_{k\to\infty}\frac{1+\frac{(k-1)\alpha_k+1}{\alpha_k+\alpha_k}}{\alpha_k}&=\frac{1}{2},\\ \lim_{k\to\infty}\left(1+\frac{(k-1)\,\alpha_k+1}{\alpha_k^2+\alpha_k}\right)&=2.\end{aligned}$$

Therefore,

$$L=\lim_{k\to\infty}\prod_{n=1}^{\infty}\left(1+\frac{(k-1)\,\alpha_k+1}{\alpha_k^{n+2}+\alpha_k}\right).$$

$$\begin{aligned}\log L&=\lim_{k\to\infty}\log\prod_{n=1}^{\infty}\left(1+\frac{(k-1)\,\alpha_k+1}{\alpha_k^{n+2}+\alpha_k}\right)\\ &=\lim_{k\to\infty}\sum_{n=1}^{\infty}\log\left(1+\frac{(k-1)\,\alpha_k+1}{\alpha_k^{n+2}+\alpha_k}\right).\end{aligned}$$

Now we want to apply Tannery's theorem, which gives sufficient conditions for the interchanging of the limit and infinite summation operations. For reference, see Tannery's theorem entry in Wikipedia.

Tannery's Theorem:

Let $S_k:=\sum_{n=1}^{\infty}a_n(k)$ *and suppose that* $\lim_{k\to\infty}a_n(k)=b_n$. *If* $|a_n(k)|\le M_n$ *and* $\sum_{n=1}^{\infty}M_n<\infty$, *then* $\lim_{k\to\infty}S_k=\sum_{n=1}^{\infty}b_n$.

Denote $a_n(k):=\log\left(1+\frac{(k-1)\alpha_k+1}{\alpha_k^{n+2}+\alpha_k}\right)$, $n\ge 1$. Since $\log(1+x)\le x$, then

$$a_n(k)\le\frac{(k-1)\,\alpha_k+1}{\alpha_k^{n+2}+\alpha_k}$$

For $k \geq 1$, $\frac{k+\sqrt{k^2+4}}{2} \geq k$ and $2k \geq \frac{k+\sqrt{k^2+4}}{2}$, hence $2k \geq \alpha_k \geq k$.
It follows that

$$\frac{(k-1)\,\alpha_k+1}{\alpha_k^{n+2}+\alpha_k} \leq \frac{(k-1)\,2k+1}{k^{n+2}+k} \leq \frac{2k^2-2k+1}{k^{n+2}} \leq \frac{2}{k^n}$$

Set $M_n := \frac{1}{2^{n-1}}$

$$a_n(k) \leq \frac{2}{k^n} \leq \frac{1}{2^{n-1}} = M_n, \quad for \;\; k>1.$$

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 2 < \infty.$$

$\lim_{k\to\infty} a_n(k) = b_n = \lim_{k\to\infty} \log\left(1+\frac{(k-1)\alpha_k+1}{\alpha_k^{n+2}+\alpha_k}\right) =$
$\lim_{k\to\infty} \log\left(1+\frac{(k-1)\frac{k+\sqrt{k^2+4}}{2}+1}{\left(\frac{k+\sqrt{k^2+4}}{2}\right)^{n+2}+\frac{k+\sqrt{k^2+4}}{2}}\right) = 0$, for all $n \geq 1$.
Therefore, by Tannery's theorem,
$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \sum_{n=1}^{\infty} \log\left(1+\frac{(k-1)\alpha_k+1}{\alpha_k^{n+2}+\alpha_k}\right) = \sum_{n=1}^{\infty} b_n = 0.$
We conclude that $L=1$.

Solution 3 by Michel Bataille, Rouen, France

We show that the required limit is 1.

It is readily checked that $\alpha_k^2 = k\alpha_k + 1$ so that

$$1+\frac{(k-1)\alpha_k+1}{\alpha_k^n+\alpha_k} = \frac{\alpha_k^n+k\alpha_k+1}{\alpha_k^n+\alpha_k} = \frac{\alpha_k^n+\alpha_k^2}{\alpha_k^n+\alpha_k} = \alpha_k \cdot \frac{\alpha_k^{n-2}+1}{\alpha_k^{n-1}+1}.$$

Now, let N be any positive integer. Then we have

$$\begin{aligned}\prod_{n=1}^{N}\left(1+\frac{(k-1)\alpha_k+1}{\alpha_k^n+\alpha_k}\right) &= \alpha_k^N \prod_{n=1}^{N} \frac{\alpha_k^{n-2}+1}{\alpha_k^{n-1}+1} \\ &= \alpha_k^N \cdot \frac{\alpha_k^{-1}+1}{\alpha_k^{N-1}+1} \\ &= \frac{1+\alpha_k}{1+\frac{1}{\alpha_k^{N-1}}}.\end{aligned}$$

But we have $\alpha_1 = \frac{1+\sqrt{5}}{2} > 1$ and for $k \geq 2$, $\alpha_k \geq \frac{2+\sqrt{k^2+4}}{2} > 1$ and therefore $\lim\limits_{N\to\infty} \alpha_k^{N-1} = \infty$ whatever $k \geq 1$. We deduce that

$$\prod_{n=1}^{\infty}\left(1+\frac{(k-1)\alpha_k+1}{\alpha_k^n+\alpha_k}\right) = \lim_{N\to\infty}\prod_{n=1}^{N}\left(1+\frac{(k-1)\alpha_k+1}{\alpha_k^n+\alpha_k}\right) = 1+\alpha_k$$

and it follows that the required limit is $\lim\limits_{k\to\infty} \frac{1+\alpha_k}{\alpha_k} = 1$ (since $\lim\limits_{k\to\infty} \alpha_k = \infty$).

Solution 4 proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA

From

$$\alpha_k = \frac{k+\sqrt{k^2+4}}{2}$$

we have

$$k\alpha_k + 1 = \alpha_k^2,$$

$$1+\frac{(k-1)\,\alpha_k+1}{\alpha_k^n+\alpha_k} = \frac{\alpha_k^n+k\alpha_k+1}{\alpha_k^n+\alpha_k} = \frac{\alpha_k^n+\alpha_k^2}{\alpha_k^n+\alpha_k} = \alpha_k\cdot\frac{\alpha_k^{n-2}+1}{\alpha_k^{n-1}+1},$$

$$\prod_{n=1}^{N}\left(1+\frac{(k-1)\,\alpha_k+1}{\alpha_k^n+\alpha_k}\right) = \prod_{n=1}^{N}\left(\alpha_k\cdot\frac{\alpha_k^{n-2}+1}{\alpha_k^{n-1}+1}\right) = \frac{1+\alpha_k}{1+\alpha_k^{1-N}},$$

$$\prod_{n=1}^{\infty}\left(1+\frac{(k-1)\,\alpha_k+1}{\alpha_k^n+\alpha_k}\right) = 1+\alpha_k \quad \text{since} \quad \alpha_k>1.$$

So

$$\lim_{k\to\infty}\frac{\prod\limits_{n=1}^{\infty}\left(1+\dfrac{(k-1)\,\alpha_k+1}{\alpha_k^n+\alpha_k}\right)}{\alpha_k} = \lim_{k\to\infty}\frac{1+\alpha_k}{\alpha_k} = 1 \quad \text{since} \quad \alpha_k\to\infty.$$

Also solved by Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; Daniel Văcaru, Pitesti, Romania, and the proposers.

5557: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $n \geq 2$ be an integer. If for all $k \in \{1,2,\ldots,n\}$ we have

$$A_k = \begin{pmatrix} k+1 & k \\ k+3 & k+2 \end{pmatrix},$$

compute the value of $\sum\limits_{1\leq i<j\leq n} \det\left(A_i+A_j\right)$.

Solution 1 by David A. Huckaby, Angelo State University, San Angelo, TX

For integers i and j,

$$A_i+A_j = \begin{pmatrix} i+1 & i \\ i+3 & i+2 \end{pmatrix} + \begin{pmatrix} j+1 & j \\ j+3 & j+2 \end{pmatrix} = \begin{pmatrix} i+j+2 & i+j \\ i+j+6 & i+j+4 \end{pmatrix},$$

so that

$$\begin{aligned}\det(A_i+A_j) &= (i+j+2)(i+j+4)-(i+j+6)(i+j)\\ &= (i+j)^2+6(i+j)+8-(i+j)^2-6(i+j)\\ &= 8.\end{aligned}$$

Since there are $n(n-1)/2$ summands in the desired sum,

$$\sum_{1\le i<j\le n} \det(A_i+A_j) = \frac{n(n-1)}{2}(8) = 4n(n-1).$$

Solution 2 by David E. Manes, Oneonta, NY

Note that if i and j are positive integers with $i<j$, then

$$A_i = \begin{pmatrix} i+1 & i \\ i+3 & i+2 \end{pmatrix}, \quad A_j = \begin{pmatrix} j+1 & j \\ j+3 & j+2 \end{pmatrix} \text{ and } A_i + A_j = \begin{pmatrix} i+j+2 & i+j \\ i+j+6 & i+j+4 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \det(A_i+A_j) &= (i+j+2)(i+j+4) - (i+j)(i+j+6) \\ &= (i+j)^2 + 6(i+j) + 8 - (i+j)^2 - 6(i+j) \\ &= 8. \end{aligned}$$

Using this equation, we will prove by induction that if $n\ge 2$, then

$$\sum_{1\le i<j\le n} \det(A_i+A_j) = 4(n^2-n).$$

If $n=2$, then $\sum\limits_{1\le i<j\le 2} \det(A_i+A_j) = \det(A_1+A_2) = 8 = 4(2^2-2)$. Therefore, the result is true for $n=2$. For the inductive step, assume that $n\ge 2$ is an integer for which the result is true. Then we want to show that the result is true for $n+1$; that is,

$$\sum_{1\le i<j\le n+1} \det(A_i+A_j) = 4(n^2+n).$$

Therefore,

$$\begin{aligned} \sum_{1\le i<j\le n+1} \det(A_i+A_j) &= \sum_{1\le i<j\le n} \det(A_i+A_j) + \sum_{1\le i<n+1} \det(A_i+A_{n+1}) \\ &= 4(n^2-n) + \det(A_1+A_{n+1}) + \det(A_2+A_{n+1} + \cdots + \det(A_n+A_{n+1}) \\ &= 4(n^2-n) + (8+8+\cdots+8) \quad \text{(n terms of 8)} \\ &= 4(n^2-n) + 8n = 4n^2+4n \\ &= 4(n^2+n). \end{aligned}$$

Hence, the result is true for $n+1$ so that by induction

$$\sum_{1\le i<j\le n} \det(A_i+A_j) = 4(n^2-n)$$

for each positive integer $n\ge 2$.

Solution 3 by Ed Gray, Highland Beach,FL

If $A_k = \begin{pmatrix} k+1 & k \\ k+3 & k+2 \end{pmatrix}$ then $A_i + A_j = \begin{pmatrix} i+j+2 & i+j \\ i+j+6 & i+j+4 \end{pmatrix}$ with

$$\det(A_i+A_j) \quad = \quad (i+j+2)\cdot(i+j+4) - (i+j)(i+j+6)$$

$$= \quad (i+j)^2 + 6\cdot(i+j) + 8 - (i+j)^2 - 6\cdot(i+j) = 8.$$

Therefore, $S = 8 \cdot (\text{number of terms in the sum})$ which is equal to
$\binom{n}{2} = \dfrac{n!}{2!(n-2)!} = \dfrac{n(n-1)}{2}$. So, $S = 4n(n-1)$.

Also solved by Ulrich Abel, Technische Hochschule Mittelhessen, Germany; Dionne Bailey, Elsie Campbell, Charles Diminnie and Karl Havlak, Angelo State University, San Angelo TX; Michel Bataille, Rouen, France; Brian Bradie, Christopher Newport University, Newport News, VA; Michael C. Faleski, Delta College, University Center, MI; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Carl Libis, Columbia Southern University, Orange Beach, AL; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece Albert Stadler, Herrliberg, Switzerland; Seán M. Stewart, Bomaderry, NSW, Australia; David Stone and John Hawkins, Georgia Southern University, Statesboro,GA; Daniel Văcaru, Piteswti, Romania, and the proposer.

5558: *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Find all continuous functions $f : \Re \to \Re$ such that

$$\int_{-x}^{0} f(t)dt + \int_{0}^{x} tf(x-t)dt = x, \forall x \in \Re.$$

Solution 1 by Moti Levy, Rehovot, Israel
Substituting $t = -u$ we get

$$\int_{-x}^{0} f(t)\,dt = \int_{0}^{x} f(-u)\,du.$$

Substituting $x - t = u$ we get

$$\int_{0}^{x} tf(x-t)\,dt = \int_{0}^{x} (x-u)\,f(u)\,du.$$

Thus the original equation can be rewritten as

$$\int_{0}^{x} \left(f(-t) + (x-t)\,f(t)\right) dt = x. \tag{1}$$

Differentiating both sides of (1) gives

$$f(-x) + \int_{0}^{x} f(t)\,dt = 1. \tag{2}$$

Equation (2) implies that $f(x)$ is differentiable. Differentiation of both sides of (2) gives

$$\begin{aligned} f'(-x) - f(x) &= 0, \\ f'(x) - f(-x) &= 0 \end{aligned}$$

$$\begin{aligned} f''(x) + f'(-x) &= 0 \\ f''(x) + f(x) &= 0 \end{aligned} \tag{3}$$

A solution of (3) is of the form

$$f(x) = a\cos x + b\sin x. \tag{4}$$

To find a particular solution, we substitute (4) in (1) and in (2) and get the following system of equations:

$$\begin{aligned} a - b + bx - a\cos x + b\cos x + a\sin x - b\sin x &= x, \\ b + a\cos x - b\cos x + a\sin x - b\sin x &= 1. \end{aligned}$$

The solution is $a = 1, b = 1$ when $-\cos x + \sin x - x\cos x - x\sin x + 1 \neq 0$. Now we check if

$$f(x) = \cos x + \sin x \tag{5}$$

is indeed a particular solution of (1) by direct substitution of (5) into the equation in the problem statement

$$\int_{-x}^{0} (\cos t + \sin t)\, dt + \int_0^x t\left(\cos(x-t) + \sin(x-t)\right) dt = x$$

and verify that (5) is a solution.
What is left to show that $\cos x + \sin x$ is the only solution.
Suppose $\cos x + \sin x + r(x)$ is a solution. Then because $\int_0^x \left(f(-t) + (x-t)f(t)\right) dt$ is linear operator then $r(x)$ must satisfy

$$\int_0^x \left(r(-t) + (x-t)r(t)\right) dt = 0. \tag{6}$$

Differentiating both sides of (6) gives

$$r(-x) + \int_0^x r(t) = 0. \tag{7}$$

By similar argument as above, $r(x)$ must satisfy $r''(x) + r(x) = 0$. The solution is

$$r(x) = \alpha\sin x + \beta\cos x.$$

$r(x)$ must satisfy (6) and (7), hence α and β must satisfy the system

$$\begin{aligned} \alpha - \beta + \beta x - \alpha\cos x + \beta\cos x + \alpha\sin x - \beta\sin x &= 0, \\ \beta + \alpha\cos x - \beta\cos x + \alpha\sin x - \beta\sin x &= 0. \end{aligned}$$

The solution is $\alpha = 0, \beta = 0$ when $-\cos x + \sin x - x\cos x - x\sin x + 1 \neq 0$, hence $r(x) = 0$.

Remark: $-\cos x + \sin x - x\cos x - x\sin x + 1 = 0$ only for discrete set of values of x. Since we require the solution to be continuous then $r(x) = 0$ for all $x \in \Re$.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

The stated equality is equivalent to

$$\int_x^0 f(t)dt + \int_0^x (x-t)f(t)dt = x.$$

We differentiate this equation and get

$$f(-x) + \int_0^x f(t)dt = 1. \tag{1}$$

This equation is equivalent to $f(x) = 1 - \int_0^{-x} f(t)dt$ and shows that $f(x)$ is differentiable infinitely often. Indeed, the right-hand side is differentiable, since f is continuous. So $f(x)$ is differentiable. If $f(x)$ is differentiable n times then $1 - \int_0^{-x} f(t)dt$ is differentiable $n+1$ times, and so is $f(x)$.

We differentiate (1) again and get $-f''(-x) + f(x) = 0$.

We replace x by $-x$ and get $-f''(x) + f(-x) = 0$. We differentiate this equation again and get

$$-f''(x) - f'(-x) = -f''(x) - f(x) = 0.$$

So $f''(x) = -f(x)$. The general solution of this differential equation is $f(x) = a\cos x + b\sin x$. Thus

$$\int_{-x}^0 f(t)dt + \int_0^x tf(x-t)dt = a - b + bx + (b-a)\cos x + (a-b)\sin x,$$

and we conclude that $a = b = 1$. Therefore the general solution is $f(x) = \cos x + \sin x$.

Solution 3 by Hatef I. Arshagi

By the properties of integrals

$$\int_{-x}^0 f(t)dt = \int_0^x f(-t)dt,$$

and by setting $x - t = \tau$, we get

$$\int_0^x tf(x-t)dt = \int_x^0 (x-\tau)f(\tau)(-d\tau) = x\int_0^x f(d\tau) = x\int_0^x f(\tau)d - \int_0^x \tau f\tau d\tau.$$

By using the above two results,

$$\int_{-x}^0 f(t)dt + \int_0^x tf(x-t)dt = \int_0^x f(-t)dt + x\int_0^x f(t)dt - \int_0^x tf(t)dt = x.$$

We differentiate both sides of

$$\int_0^x f(-t)dt + x\int_0^x f(t)dt - \int_0^x tf(t)dt = x,$$

using the Fundamental Theorem of Calculus, then

$$f(-x) + \int_0^x f(t)dt + xf(x) - xf(x) = 1,$$

from this we have

$$f(-x) = 1 - \int_0^x f(t)dt,$$

and this implies that $f(0) = 1$, and $f(x)$ is differentiable and $f''(x)$ exists. So,

$$\begin{cases} f'(-x) &= f(x) \\ f'(x) &= f(-x) \\ f''(x) &= -f'(-x)' \\ f'(0) &= 1. \end{cases}$$

We summarize this in

$$\begin{cases} f''(x) + f(x) &= 0 \\ f(0) &= 1 \\ f'(0) &= 1. \end{cases}$$

The general solution of $f''(x) + f(x) = 0$, is $f(x) = A \sin x + B \cos x$, and using $f(0) = 1$ and $f'(0) = 1$, we conclude that the solution is $f(x) = \sin x + \cos x$.

Solution 4 by Michel Bataille, Rouen, France

We show that the the function f_0 defined by $f_0(x) = \sin x + \cos x$ is the unique solution to the functional equation.
First, we observe that a change of variables yields

$$\int_0^x tf(x-t)\,dt = \int_0^x (x-u)f(u)\,du = x\int_0^x f(u), du - \int_0^x uf(u)\,du.$$

Let $F(x) = \int_0^x f(t)\,dt$ and $G(x) = \int_0^x F(t)\,dt$ so that $F(0) = G(0) = 0$ and $F'(x) = f(x),\ G'(x) = F(x)$ for all x. By integration by parts, we deduce that

$$\int_0^x uf(u)\,du = [uF(u)]_0^x - \int_0^x F(u)\,du = xF(x) - G(x)$$

and so

$$\int_0^x tf(x-t)\,dt = xF(x) - xF(x) + G(x) = G(x).$$

Therefore the equation can be rewritten as

$$G(x) - F(-x) = x. \qquad (1)$$

Now, we have $F_0(x) = \int_0^x f_0(t)\,dt = \sin x - \cos x + 1$ and $G_0(x) = \int_0^x F_0(t)\,dt = -\cos x - \sin x + x + 1$, hence $G_0(x) - F_0(-x) = x$ for all x and therefore f_0 is a solution to (1).
Conversely, let f be an arbitrary solution. Then (1) is satisfied for all x and so is $F(x) + F'(-x) = 1$ obtained by differentiation. Thus, for all x we have

$$F(x) + f(-x) = 1 \quad \text{and} \quad F(-x) + f(x) = 1. \qquad (2)$$

Now, let $g(x) = F(x) + F(-x),\ h(x) = F(x) - F(-x)$. These functions g, h satisfy $g(0) = 0 = h(0)$ and $g'(x) = f(x) - f(-x),\ h'(x) = f(x) + f(-x)$, hence $g'(0) = 0$ and from (2) by subtraction and addition,

$$h(x) - g'(x) = 0, \qquad g(x) + h'(x) = 2$$

for all x. Since h is differentiable, the former shows that the same is true of g' and $g''(x) = h'(x)$; the latter then gives $g''(x) + g(x) = 2$ and g is a solution to the differential equation $y'' + y = 2$ with initial conditions $y(0) = y'(0) = 0$. We classically obtain $g(x) = 2 - 2\cos x$ and then successively deduce that $h(x) = 2\sin x$, $\quad F(x) = \frac{1}{2}(g(x) + h(x)) = 1 + \sin x - \cos x$, $\quad f(x) = F'(x) = \sin x + \cos x = f_0(x)$ for all x. This completes the proof.

Solution 5 by Albert Natian, Los Angeles Valley College, Valley Glen, CA

In the second integral above, let $u = x - t$. Then $t = x - u$ and $dt = -du$ so that

$$\int_0^x tf(x-t)\ dt = \int_x^0 (x-u)\, f(u)\ (-du)$$

$$= \int_0^x (x-u)\, f(u)\ du$$

$$= x\int_0^x f(u)\ du - \int_0^x uf(u)\ du.$$

We thus have the integral equation

$$\int_{-x}^0 f(t)\ dt + x\int_0^x f(u)\ du - \int_0^x uf(u)\ du = x$$

both sides of which we differentiate to get

$$f(-x) + \int_0^x f(u)\ du = 1 \quad \text{or} \quad f(x) = 1 - \int_0^{-x} f(u)\ du \quad \text{with} \quad f(0) = 1.$$

Differentiating the latter integral equation twice, we see that

$$f'(x) = f(-x), \quad f'(-x) = f(x), \quad f''(x) = -f'(-x) = -f(x)$$

which has a general solution

$$f(x) = A\cos x + B\sin x \quad \text{with} \quad f(0) = f'(0) = 1.$$

Since (it's immediate that) $A = B = 1$, then

$$f(x) = \cos x + \sin x.$$

Solution 6 by Seán M. Stewart, Bomaderry, NSW, Australia

We wish to find all continuous function $f : R \to R$ such that

$$\int_{-x}^0 f(t)\, dt + \int_0^x tf(x-t)\, dt = x. \tag{1}$$

Here $x \in R$. In the second of the integrals appearing on the left of (1), enforcing a substitution of $t \mapsto x - t$ gives

$$\int_{-x}^0 f(t)\, dt + x\int_0^x f(t)\, dt - \int_0^x tf(t)\, dt = x. \tag{2}$$

Differentiating (2) with respect to x using Leibniz' rule yields

$$\int_0^x f(t)\,dt = 1 - f(-x). \tag{3}$$

Setting $x = 0$ in (3) we immediately see that $f(0) = 1$. Differentiating (3) with respect to x again using Leibniz' rule, one obtains

$$f(x) = f'(-x). \tag{4}$$

From (4) we immediately see that $f'(0) = f(0) = 1$. Next, as $x \in R$, interchanging x with $-x$ leads to

$$f'(x) = f(-x). \tag{5}$$

Differentiating (5) with respect to x gives

$$f''(x) = -f'(-x) = -f(x),$$

where the result in (4) has been used, or

$$f''(x) + f(x) = 0,$$

a second-order, linear, homogeneous differential equation with constant coefficients. Solving, for the non-trivial solution one has

$$f(x) = c_1 \cos x + c_2 \sin x,$$

where c_1 and c_2 are constants to be determined. Note the trivial solution of $f(x) = 0$ is not a solution of (1). The two constants can be found from the initial conditions $f(0) = f'(0) = 1$. Doing so we find $c_1 = c_2 = 1$. Thus all continuous functions f that satisfy (1) are: $f(x) = \sin x + \cos x$.

Also solved by Ulrich Abel, Technische Hochschule Mittelhessen, Germany; G. C. Greuber, Newport News, VA; Kee-Wai Lau, Hong Kong, China; Daniel Văcaru, Pitesti, Romania, and the proposers.

Mea Culpa

Michel Bataille of Rouen, France should have been credited for having solved problem 5552.