Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before February 15, 2018

- **5469:** *Proposed by Kenneth Korbin, New York, NY*

  Let x and y be positive integers that satisfy the equation $3x^2 = 7y^2 + 17$. Find a pair of larger integers that satisfy this equation expressed in terms of x and y.

- **5470:** *Proposed by Moshe Stupel, "Shaanan" Academic College of Education and Gordon Academic College of Education, and Avi Sigler, "Shaanan" Academic College of Education, Haifa, Israel*

  Prove that there are an infinite number of Heronian triangles (triangles whose sides and area are natural numbers), whose side lengths are three consecutive natural numbers.

- **5471:** *Proposed by Arkady Alt, San Jose, CA*

  For natural numbers p and n where $n \geq 3$ prove that

$$n^{\frac{1}{n^p}} > (n+p)^{\frac{1}{(n+1)(n+2)(n+3)\cdots(n+p)}}.$$

- **5472:** *Proposed by Francisco Perdomo and Ángel Plaza, both at Universidad Las Palmas de Gran Canaria, Spain*

  Let α, β, and γ be the three angles in a non-right triangle. Prove that

$$\frac{1+\sin^2\alpha}{\cos^2\alpha} + \frac{1+\sin^2\beta}{\cos^2\beta} + \frac{1+\sin^2\gamma}{\cos^2\gamma} \geq \frac{1+\sin\alpha\sin\beta}{1-\sin\alpha\sin\beta} + \frac{1+\sin\beta\sin\gamma}{1-\sin\beta\sin\gamma} + \frac{1+\sin\gamma\sin\alpha}{1-\sin\gamma\sin\alpha.}$$

- **5473:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $x_1, \cdots, x_n$ be positive real numbers. Prove that for $n \geq 2$, the following inequality holds:

$$\left(\sum_{k=1}^{n} \frac{\sin x_k}{((n-1)x_k + x_{k+1})^{1/2}}\right)\left(\sum_{k=1}^{n} \frac{\cos x_k}{((n-1)x_k + x_{k+1})^{1/2}}\right) \leq \frac{1}{2}\sum_{k=1}^{n}\frac{1}{x_k.}$$

  (Here the subscripts are taken modulo n)

- **5474:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a, b \in \Re, b \neq 0$. Calculate

$$\lim_{n\to\infty} \begin{pmatrix} 1-\dfrac{a}{n^2} & \dfrac{b}{n} \\ \\ \dfrac{b}{n} & 1+\dfrac{a}{n^2}. \end{pmatrix}^n.$$

Solutions

- **5451:** *Proposed by Kenneth Korbin, New York, NY*

Given triangle ABC with sides $a = 8, b = 19$ and $c = 22$. The triangle has an interior point P where $\overline{AP}$, $\overline{BP}$, and $\overline{CP}$ each have positive integer length. Find $\overline{AP}$ and $\overline{BP}$, if $\overline{CP} = 4$.

Solution 1 by David E. Manes, Oneonta, NY

We will show that $\overline{BP} = 6$ and $\overline{AP} = 17$.
Using the law of cosines in ΔABC, one obtains

$$\cos\angle C = \frac{8^2+4^2-22^2}{2\cdot 8\cdot 19} = \frac{-59}{304}$$

so that $\angle C = \arccos\left(\dfrac{-59}{304}\right)$. Let $x = \overline{BP}$ and $y = \overline{AP}$. By the triangle inequality in ΔPCB, it follows that $5 \leq x \leq 11$. If $x = 5$, then

$$\cos\angle BCP = \frac{8^2+4^2-5^2}{2\cdot 8\cdot 4} = \frac{55}{64}.$$

Therefore, $\angle BCP = \arccos\left(\dfrac{55}{64}\right)$ and
$\angle PCA = \angle C - \angle BCP = \arccos\left(\dfrac{-59}{304}\right) - \arccos\left(\dfrac{55}{64}\right)$. Using the identity
$\cos(\alpha - \beta) = \cos\alpha\cdot\cos\beta + \sin\alpha\cdot\sin\beta$, we get

$$\cos\angle PCA = \left(\frac{-59}{304}\right)\left(\frac{55}{64}\right) + \left(\frac{77\sqrt{15}}{304}\right)\left(\frac{3\sqrt{7\cdot 17}}{64}\right) = \frac{-3245+231\sqrt{3\cdot 5\cdot 7\cdot 17}}{304\cdot 64}.$$

Thus,

$$y^2 = 4^2+19^2-2\cdot 4\cdot 19\cos\angle PCA = 377 - 19\left(\frac{-3245+231\sqrt{15\cdot 119}}{304\cdot 8}\right)$$
$$= \frac{916864+61655-4389\sqrt{1785}}{2432}.$$

Therefore,

$$y = \sqrt{\frac{978519-4389\sqrt{1785}}{2432}} \approx 18.058$$

is not an integer. Hence, $x \neq 5$.
However, if $x = 6$, then

$$\cos \angle BCP = \frac{8^2 + 4^2 - 6^2}{64} = \frac{11}{16}$$

so that $\angle BCP = \arccos\left(\frac{11}{16}\right)$ and
$\angle PCA = \angle C - \angle BCP = \arccos\left(\frac{-59}{304}\right) - \arccos\left(\frac{11}{16}\right)$. Thus,

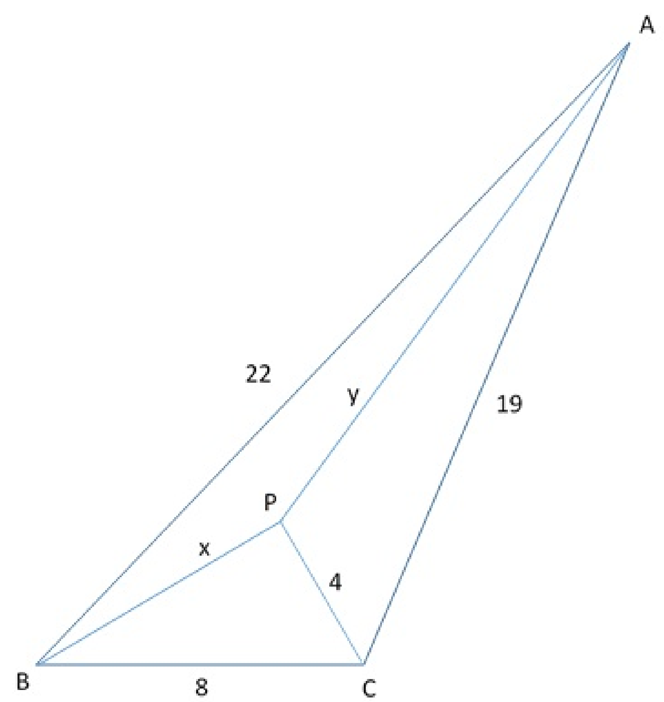
$$\begin{aligned}\cos \angle PCA &= \cos\left[\arccos\left(\frac{-59}{304}\right) - \arccos\left(\frac{11}{16}\right)\right] \\ &= \left(\frac{-59}{304}\right)\left(\frac{11}{16}\right) + \left(\frac{77\sqrt{15}}{304}\right)\left(\frac{3\sqrt{15}}{16}\right) = \frac{-649 + 3465}{4864} \\ &= \frac{11}{19}.\end{aligned}$$

Therefore,

$$y^2 = 4^2 + 19^2 - 2 \cdot 4 \cdot 19\left(\frac{11}{19}\right) = 289$$

whence $y = 17$. Hence, $x = \overline{BP} = 6$ and $y = \overline{AP} = 17$. The solution is unique since $x = 7$ does not yield an integer value for y while each of the values $x = 8, 9, 10, 11$ does not yield a triangle for ΔBPA.

Solution 2 by Michael N. Fried, Ben-Gurion University of the Negev, Beer-Sheva, Israel



Let $\overline{BP} = x$ and $\overline{AP} = y$. Because of the triangle inequality, $8 < x + 4, x < 8 + 4$ or $5 \leq x \leq 11$. Similarly, we have $16 \leq y \leq 22$.

These inequalities can be improved slightly using Stewart's formula for the length of cevians: if ABC is a triangle with sides $\overline{AC} = b$ and $\overline{BC} = a$ and if d is the length of a cevian from A which divides $\overline{AB}$ into segments of lengths $\overline{AB} = m$ and $\overline{AB} = n$, then:

$$d^2 = \frac{ma^2 + nb^2}{m+n} - mn$$

(this is just an easy consequence of the law of cosines). Since the maximum value of y occurs when P lies on $\overline{BC}$, by Stewart's formula, $y_{max}^2 = \frac{22^2+19^2}{2} - 4^2 = 406.5 = 20.16^2$, so $y \leq 20$. Similarly, the maximum value of x occurs when P lies on $\overline{AC}$, so that P divides $\overline{AC}$ into segments of lengths 4 and 15. Thus, again by Stewart's formula $x_{max}^2 = \frac{15 \cdot 8^2 + 4 \cdot 22^2}{19} - 4 \cdot 15 = 92.42 \approx 9.61^2$, so that $x \leq 9$. Hence:

$$5 \leq x \leq 9$$

$$16 \leq y \leq 20$$

Since P lies on a circle centered at C, and the lines $\overline{BP}$ all lie on one side of $\overline{BC}$, each length x of $\overline{BP}$ corresponds to a unique P and, therefore, to a unique value of y.
To find y for a given value of x, let $\angle BCP = \theta, \angle PCA = \phi$, and $\angle BCA = \gamma$. The cosine of γ is fixed and given by the law of cosines:

$$\cos\gamma = \frac{19^2 + 8^2 - 22^2}{2 \cdot 8 \cdot 19} = -\frac{59}{304}$$

The sine of γ is just $\sqrt{1 - \cos^2\gamma}$, that is:

$$\sin\gamma = \sqrt{1 - \frac{59^2}{304^2}} = \frac{77}{304}\sqrt{15}$$

The cosine of θ for a given value of x is also given by the law of cosines:

$$\cos\theta = \frac{8^2 + 4^2 - x^2}{2 \cdot 8 \cdot 4} = \frac{80 - x^2}{64}$$

And again, $\sin\theta$ is given by $\sqrt{1 - \cos^2\theta}$. Hence, the cosine of ϕ is given:

$$\cos\phi = \cos(\gamma - \theta) = -\frac{59}{304}\cos\theta + \frac{77}{304}\sqrt{15}\sin\theta$$

Thus, for any x we can calculate y, once again by the law of cosines:

$$y^2 = 4^2 + 19^2 - 2 \cdot 4 \cdot 19 \cdot \cos\phi = 377 - 152\cos\phi$$

Calculating y for $x = 5, 6, 7, 8, 9$ we find one integral value for y: $y = 17$ corresponding to $x = 6$.
So we have our answer:

$$\overline{AP} = 17$$

$$\overline{BP} = 6$$

Solution 3 by Brian D. Beasley, Presbyterian College, Clinton, SC

We model the given triangle in the Cartesian plane by first placing A at $(19, 0)$ and C at $(0, 0)$. Then B must lie on the circles with equations

$$x^2 + y^2 = 64 \quad \text{and} \quad (x-19)^2 + y^2 = 484,$$

so we place B in the second quadrant at (d,e), where $d = -59/38$ and $e = 77\sqrt{15}/38$. Next, we seek an interior point $P = (x,y)$ such that $x^2 + y^2 = 16$, $(x-19)^2 + y^2 = m^2$, and $(x-d)^2 + (y-e)^2 = n^2$ for positive integers $m = \overline{AP}$ and $n = \overline{BP}$. Since P is interior to triangle ABC and lies on the circle with equation $x^2 + y^2 = 4$, we have $m \in \{16, 17, 18, 19, 20\}$ and $n \in \{5, 6, 7, 8, 9\}$. Solving the system

$$\begin{cases} x^2 + y^2 = 16 \\ (x-19)^2 + y^2 = m^2 \end{cases}$$

yields $x = (377 - m^2)/38$ and $y = \sqrt{-m^4 + 754m^2 - 119025}/38$. Substituting these values for x and y into $(x-d)^2 + (y-e)^2 = n^2$ for $m \in \{16, 17, 18, 19, 20\}$, we find that only $m = 17$ produces a positive integer value for n, namely $n = 6$. Hence $P = (44/19, 16\sqrt{15}/19)$ with $\overline{AP} = 17$ and $\overline{BP} = 6$.

Comment by Albert Stadler, Herrliberg, Switzerland: There is no other interior point even if we get rid of the condition that $\overline{CP} = 4$. However, letting $u = \overline{AP}, v = \overline{BP}$ and $w = \overline{CP}$ and if we permit P to lie on a side of the triangle, then $(u, v, w) = (16, 6, 7)$ is the only additional point.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Charles McCracken, Dayton, OH; Vijaya Prasad Nalluri, Rajahmundry, India; Valentin Shopov, Munich, Germany; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Students at Taylor University, Upland, IN,ne
Team 1: {Hannah Peters, Ben Robison, Stevanni McCray}
Team 2: {Hannah King, Deborah Settles, Jackson Bronkema}
Team 3: {Gwyneth Terrett, Samantha Korn, Elissa Grace Moore}, and the proposer.

- **5452:** *Proposed by Roger Izard, Dallas, TX*

Let point O be the orthocenter of a given triangle ABC. In triangle ABC let the altitude from B intersect line segment AC at E, and the altitude from C intersect line segment AB at D. If AC and AB are unequal, derive a formula which gives the square of BC in terms of AC, AB, EO, and OD.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

Let $a = BC,\ b = CA,\ c = AB,\ d = OD,\ e = EO,\ f = EA$, and $g = AD$. Applying the Pythagorean Theorem to $\triangle ABE,\ \triangle BCE,\ \triangle OEA$ and $\triangle OAD$, and using the fact that $\triangle ABE \sim \triangle CAD$, because they are both right triangles with common angle at vertex A, we obtain:

$$c^2 = AB^2 = BE^2 + EA^2 = BC^2 - CE^2 + EA^2 = a^2 - (b-f)^2 + f^2 = a^2 - b^2 + 2bf,$$

$$e^2 + f^2 = EO^2 + EA^2 = OA^2 = OD^2 + AD^2 = d^2 + g^2, \text{ and}$$

$$\frac{b}{g} = \frac{CA}{AD} = \frac{AB}{EA} = \frac{c}{f}.$$

From these two last lines, we obtain

$$\begin{aligned} e^2 + f^2 &= d^2 + \frac{b^2 f^2}{c^2} \\ c^2e^2 + c^2f^2 &= c^2d^2 + b^2f^2, \end{aligned}$$

and since $b \neq c$ by hypothesis, we see that $f^2 = \dfrac{c^2\left(e^2 - d^2\right)}{b^2 - c^2}$, and from the equality $c^2 = a^2 - b^2 + 2bf$ gives us a^2 in terms b, c, e and d. Namely,

$$a^2 = \left(b^2 + c^2 - 2bf\right) = b^2 + c^2 - 2bc\sqrt{\frac{e^2 - d^2}{b^2 - c^2}}.$$

Solution 2 by Kee-Wai Lau,Hong Kong, China

By the cosine formula, we have

$\dfrac{EO}{OA} = \sin\angle OAE = \cos\angle ACB = \dfrac{AC^2 + BC^2 - AB^2}{2(AC)(BC)}$, and similarly

$\dfrac{OD}{OA} = \sin\angle OAD = \cos\angle ABC = \dfrac{AB^2 + BC^2 - AC^2}{2(AB)(BC)}$. Hence,

$$\frac{EO}{OD} = \frac{AB(AC^2 + BC^2 - AB^2)}{AC(AB^2 + BC^2 - AC^2)}. \qquad (1)$$

Since $AC \neq AB$, so $\dfrac{EO}{OD} \neq \dfrac{AB}{AC}$ or $(AB)(OD) - (AC)(EO) \neq 0$. Solving (1) for BC^2 we obtain

$$BC^2 = \frac{(AB + AC)(AB - AC)\left((AB)(OD) + (AC)(EO)\right)}{(AB)(OD) - (AC)(EO)}.$$

Also solved by Ed Gray, Highland Beach, FL; David E. Manes, Oneonta, NY; Vijaya Prasad Nalluri, Rajahmundry, India; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5453:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

If $a, b, c \in (0, 1)$ or $a, b, c \in (1, \infty)$ and m, n are positive real numbers, then prove that

$$\frac{\log_a b + \log_b c}{m + n\log_a c} + \frac{\log_b c + \log_c a}{m + n\log_b a} + \frac{\log_c a + \log_a b}{m + n\log_c b} \geq \frac{6}{m+n}$$

Solution 1 by Moti Levy, Rehovot, Israel

Let $x := \log_a b$, $y := \log_b c$, $z := \log_c a$. Then $xyz = \log_a b \log_b c \log_c a = 1$, and the condition $a, b, c \in (0,1)$ or $a, b, c \in (1, \infty)$ implies that $x,\ y,\ z > 0$.

The original inequality may be rephrased as:

$$\frac{x+y}{m+z^{-1}n} + \frac{y+z}{m+x^{-1}n} + \frac{z+x}{m+y^{-1}n} \geq \frac{6}{m+n}, \quad xyz = 1, \quad x,y,z > 0, \tag{1}$$

or as

$$\frac{3}{\sum_{cyc}\left(\frac{m+z^{-1}n}{x+y}\right)^{-1}} \leq \frac{m+n}{2}.$$

Since the harmonic mean is less than or equal to the geometric mean,

$$\frac{3}{\sum_{cyc}\left(\frac{m+z^{-1}n}{x+y}\right)^{-1}} \leq \sqrt[3]{\frac{m+z^{-1}n}{x+y}\frac{m+x^{-1}n}{y+z}\frac{m+y^{-1}n}{z+x}}.$$

Hence it is enough to prove (2):

$$\frac{m+z^{-1}n}{x+y}\frac{m+x^{-1}n}{y+z}\frac{m+y^{-1}n}{z+x} \leq \frac{(m+n)^3}{8},$$

$$\frac{1}{xyz}\frac{(n+mz)(n+mx)(n+my)}{(x+y)(x+z)(y+z)} \leq \frac{(m+n)^3}{8},$$

$$\frac{(n+mz)(n+mx)(n+my)}{(x+y)(x+z)(y+z)} \leq \frac{(m+n)^3}{8}. \tag{2}$$

Further simplification of (2) results in

$$\frac{n^3 + mn^2x + mn^2y + mn^2z + m^2nxy + m^2nxz + m^2nyz + m^3xyz}{(x+y)(x+z)(y+z)} \leq \frac{(m+n)^3}{8}$$

$$\frac{n^3 + mn^2(x+y+z) + m^2n(xy+yz+xz) + m^3}{(x+y)(x+z)(y+z)} \leq \frac{(m+n)^3}{8} \tag{3}$$

Equating the left and right sides of (3) shows that the inequality (3) is equivalent to (4) and (5):

$$\frac{x+y+z}{(x+y)(x+z)(y+z)} \leq \frac{3}{8}, \tag{4}$$

$$\frac{xy+yz+xz}{(x+y)(x+z)(y+z)} \leq \frac{3}{8}. \tag{5}$$

We now use the p, q, r notation:

$$p := x+y+z,$$
$$q := xy+yz+zx,$$
$$r := xyz.$$

In this notation, (4) and (5) become

$$\frac{p}{pq-r} \leq \frac{3}{8}, \tag{6}$$

$$\frac{q}{pq-r} \leq \frac{3}{8}. \tag{7}$$

In our case $r = 1$, which implies (by AM-GM inequality) that $p \geq 3$ and $q \geq 3$. Now proving (4) and (5) is straightforward:

$$\begin{aligned}\frac{p}{pq-1} &\leq \frac{3}{8},\\ 3pq - 3 - 8p &\geq 0,\\ 3pq - 3 - 8p &\geq p - 3 \geq 0.\end{aligned}$$

$$\begin{aligned}\frac{q}{pq-1} &\leq \frac{3}{8},\\ 3pq - 3 - 8q &\geq 0,\\ 3pq - 3 - 8q &\geq q - 3 \geq 0.\end{aligned}$$

Solution 2 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Note that since $\log_a b = \dfrac{\ln b}{\ln a}$ and $a, b, c \in (0,1)$ or $a, b, c \in (0,1)$, all the logarithms in the proposed inequality are positive, so the right-hand side is positive.

We will apply the following parametrized Nesbitt's inequality (see reference 1, theorem 7).

Let $x,\ y,\ z,\ tx + ky + lz,\ ty + kz + lx,\ tz + kx + ly$ be positive real numbers and let $-k - l < t < \dfrac{k+l}{2}$.

Then $\dfrac{x}{tx + ky + lz} + \dfrac{y}{ty + kz + lx} + \dfrac{z}{tz + kx + ly} \geq 3t + k + l.$ (1)

We will consider two inequalities, from which the stated problem will follow.

$$\frac{\log_a b}{m + n\log_a c} + \frac{\log_b c}{m + n\log_b a} + \frac{\log_c a}{m + n\log_c b} \geq \frac{3}{m+n} \qquad (2)$$

$$\frac{\log_b c}{m + n\log_a c} + \frac{\log_c a}{m + n\log_b a} + \frac{\log_a b}{m + n\log_c b} \geq \frac{3}{m+n}. \qquad (3)$$

Notice that the right-hand side of (2) is

$$RHS = \frac{\ln b}{m\ln a + n\ln c} + \frac{\ln c}{m\ln b + n\ln a} + \frac{\ln a}{m\ln c + n\ln b} \geq \frac{3}{m+n}$$

by the parametrized Nesbitt's inequality with $t = 0$, $k = m$ and $l = n$, and $x = \ln b$, $y = \ln c$, and $z = \ln a$. It also should be noticed that in the last expression we may assume that all the ln's are positive.

Now, the right-hand side of (3) is

$$RHS = \frac{\ln a \ln c}{m\ln a\ln b + n\ln b\ln c} + \frac{\ln a\ln b}{m\ln b\ln c + n\ln a\ln c} + \frac{\ln b\ln c}{m\ln a\ln c + n\ln a\ln b} \geq \frac{3}{m+n}$$

by the parametrized Nesbitt's inequality with $t = 0$, $k = m$ and $l = n$, and $x = \ln a\ln c$, $y = \ln a\ln b$, and $z = \ln b\ln c$.

References:

(1) Shanhe Wu and Ovidiu Furdui, *A note on a conjectured Nesbitt type inequality*, Taiwanese Journal of Mathematics, 15 (2) (2011), 449-456.

Solution 3 by Soumitra Mandal, Chandar Nagore, India

$$\sum_{cyc}\frac{\log_a b+\log_b c}{m+n\log_a c}=\sum_{cyc}\frac{\log b+\frac{\log a\cdot\log c}{\log b}}{m\log a+n\log c}$$

$$=\sum_{cyc}\frac{\log b}{m\log a+n\log c}+\sum_{cyc}\frac{\frac{\log a\cdot\log c}{\log b}}{m\log a+n\log c}$$

$$=\sum_{cyc}\frac{(\log b)^2}{n\log a\cdot\log b+n\log c\cdot\log b}+\sum_{cyc}\frac{\left(\frac{1}{\log b}\right)^2}{\frac{m}{\log b\cdot\log c}+\frac{n}{\log b\cdot\log a}}$$

$$\overbrace{\geq}^{BERGSTROM}\frac{(\log a+\log b+\log c)^2}{(m+n)(\log a\cdot\log b+\log b\cdot\log c+\log c\cdot\log a)}+$$

$$+\frac{\left(\frac{1}{\log a}+\frac{1}{\log b}+\frac{1}{\log c}\right)^2}{(m+n)\left(\frac{1}{\log a\cdot\log b}+\frac{1}{\log b\cdot\log c}+\frac{1}{\log c\cdot\log a}\right)}\geq\frac{3}{m+n}+\frac{3}{m+n}=\frac{6}{m+n}$$

Editor's Comments: **Anna V. Tomova of Varna, Bulgaria** approached the solution as follows: She showed that the left hand side of the inequality can be put into the canonical form of $X+Y+\dfrac{1}{XY}$. She then showed that this canonical form has a global minimum at $(1,1)$, forcing it to have a minimal value of 3, and working with this she produced the final result.

Bruno Salgueiro Fanego of Viveiro, Spain noted that the stated problem is a specific case of a more general result. Namely: If $x,y,z\in(0,\infty)$ and $xyz=1$, then

$$\frac{x+y}{m+\dfrac{n}{z}}+\frac{y+z}{m+\dfrac{n}{x}}+\frac{z+x}{m+\dfrac{n}{y}}\geq\frac{6}{m+n}.$$

He proved the more general result, and applied it to the specific case.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego of Viveiro, Spain; Ed Gray of Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Shravan Sridhar, Udupi, India; Albert Stadler, Herrliberg, Switzerland; Anna V. Tomova of Varna, Bulgaria, and the proposer.

5454: *Proposed by Arkady Alt, San Jose, CA*

Prove that for integers k and l, and for any $\alpha,\beta\in\left(0,\frac{\pi}{2}\right)$, the following inequality holds:

$$k^2\tan\alpha+l^2\tan\beta\geq\frac{2kl}{\sin(\alpha+\beta)}-\left(k^2+l^2\right)\cot(\alpha+\beta).$$

Solution 1 by Ed Gray, Highland Beach, FL

We rewrite the inequality by transposing

1) $k^2\left(\frac{\sin a}{\cos a}+\frac{\cos(a+b)}{\sin(a+b)}\right)+t^2\left(\frac{\sin b}{\cos b}+\frac{\cos(a+b)}{\sin(a+b)}\right)\geq\frac{2kt}{\sin(a+b)}$

Multiplying by $\sin(a+b)$

2) $k^2\left(\frac{\sin a(\sin(a+b))}{\cos a}+\cos(a+b)\right)+t^2\left(\frac{\sin b\sin(a+b)}{\cos b}+\cos(a+b)\right)\geq 2kt$

3) $k^2\left(\frac{\sin a\sin(a+b)+\cos a\cos(a+b)}{\cos a}\right)+t^2\left(\frac{\sin b\sin(a+b)+\cos b\cos(a+b)}{\cos b}\right)\geq 2kt$

4) $k^2\left(\frac{\cos b}{\cos a}\right)+t^2\left(\frac{\cos a}{\cos b}\right)\geq 2kt$

5) $\frac{k^2\cos^2 b+t^2\cos^2 a}{\cos a\cos b}\geq 2kt$

6) $k^2\cos^2 b+t^2\cos^2 a\geq 2kt\cos a\cos b$, and transposing,

7) $(k\cos b-t\cos a)^2\geq 0$..

So we retrace our steps to obtain the original inequality.

Solution 2 by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC

First we consider the case when $\alpha+\beta=\frac{\pi}{2}$, then $\sin(\alpha+\beta)=1$, $\cot(\alpha+\beta)=0$, and $\tan\beta=\cot\alpha$. From these we have

$$k^2\tan\alpha+t^2\tan\beta-\frac{2kl}{\sin(\alpha+\beta)}+\left(k^2+l^2\right)\cot(\alpha+\beta)=k^2\tan\alpha+l^2\cot\alpha-2lk=\left(k\sqrt{\tan\alpha}-l\sqrt{\cot\alpha}\right)^2\geq 0,$$

which completes the proof when $\alpha+\beta=\frac{\pi}{2}$.

Now suppose that $\alpha+\beta\neq\frac{\pi}{2}$. By using the identity $\cot(\alpha+\beta)=\frac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$, we have

$$k^2\tan\alpha+l^2\tan\beta+(k^2+l^2)\cot(\alpha+\beta)-\frac{2kl}{\sin(\alpha+\beta)}$$

$$=\quad k^2\tan\alpha+l^2\tan\beta+(k^2+l^2)\frac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}-\frac{2kl}{\sin(\alpha+\beta)}$$

$$=\quad\frac{k^2\tan^2\alpha+k^2\tan\alpha\tan\beta+l^2\tan\beta\tan\alpha+l^2\tan^2\beta+(k^2+l^2)-\left(k^2+l^2\right)\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}-\frac{2kl}{\sin(\alpha+\beta)}$$

$$=\quad\frac{k^2\tan^2\alpha+l^2\tan^2\beta+(k^2+l^2)}{\tan\alpha+\tan\beta}-\frac{2kl}{\sin(\alpha+\beta)}$$

$$=\quad\frac{k^2(1+\tan^2\alpha)+l^2(1+\tan^2\beta)}{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}}-\frac{2kl}{\sin(\alpha+\beta)}$$

$$= \frac{\dfrac{k^2 \sec^2\alpha + l^2\sec^2\beta}{\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}}}{} - \frac{2kl}{\sin(\alpha+\beta)}$$

$$= \frac{k^2\dfrac{\cos\beta}{\cos\alpha} + l^2\dfrac{\cos\alpha}{\cos\beta}}{\sin(\alpha+\beta)} - \frac{2kl}{\sin(\alpha+\beta)}$$

$$= \frac{\left(\sqrt{k\dfrac{\cos\beta}{\cos\alpha}} - l\sqrt{k\dfrac{\cos\alpha}{\cos\beta}}\right)^2}{\sin(\alpha+\beta)} \geq 0.$$

Editor's Note: Most of the solvers mentioned that the inequality holds for all real values of k and l. **David Stone and John Hawkins of Georgia Southern University** when a bit further. They stated: "the conditions that α and β be first quadrant angles is an easy way to make $\sin(\alpha+\beta) \neq 0$ and $\tan\alpha, \tan\beta, \cot(\alpha+\beta)$ be defined and guarantee that $\cos\alpha\cos\beta\sin(\alpha+\beta) > 0$." But the proof shows that the inequality would be true for any values of α and β which satisfy these conditions.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Boris Rays, Brooklyn, NY; Daniel Sitaru, "Theodor Costescu" National Economic College, Severin Mehedinti; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Anna V. Tomova, Varna, Bulgaria, and the proposer.

5455: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Find all real solutions to the following system of equations:

$$\begin{aligned} \frac{1}{a} + \frac{1}{b} + \frac{1}{c} &= \frac{1}{abc} \\ a+b+c &= abc + \frac{8}{27}(a+b+c)^3 \end{aligned}$$

Solution 1 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

Suppose a, b, c are real numbers satisfying our system. Consider the polynomial

$$\begin{aligned} g(x) &= (x-a)(x-b)(x-c) \\ &= x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc. \end{aligned}$$

The first equation of our original system implies $ab+ac+bc = 1$. So

$$g(x) = x^3 - \lambda x^2 + x - \mu$$

where $\lambda = a+b+c$ and $\mu = abc$. Note that the second equation of our original system can be written as $\lambda = \mu + \frac{8}{27}\lambda^3$. We make the usual substitution to get a depressed cubic:

$g(x+\lambda/3) = x^3 + px + q$ where

$$p = 1 - \frac{1}{3}\lambda^2 \text{ and } q = \frac{-2}{27}\lambda^3 + \frac{1}{3}\lambda - \mu.$$

Using $\lambda = \mu + \frac{8}{27}\lambda^3$ we have

$$q = \frac{2}{9}\lambda^3 - \frac{2}{3}\lambda$$

which we factor to get

$$q = \frac{-2}{3}\lambda\left(1 - \frac{1}{3}\lambda^2\right) = \frac{-2}{3}\lambda p.$$

The discriminant of $g(x+\lambda/3)$ is

$$\begin{aligned} D &= -4p^3 - 27q^2 \\ &= -4p^3 - 12\lambda^2 p^2 \\ &= -4p^2(p + 3\lambda^2) \\ &= -4p^2\left(1 + \frac{8}{3}\lambda^2\right) \end{aligned}$$

Note that $D \geq 0$ if and only if $p = 0$. Recall that a real cubic polynomial has three real roots if and only if its discriminant is ≥ 0. Thus $g(x+\lambda/3)$ has three real roots if and only if $p = 0$ if and only if $\lambda = \pm\sqrt{3}$. Note that when $\lambda = \pm\sqrt{3}$ we have $g(x+\lambda/3) = x^3$, and hence $g(x) = (x - \lambda/3)^3$. Therefore the only solutions to the original system are

$$a = b = c = \frac{\sqrt{3}}{3} \text{ and } a = b = c = \frac{-\sqrt{3}}{3}.$$

Solution 2 by Moti Levy, Rehovot, Israel

$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{abc}$ implies $ab + bc + ca = 1$.

Substitution of $abc = \dfrac{1}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}$ in the second equation gives

$$\begin{aligned} a + b + c - \frac{1}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} - \frac{8}{27}(a+b+c)^3 &= 0, \\ \frac{(a+b)(a+c)(b+c)}{ab + ac + bc} - \frac{8}{27}(a+b+c)^3 &= 0, \\ (a+b)(a+c)(b+c) - \frac{8}{27}(a+b+c)^3 &= 0. \qquad (1) \end{aligned}$$

Let $x = a + b$, $y = b + c$ and $z = c + a$ then (1) becomes

$$xyz - \left(\frac{x+y+z}{3}\right)^3 = 0,$$

or $\sqrt[3]{xyz} = \dfrac{x+y+z}{3}$. The geometric mean is equal to the arithmetic mean if and only if $x = y = z$ which implies that $a = b = c$.

Therefore the system of equation has only two solutions:

$$a = b = c = \frac{1}{\sqrt{3}}, \quad a = b = c = -\frac{1}{\sqrt{3}}.$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

Let $p = a+b+c,\ q = ab+bc+ca$, and $r = abc$. The first given equation becomes

$$q = 1 \tag{1}$$

and the second equations becomes

$$r = p - \frac{8p^3}{27}. \tag{2}$$

It can be checked readily that

$$p^2q^2 - 4p^3r + 18pqr - 4q^3 - 27r^2 = (a-b)^2(b-c)^2(c-a)^2. \tag{3}$$

Using (1) and (2) we reduce the left side of (3) to $\dfrac{-4(p^2-3)^2(8p^2+3)}{27}$, which is non-positive. Since the right side of (3) is nonnegative, so both sides of (3) equal to zero. It follows that $p^2 = 3$ and by (2), $r = \dfrac{p}{9}$. Moreover, either $a = b$ or $b = c$ or $c = a$. By symmetry we only consider the case $a = b$. Hence either $2a+c = \sqrt{3},\ a^2+2ac = 1$, or $2a+c = -\sqrt{3},\ a^2+2ac = 1$, giving respectively $a = c = \dfrac{1}{\sqrt{3}}$ and $a = c = \dfrac{-1}{\sqrt{3}}$. Thus the solutions to the original system are precisely $(a,b,c) = \left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right),\ \left(\dfrac{-1}{\sqrt{3}}, \dfrac{-1}{\sqrt{3}}, \dfrac{-1}{\sqrt{3}}\right)$.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Le Van, Ho Chi Minh City, Vietnam; Albert Stadler, Herrliberg, Switzerland; Anna V. Tomova, Varna, Bulgaria, and the proposer.

5456: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let k be a positive integer. Calculate

$$\lim_{x\to\infty} \mathrm{e}^{-x} \sum_{n=k}^{\infty} (-1)^n \binom{n}{k} \left(\mathrm{e}^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}\right).$$

Solution 1 Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany

By the Taylor formula we have

$$e^x = 1 + x + x^2/2! + \cdots + x^n/n! + \int_0^x \frac{(x-t)^n}{n!} e^t dt.$$

It follows that

$$\begin{aligned}
& e^{-x} \sum_{n=k}^{\infty} (-1)^n \binom{n}{k} \left(e^x - 1 - x - x^2/2! - \cdots - x^n/n!\right) \\
= & \sum_{n=k}^{\infty} (-1)^n \binom{n}{k} \int_0^x \frac{(x-t)^n}{n!} e^{t-x} dt \\
= & \frac{1}{k!} \sum_{n=k}^{\infty} (-1)^n \int_0^x \frac{t^n}{(n-k)!} e^{-t} dt \\
= & \frac{1}{k!} \sum_{n=0}^{\infty} (-1)^{n+k} \int_0^x \frac{t^{n+k}}{n!} e^{-t} dt \\
= & \frac{(-1)^k}{k!} \int_0^x t^k e^{-2t} dt.
\end{aligned}$$

The limit as $x \to +\infty$ is given by

$$\frac{(-1)^k}{k!}\int_0^\infty t^k e^{-2t}dt = \frac{(-1)^k}{2^{k+1}}.$$

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

$$\lim_{x\to} e^{-x}\sum_{n=k}^{\infty}(-1)^n\binom{n}{k}\left(e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}\right)$$

$$= \lim_{x\to\infty} e^{-x}\sum_{n=k}^{\infty}(-1)^n\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\sum_{p=n+1}^{\infty}\frac{x^p}{p!}$$

$$= \lim_{x\to\infty} e^{-x}\frac{1}{k!}\sum_{n=k}^{\infty}(-1)^n\left(\frac{d^k}{dt^k}t^n\right)\bigg|_{t=1}\sum_{p=n+1}^{\infty}\frac{x^p}{p!} = \frac{1}{k!}\lim_{x\to\infty} e^{-x}\frac{d^k}{dt^k}\left(\sum_{n=k}^{\infty}(-1)^n t^n\sum_{p=n+1}^{\infty}\frac{x^p}{p!}\right)\bigg|_{t=1}$$

$$= \frac{1}{k!}\lim_{x\to\infty} e^{-x}\frac{d^k}{dt^k}\left(\sum_{n=1}^{\infty}\frac{x^n}{n!}\sum_{p=0}^{n-1}(-1)^p t^p\right)\bigg|_{t=1} = \frac{1}{k!}\lim_{x\to\infty} e^{-x}\frac{d^k}{dt^k}\left(\frac{1}{1+t}\left(\sum_{n=1}^{\infty}\frac{x^n}{n!} - \sum_{n=1}^{\infty}\frac{(-xt)^n}{n!}\right)\right)\bigg|_{t=1}$$

$$= \frac{1}{k!}\lim_{x\to\infty}\frac{d^k}{dt^k}\left(\frac{1}{1+t}e^{-x}\left(e^x - 1 - \left(e^{-x} - 1\right)\right)\right)\bigg|_{t=1} = \frac{1}{k!}\lim_{x\to\infty}\frac{d^k}{dt^k}\left(\frac{1}{1+t}\left(1 - e^{-x(1+t)}\right)\right)\bigg|_{t=1}$$

$$= \frac{1}{k!}\frac{d^k}{dt^k}\left(\frac{1}{1+t}\lim_{x\to\infty}\left(1 - e^{-x(1+t)}\right)\right)\bigg|_{t=1} = \frac{1}{k!}\left(\frac{d^k}{dt^k}\frac{1}{1+t}\right)\bigg|_{t=1} = \frac{(-1)^k}{(1+t)^{k+1}}\bigg|_{t=1} = \frac{(-1)^k}{2^{k+1}}.$$

Solution 3 by Albert Stadler, Herrliberg, Switzerland

Repeated integration by parts yields

$$\int_0^x e^{-t}\frac{t^n}{n!}dt = -e^{-x}\frac{x^n}{n!} + \int_0^x d^{-t}\frac{t^{n-1}}{(n-1)!}dt = -e^{-x}\left(\frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + \cdots + x\right) + \int_0^x e^{-t}dt$$

$$= -e^{-x}\left(\frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + \cdots + x + 1\right) + 1.$$

So,

$$e^{-x} - \left(\frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + \cdots + x + 1\right) = e^x\int_0^x e^{-t}\frac{t^n}{n!}dt$$

and

$$e^{-x}\sum_{n=k}^{\infty}(-1)^n\binom{n}{k}\left(e^x - \left(\frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + \cdots + x + 1\right)\right) = \frac{1}{k!}\sum_{n=k}^{\infty}\frac{(-1)^n}{(n-k)!}\left(\int_0^x e^{-t}t^n dt\right)$$

$$= \frac{(-1)^k}{k!}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\int_0^x e^{-t}t^{n+k}dt\right)$$

$$= \frac{(-1)^k}{k!}\int_0^x e^{-2t}t^k dt \to \frac{(-1)^k}{k!}\int_0^\infty e^{-2t}t^k dt = \frac{(-1)k!}{k!2^{k+1}} = \frac{(-1)^k}{2^{k+1}}, \text{as } x \to \infty.$$

Solution 4 by Kee-Wai Lau, Hong Kong, China

We show that the given limit equals $(-1)^k 2^{-(k+1)}$.

For real x let $f(x) = e^{-x}x^k = \sum_{m=k}^{\infty}(-1)^{m-k}\frac{x^m}{m-k)!}$ so that

$$f^n(0) = \begin{cases} 0, & 0 \le n \le k-1 \\ (-1)^{n-k}n(n-1)\cdots(n-k+1), & n \ge k \end{cases}.$$

where $f^n(x)$ is the n th derivative of $f(x)$.

According to problem $3.89(a)$ on pp$124, 227$ of the book [Ovidiu Furdui; Limits, Series, and Fractional Part Integrals, Springer 2013] we have

$$\sum_{n=0}^{\infty} f^n(0)\left(e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}\right) = \int_0^x e^{x-t}f(t)dt.$$

Hence,

$$\begin{aligned} e^{-x}\sum_{n=k}^{\infty}(-1)^n\binom{n}{k}\left(e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}\right) &= \frac{(-1)^k}{k!}\int_0^x e^{-2t}t^k dt \\ &= \frac{(-1)^k}{2^{k+1}k!}\int_0^{2x} e^{-t}t^k dt. \end{aligned}$$

Now our result for the limit follows from the well-known fact that $\int_0^\infty e^{-t}t^k dt = k!$.

Also solved by Moti Levy, Rehovot, Israel; Anna V. Tomova, Varna, Bulgaria, and the proposer.

Editor's Comment: In Anna's solution to 5456 she acknowledged contributing conversations with **Peter Breuer and Joachim Domsta of Bulgaria.**