Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before April 15, 2012

- **5194:** *Proposed by Kenneth Korbin, New York, NY*

  Find two pairs of positive integers (a, b) such that,

  $$\frac{14}{a}+\frac{a}{b}+\frac{b}{14}=41.$$

- **5195:** *Proposed by Kenneth Korbin, New York, NY*

  If N is a prime number or a power of primes congruent to 1 (mod 6), then there are positive integers a and b such that $3a^2+3ab+b^2=N$ with $(a,b)=1$.

  Find a and b if $N=2011$, and if $N=2011^2$, and if $N=2011^3$.

- **5196:** *Proposed by Neculai Stanciu, Buzău, Romania*

  Determine the last six digits of the product $(2010)\left(5^{2014}\right)$.

- **5197:** *Proposed by Pedro H. O. Pantoja, UFRN, Brazil*

  Let x, y, z be positive real numbers such that $x^2+y^2+z^2=4$. Prove that,

  $$\frac{1}{6-x^2}+\frac{1}{6-y^2}+\frac{1}{6-z^2}\leq\frac{1}{xyz}.$$

- **5198:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let m, n be positive integers. Calculate,

  $$\sum_{k=1}^{2n}\prod_{i=0}^{m}\left(\lfloor k+\frac{1}{2}\rfloor+a+i\right)^{-1},$$

  where a is a nonnegative number and $\lfloor x\rfloor$ represents the greatest integer less than or equal to x.

- **5199:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $k > 0$ and $n \geq 0$ be real numbers. Calculate,

$$\int_0^1 x^n \ln\left(\sqrt{1+x^k} - \sqrt{1-x^k}\right) dx.$$

Solutions

- **5176:** *Proposed by Kenneth Korbin, New York, NY*

Solve:

$$\begin{cases} x^2 + xy + y^2 = 3^2 \\ y^2 + yz + z^2 = 4^2 \\ z^2 + xz + x^2 = 5^2. \end{cases}$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

Let

$$\begin{cases} A = x^2 + xy + y^2 - 9 \\ B = y^2 + yz + z^2 - 16 \\ C = z^2 + xz + x^2 - 25 \end{cases}$$

By assumption $A = B = C = 0$. So, $0 = A + B - C = xy + yz - xz + 2y^2$ or equivalently $z(x-y) = y(x+2y)$. Obviously $x \neq y$, since if $x = y$ then $0 = B = x^2 + xz + z^2 - 16$ and $0 = C = z^2 + xz + x^2 - 25$ which is a contradiction. So,

$$z = \frac{y\,(x+2y)}{x-y}. \qquad (1)$$

We insert this value of z into the equation $B = 0$ and obtain

$$\begin{aligned} 16 &= y^2 + y \cdot \frac{y(x+2y)}{x-y} + \left(\frac{y(x+2y)}{x-y}\right)^2 \\ &= y^2 \cdot \frac{(x-y)^2 + (x-y)(x+2y) + (x+2y)^2}{(x-y)^2} \\ &= y^2 \cdot \frac{x^2 - 2xy + y^2 + x^2 + xy - 2y^2 + x^2 + 4xy + 4y^2}{(x-y)^2} \\ &= y^2 \cdot \frac{3x^2 + 3xy + 3y^3}{(x-y)^2} = \frac{27y^2}{(x-y)^2}. \end{aligned}$$

So,

$$4(x-y) = \pm 3\sqrt{3}y \qquad \text{or equivalently,}$$

$$x = \left(1 + \frac{3\sqrt{3}}{4}\right) y \qquad \text{or } x = \left(1 - \frac{3\sqrt{3}}{4}\right) y. \qquad (2)$$

$A = 0$ then implies

$$\left\{\left(1 \pm \frac{3\sqrt{3}}{4}\right)^2 + \left(1 \pm \frac{3\sqrt{3}}{4}\right) + 1\right\} y^2 = 9.$$

Taking into account (1) and (2) we conclude that

$$\begin{aligned}
(x, y, z) \in \quad & \left\{\left(\frac{9 + 4\sqrt{3}}{\sqrt{25 + 12\sqrt{3}}}, \frac{4\sqrt{3}}{\sqrt{25 + 12\sqrt{3}}}, \frac{4(4 + \sqrt{3})}{\sqrt{25 + 12\sqrt{3}}}\right),\right. \\
& \left(-\frac{9 + 4\sqrt{3}}{\sqrt{25 + 12\sqrt{3}}}, -\frac{4\sqrt{3}}{\sqrt{25 + 12\sqrt{3}}}, -\frac{4(4 + \sqrt{3})}{\sqrt{25 + 12\sqrt{3}}}\right), \\
& \left(\frac{-9 + 4\sqrt{3}}{\sqrt{25 - 12\sqrt{3}}}, \frac{4\sqrt{3}}{\sqrt{25 - 12\sqrt{3}}}, \frac{4(-4 + \sqrt{3})}{\sqrt{25 - 12\sqrt{3}}}\right), \\
& \left.\left(\frac{9 - 4\sqrt{3}}{\sqrt{25 - 12\sqrt{3}}}, \frac{-4\sqrt{3}}{\sqrt{25 - 12\sqrt{3}}}, \frac{4(4 - \sqrt{3})}{\sqrt{25 - 12\sqrt{3}}}\right)\right\}.
\end{aligned}$$

The system of equations in the statement of the problem has an interesting geometric interpretation. Let ABC be a triangle all of whose angles are smaller than $120°$. The Fermat point (or Torricelli point) of the triangle ABC is a point P such that the total distance from the three vertices of the triangle to the point is the minimum possible (see http://en.wikipedia.org/wiki/Fermat point).

Let $AB = c, BC = a, CA = b, AP = x, BP = y, CP = z$. Then

$$\begin{aligned}
\angle APB = \angle APC &= \angle BPC = 120° \text{ and} \\
x^2 + xy + y^2 &= c^2, \\
y^2 + yz + z^2 &= a^2, \\
z^2 + xz + x^2 &= b^2,
\end{aligned}$$

by the law of cosines. So x, y and z are the distances from the three vertices of the triangle to the Fermat point of the triangle.

- **Solution 2 by José Luis Díaz-Barrero, Barcelona, Spain**

Subtracting the equations term by term, we obtain

$$\begin{array}{c} (x^2 - y^2) + z(x - y) = 9, \\ (x^2 - z^2) + y(x - z) = -7, \end{array} \quad \Leftrightarrow \quad \begin{array}{c} (x - y)(x + y + z) = 9, \\ (x - z)(x + y + z) = -7. \end{array}$$

Putting $u = x + y + z$, then we obtain $(x - y)u = 9$ and $(x - z)u = -7$. Adding both equations yields $(3x - (x + y + z))u = 2$ from which follows $x = \dfrac{u^2 + 2}{3u}$. Likewise, we obtain $y = \dfrac{u^2 - 25}{3u}$, and $z = \dfrac{u^2 + 23}{3u}$. Substituting the values of x, y, z into one of the equations of the given system, yields

$$\left(\frac{u^2+2}{3u}\right)^2 + \left(\frac{u^2+2}{3u}\right)\left(\frac{u^2-25}{3u}\right) + \left(\frac{u^2-25}{3u}\right)^2 = 3^2$$

or equivalently,

$$3u^4 - 150u^2 + 579 = 0.$$

Solving the preceding equation, we have the solutions:

$$\pm\sqrt{25 - 12\sqrt{3}}, \quad \pm\sqrt{25 + 12\sqrt{3}}.$$

Substituting these values in the expressions of x, y, z yields four triplets of solutions for the system. Namely,

$$\begin{aligned}(x_1, y_1, z_1) &= \left(\frac{27 - 12\sqrt{3}}{3\sqrt{25 - 12\sqrt{3}}}, \frac{-4\sqrt{3}}{\sqrt{25 - 12\sqrt{3}}}, \frac{48 - 12\sqrt{3}}{3\sqrt{25 - 12\sqrt{3}}}\right)\\ &= (1.009086173, -3.374440097, 4.418495493)\end{aligned}$$

$$\begin{aligned}(x_2, y_2, z_2) &= \left(-\frac{27 - 12\sqrt{3}}{3\sqrt{25 - 12\sqrt{3}}}, \frac{4\sqrt{3}}{\sqrt{25 - 12\sqrt{3}}}, -\frac{48 - 12\sqrt{3}}{3\sqrt{25 - 12\sqrt{3}}}\right)\\ &= (-1.009086173, 3.374440097, -4.418495493)\end{aligned}$$

$$\begin{aligned}(x_3, y_3, z_3) &= \left(\frac{27 + 12\sqrt{3}}{3\sqrt{25 + 12\sqrt{3}}}, \frac{4\sqrt{3}}{\sqrt{25 + 12\sqrt{3}}}, \frac{48 + 12\sqrt{3}}{3\sqrt{25 - 12\sqrt{3}}}\right)\\ &= (2.354003099, 1.023907822, 3.388521646)\end{aligned}$$

$$\begin{aligned}(x_4, y_4, z_4) &= \left(-\frac{27 + 12\sqrt{3}}{3\sqrt{25 + 12\sqrt{3}}}, -\frac{4\sqrt{3}}{\sqrt{25 + 12\sqrt{3}}}, -\frac{48 + 12\sqrt{3}}{3\sqrt{25 - 12\sqrt{3}}}\right)\\ &= (-2.354003099, -1.023907822, -3.388521646)\end{aligned}$$

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy; Boris Rays, Brooklyn, NY; Titu Zvonaru, Comănesti, Romania jointly with Neculai Stanciu, Buzău, Romania, and the proposer.

- **5177:** *Proposed by Kenneth Korbin, New York, NY*

  A regular nonagon $ABCDEFGHI$ has side 1.

  Find the area of $\triangle ACF$.

Solution 1 by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX

We begin with the following known facts:

1. Each angle in a regular nonagon is 140°.
2. $\cos 140^\circ = \cos(180^\circ - 40^\circ) = -\cos 40^\circ$.
3. $\cos 100^\circ = -\cos 80^\circ$.
4. $1 + \cos 2\theta = 2\cos^2\theta$.
5. $\mathcal{A} = \frac{1}{2}ab\sin C$ in $\triangle ABC$.

Hence, $\triangle ABC \cong \triangle HIA \cong \triangle HGF$ by SAS. Using Fact 1, since $\angle B = \angle I = \angle G = 140^\circ$, it follows that $\angle BAC = \angle IAH = \angle IHA = \angle GHF = \angle GFH = 20^\circ$. Thus, $\angle AHF = 100^\circ$. Since $\triangle AHF$ is an isosceles triangle, $\angle HAF = \angle HFA = 40^\circ$. Therefore, $\angle CAF = 60^\circ$. In $\triangle ABC$, using the Law of Cosines and Facts 2 and 4,

$$\begin{aligned} AC^2 &= 1 + 1 - 2\cos 140^\circ \\ &= 2(1 - \cos 140^\circ) \\ &= 2(1 + \cos 40^\circ) \\ &= 4\cos^2 20^\circ \quad \text{Then,} \\ AC &= 2\cos 20^\circ. \end{aligned}$$

Similarly, since $AC = HA = HF = 2\cos 20^\circ$, using the Law of Cosines and Facts 3 and 4 in $\triangle HAF$,

$$\begin{aligned} AF^2 &= (2\cos 20^\circ)^2 + (2\cos 20^\circ)^2 - 2(2\cos 20^\circ)^2\cos 100^\circ \\ &= 8\cos^2 20^\circ(1 - \cos 100^\circ) \\ &= 8\cos^2 20^\circ(1 + \cos 80^\circ) \\ &= 16\cos^2 20^\circ\cos^2 40^\circ \quad \text{Thus,} \\ AF &= 4\cos 20^\circ\cos 40^\circ. \end{aligned}$$

In $\triangle ACF$, using Fact 5,

$$\begin{aligned} A &= \frac{1}{2}(AC)(AF)\sin 60^\circ \\ &= \frac{1}{2}(2\cos 20^\circ)(4\cos 20^\circ\cos 40^\circ)\left(\frac{\sqrt{3}}{2}\right) \\ &= 2\sqrt{3}\cos^2 20^\circ\cos 40^\circ \\ &\approx 2.343237. \end{aligned}$$

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Denote the circumcenter and the circumradius of the nonagon by O and r, respectively.

The nonagon can be oriented within the Cartesian plane so that its vertices are

$$A\left(r\cos 0^\circ, r\sin 0^\circ\right) \qquad B\left(r\cos 40^\circ, r\sin 40^\circ\right) \qquad C\left(r\cos 80^\circ, r\sin 80^\circ\right)$$

$$D\left(r\cos 120^\circ, r\sin 120^\circ\right) \quad E\left(r\cos 160^\circ, r\sin 160^\circ\right) \quad F\left(r\cos 200^\circ, r\sin 200^\circ\right)$$

$$G\left(r\cos 240^\circ, r\sin 240^\circ\right) \quad H\left(r\cos 280^\circ, r\sin 2800^\circ\right) \quad I\left(r\cos 320^\circ, r\sin 320^\circ\right).$$

Then,

$$\begin{aligned}
1^2 = \overline{AB}^2 &= \left(r\cos 40^\circ - r\cos 0^\circ\right)^2 + \left(r\sin 40^\circ - r\sin 0^\circ\right)^2 \\
&= r^2\left(\cos^2 40^\circ - 2\cos 40^\circ + 1 + \sin^2 40^\circ\right)^2 \\
&= 2r^2\left(1-\cos 40^\circ\right) \Rightarrow r^2 = \frac{1}{2\left(1-\cos 40^\circ\right)}.
\end{aligned}$$

The area of $\triangle ACF$ is

$$\begin{aligned}
[\triangle ACF] &= \frac{1}{2}\left| det \begin{pmatrix} 1 & 1 & 1 \\ r & r\cos 80^\circ & r\cos 200^\circ \\ 0 & r\sin 80^\circ & r\sin 200^\circ \end{pmatrix} \right| \\
&= \frac{1}{2}\left| r^2\cos 80^\circ \sin 200^\circ + r^2\sin 80^\circ - r^2\cos 200^\circ \sin 80^\circ - r^2 \sin 200^\circ \right| \\
&= \frac{1}{2}\left| r^2\left(\cos 80^\circ \sin 200^\circ - \sin 80^\circ \cos 200^\circ\right) + r^2\sin 80^\circ - r^2\sin 200^\circ \right| \\
&= \frac{r^2}{2}\left|(\sin(200^\circ - 80^\circ) + \sin 80^\circ - \sin 200^\circ\right| \\
&= \frac{1}{4\left(1-\cos 40^\circ\right)}\left|\sin 120^\circ + \sin 80^\circ - \sin 200^\circ\right| \\
&\approx 2.343237.
\end{aligned}$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

It is easy to check that $\angle BAC = 20^\circ, \angle IAF = \angle FAC = 60^\circ$ and $AC = 2\cos 20^\circ$. Suppose that the perpendicular from I to AF meets AF at J, the perpendicular from H to AF meets AF at K, and the perpendicular from I to HK meets HK at L. Then $\angle HIL = 20^\circ$ and

$$AF = 2(AJ + JK) = 2(AJ + IL) = 2(\cos 60^\circ + \cos 20^\circ) = 1 + 2\cos 20^\circ.$$

Hence the area of $\triangle ACF$ equals

$$\frac{(AC)(AF)\sin\angle FAC}{2}$$

$$
\begin{aligned}
&= \frac{\cos 20^\circ (1+2\cos 20^\circ)\sqrt{3}}{2} \\
&= \frac{(1+\cos 20^\circ + \cos 40^\circ)\sqrt{3}}{2} \\
&= \frac{\sqrt{3}(1+\sqrt{3}\cos 10^\circ)}{2} \\
&\approx 2.343237.
\end{aligned}
$$

Solution 4 by proposer

$$
\begin{aligned}
\text{Area of } \triangle ACF &= \frac{\sin 40^\circ \cdot \sin 60^\circ \cdot \sin 80^\circ}{2\sin^2 20^\circ} \\
&= \frac{\sqrt{3}}{16}\left[3\tan^2 70^\circ - 1\right] \\
&\approx 2.343237.
\end{aligned}
$$

Comment by editor: Sines and cosines of angles of $10^\circ, 20^\circ, 40^\circ$ and their complements often appear in the above solutions. **David Stone and John Hawkins of Statesboro, GA** noted in their solution that: "It may be possible to express the result ($\sqrt{3}\cos 40^\circ\,(1+\cos 40^\circ)$) in terms of radicals, even though $\cos 40^\circ$ itself cannot be expressed in terms of surds; it (along with $\sin 10^\circ$ and $-\sin 70^\circ$) is a zero of the famous *casus irreducibilis* cubic $8x^3 - 6x + 1 = 0$."

Also solved by Scott H. Brown, Montgomery, AL; Brian D. Beasley, Clinton, SC; Kenneth Day and Michael Thew (jointly, students at Saint George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; John Nord, Spokane, WA; Boris Rays, Brooklyn, NY, and Albert Stadler, Herrliberg, Switzerland.

- **5178:** *Proposed by Neculai Stanciu, Buzău, Romania*

Prove: If x, y and z are positive real numbers such that $xyz \geq 7+5\sqrt{2}$, then

$$x^2+y^2+z^2-2(x+y+z) \geq 3.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

By the AM-GM inequality, $\dfrac{x+y+z}{3} \geq \sqrt[3]{xyz} \geq \sqrt[3]{7+5\sqrt{2}} = 1+\sqrt{2}$. Let $f(x) = x^2-2x-1$. $f(x)$ is a convex function that is monotonically increasing for $x \geq 1$. By Jensen's inequality,

$$x^3+y^3+z^3-2\,(x+y+z)-3 = f(x)+f(y)+f(z) \geq 3f\left(\frac{x+y+z}{3}\right) \geq 3f\left(1+\sqrt{2}\right) = 0.$$

Solution 2 by David E. Manes, Oneonta, NY

Note that for positive real numbers if $x \geq 1+\sqrt{2}$, then $(x-1)^2 \geq 2$ with equality if and only if $x = 1+\sqrt{2}$. Therefore, if $x, y, z \geq 1+\sqrt{2}$, then $xyz \geq 7+5\sqrt{2}$ and $(x-1)^2 + (y-1)^2 + (z-1)^2 \geq 6$. Expanding this inequality yields $x^2+y^2+x^2-2(x+y+z) \geq 3$ with equality if and only if $x = y = z = 1+\sqrt{2}$.

Solution 3 by Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy

We know tht $x^2+y^2+z^2 \geq \dfrac{(x+y+z)^2}{3}$ thus the inequality is implied by

$$S^2 - 6S - 9 \geq 0, \qquad S = x+y+z$$

yielding $S \geq 3(1+\sqrt{2})$. Moreover by the AGM we have $S \geq 3(xyz)^{1/3} \geq 3(7+5\sqrt{2})^{1/3}$, thus we need to check that $3(7+5\sqrt{2})^{1/3} \geq 3(1+\sqrt{2})$ or $7+5\sqrt{2} \geq (1+\sqrt{2})^3$ which is actually an equality, and we are done.

Also solvled by Arkady Alt, San Jose California; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Boris Rays, Brooklyn, NY, and the proposer.

- **5179:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Find all positive real solutions $(x_1, x_2, \ldots, x_n)$ of the system

$$\begin{cases} x_1 + \sqrt{x_2+11} = \sqrt{x_2+76}, \\ x_2 + \sqrt{x_3+11} = \sqrt{x_3+76}, \\ \cdots\cdots\cdots \\ x_{n-1} + \sqrt{x_n+11} = \sqrt{x_n+76}, \\ x_n + \sqrt{x_1+11} = \sqrt{x_1+76}. \end{cases}$$

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

If $f(t) = \sqrt{t+76} - \sqrt{t+11}$ on $(0, \infty)$, then

$$f'(t) = \frac{1}{2}\left(\frac{1}{\sqrt{t+76}} - \frac{1}{\sqrt{t+11}}\right), \text{ and hence,}$$

$$\begin{aligned} |f'(t)| &= \frac{1}{2}\left(\frac{1}{\sqrt{t+11}} - \frac{1}{\sqrt{t+76}}\right) \\ &< \frac{1}{2}\frac{1}{\sqrt{t+11}} \\ &< \frac{\sqrt{11}}{22} \\ &< 1 \end{aligned}$$

for $t > 0$. It follows that $f(t)$ is a contraction mapping on $(0, \infty)$ and therefore, $f(t)$ has a unique fixed point $t^* \in (0, \infty)$. Further, it is well-known that for any $\bar{t} \in (0, \infty)$, the sequence defined recursively by $t_1 = \bar{t}$ and $t_{k+1} = f(t_k)$ for $k \geq 1$ must converge to t^*. By trial and error, we find that $t^* = 5$.

In this problem,

$$\begin{aligned} x_1 &= f(x_2), \\ x_2 &= f(x_3), \\ &\vdots \\ x_{n-1} &= f(x_n), \\ x_n &= f(x_1). \end{aligned}$$

If we let $t_1 = x_1$ and define $t_{k+1} = f(t_k)$ for $k \geq 1$, then $(x_1, x_n, \ldots, x_3, x_2)$ is a cycle in the sequence $\{t_k\}$. However, as described above, $t_k \to 5$ as $k \to \infty$. These conditions force $x_1 = x_2 = \cdots = x_n = 5$ and therefore, this must be the unique solution for this system.

Also solved by Arkady Alt, San Jose, CA; Scott H. Brown, Montgomery, AL; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; Neculai Stanciu, Buzău Romania, jointly with Titu Zvonaru, Comănesti, Romania, and the proposer.

- **5180:** *Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy*

Let a, b and c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{1+a}{bc} + \frac{1+b}{ac} + \frac{1+c}{ab} \geq \frac{4}{\sqrt{a^2+b^2-ab}} + \frac{4}{\sqrt{b^2+c^2-bc}} + \frac{4}{\sqrt{a^2+c^2-ac}}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

Multiplying both sides of the desired inequality by abc, we see that it is equivalent to

$$1 + a^2 + b^2 + c^2 \geq 4abc\left(\frac{1}{\sqrt{a^2+b^2-ab}} + \frac{1}{\sqrt{b^2+c^2-bc}} + \frac{1}{\sqrt{a^2+c^2-ac}}\right). \qquad (1)$$

Since

$$a^2 + b^2 - ab = (a-b)^2 + ab \geq ab, \quad b^2 + c^2 - bc \geq bc, \quad a^2 + c^2 - ac \geq ac,$$

the right hand side of (1) is less than or equal to

$$\begin{aligned} & 4\left(\sqrt{abc} + \sqrt{bca} + \sqrt{cab}\right) \\ \leq\ & 2\left((a+b)c + (b+c)a + (c+a)b\right) \\ =\ & 4(ab + bc + ca) \\ =\ & 2\left((a+b+c)^2 - a^2 - b^2 - c^2\right) \end{aligned}$$

$$= \quad 2 - 2\left(a^2 + b^2 + c^2\right).$$

Now

$$a^2 + b^2 + c^2 = \left(a - \frac{1}{3}\right)^2 + \left(b - \frac{1}{3}\right)^2 + \left(c - \frac{1}{3}\right)^2 + \frac{2(a+b+c)}{3} - \frac{1}{3} \geq \frac{1}{3},$$

so that $1 + a^2 + b^2 + c^2 \geq 2 - 2\left(a^2 + b^2 + c^2\right)$.
This proves (1) and completes the solution.

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

By the AM-GM inequality,

$$\begin{aligned}
\frac{1+a}{bc} + \frac{1+b}{ca} + \frac{1+c}{ab} &= \frac{a + a^2 + b + b^2 + c + c^2}{abc} \\
&= \frac{1 + a^2 + b^2 + c^2}{abc} \\
&= \frac{(a+b+c)^2 + a^2 + b^2 + c^2}{abc} \\
&= \frac{(2a^2 + 2bc) + (2b^2 + 2ca) + (2c^2 + 2ab)}{abc} \\
&\geq 4\frac{a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}}{abc} \\
&= \frac{4}{\sqrt{bc}} + \frac{4}{\sqrt{ca}} + \frac{4}{\sqrt{ab}}.
\end{aligned}$$

The conclusion follows since

$$\frac{1}{\sqrt{xy}} \geq \frac{1}{\sqrt{x^2 + y^2 - xy}}.$$

(Note that this inequality is equivalent to $x^2 + y^2 - xy \geq xy$ which is obviously true.)

Also solved by Arkady Alt, San Jose, CA; Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie (jointly), San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; David E. Manes, Oneonta, NY; Titu Zvonaru, Comănesti, Romania jointly with Neculai Stanciu, Buzău, Romania, and the proposer.

- **5181:** *Proposed by Ovidiu Furdui, Cluj, Romania*

  Calculate:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n \cdot m}{(n+m)!}.$$

Solution 1 by Anastasios Kotronis, Athens, Greece The summands being all positive we can sum by triangles :

$$\begin{aligned}\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{nm}{(n+m)!} &= \sum_{k,\ell,n\in\wedge\, k+\ell=n}\frac{nm}{(n+m)!}=\sum_{n=2}^{+\infty}\frac{\sum_{\ell=1}^{n-1}(n-\ell)\ell}{n!}\\ &= \frac{1}{6}\sum_{n=2}^{+\infty}\frac{(n-1)n(n+1)}{n!}=\frac{1}{6}\sum_{n=2}^{+\infty}\frac{(n+1)}{(n-2)!}\\ &= \frac{1}{6}\sum_{n=0}^{+\infty}\frac{(n+3)}{n!}=\frac{1}{6}\sum_{n=0}^{+\infty}\frac{1}{n!}\frac{dx^{n+3}}{dx}\Big|_{x=1}\\ &= \frac{1}{6}\frac{d}{dx}\left(\sum_{n=0}^{+\infty}\frac{x^{n+3}}{n!}\right)\Big|_{x=1}=\frac{1}{6}\frac{d\left(x^3e^x\right)}{dx}\Big|_{x=1}\\ &= \frac{2e}{3}.\end{aligned}$$

Solution 2 by Arkady Alt, San Jose, CA

Let $k=m+n$. Then $m=k-n$ and domain of summation $\begin{cases}1\le n\\ 1\le m\end{cases}$ can be represented as $\begin{cases}2\le k\\ 1\le n\le k-1\\ m=k-n\end{cases}$. Hence,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{nm}{(n+m)!}=\sum_{k=2}^{\infty}\sum_{n=1}^{k-1}\frac{n(k-n)}{k!}=\sum_{k=2}^{\infty}\frac{1}{k!}\sum_{n=1}^{k-1}n(k-n)=\sum_{k=2}^{\infty}\frac{1}{k!}\sum_{n=1}^{k-1}n(k-n).$$

Since

$$\begin{aligned}\sum_{n=1}^{k-1}n(k-n) &= \frac{k^2(k-1)}{2}-\frac{(k-1)k(2k-1)}{6}\\ &= \frac{k}{6}\left(3k^2-3k-2k^2+3k-1\right)\\ &= \frac{k\left(k^2-1\right)}{6},\end{aligned}$$

then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{nm}{(n+m)!} = \frac{1}{6}\sum_{k=2}^{\infty}\frac{k+1}{(k-2)!}$$

$$
\begin{aligned}
&= \frac{1}{6}\sum_{k=0}^{\infty}\frac{k+3}{k!} \\
&= \frac{1}{6}\left(\sum_{k=0}^{\infty}\frac{3}{k!}+\frac{k}{k!}\right) \\
&= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{k!}+\frac{1}{6}\sum_{k=1}^{\infty}\frac{1}{(k-1)!} \\
&= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{k!}+\frac{1}{6}\sum_{k=0}^{\infty}\frac{1}{k!} \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{2}+\frac{1}{6}\right) \\
&= \frac{2}{3}\sum_{k=0}^{\infty}\frac{1}{k!} \\
&= \frac{2e}{3}.
\end{aligned}
$$

Solution 3 by the proposer

The series equals $\frac{2e}{3}$. First we note that for $m \geq 0$ and $n \geq 1$ one has that

$$\int_0^1 x^m(1-x)^{n-1}dx = B(m+1,n) = \frac{m!\cdot(n-1)!}{(n+m)!}.$$

Thus,

$$
\begin{aligned}
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n\cdot m}{(n+m)!} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n}{(n-1)!}\cdot\frac{1}{(m-1)!}\int_0^1 x^m(1-x)^{n-1}dx \\
&= \int_0^1\left(\sum_{n=1}^{\infty}\frac{n}{(n-1)!}(1-x)^{n-1}\right)\cdot\left(\sum_{m=1}^{\infty}\frac{x^m}{(m-1)!}\right)dx \\
&= \int_0^1\left(1+\sum_{n=2}^{\infty}\frac{n}{(n-1)!}(1-x)^{n-1}\right)\cdot xe^x dx \\
&= \int_0^1\left(1+\sum_{n=2}^{\infty}\frac{(1-x)^{n-1}}{(n-2)!}+\sum_{n=2}^{\infty}\frac{(1-x)^{n-1}}{(n-1)!}\right)\cdot xe^x dx \\
&= \int_0^1\left(1+(1-x)e^{1-x}+e^{1-x}-1\right)\cdot xe^x dx \\
&= e\int_0^1(2-x)xdx = \frac{2e}{3},
\end{aligned}
$$

and the problem is solved.

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