Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
April 15, 2016

- **5385:** *Proposed by Kenneth Korbin, New York, NY*

  A triangle with integer length sides and integer area has perimeter $P = 6^6$. Find the sides of the triangle when the area is minimum.

- **5386:** *Proposed by Michael Brozinsky, Central Islip, NY.*

  Determine whether or not there exit nonzero constants a and b such that the conic whose polar equation is
  $$r = \sqrt{\frac{a}{\sin(2\theta) - b\cos(2\theta)}}$$
  has a rational eccentricity.

- **5387:** *Proposed by Arkady Alt, San Jose, CA*

  Let $D := \{(x, y) \mid x, y \in R_+,\ x \neq y \text{ and } x^y = y^x\}$.(Obviously $x \neq 1$ and $y \neq 1$).
  Find $\sup\limits_{(x,y)\in D} \left(\dfrac{x^{-1} + y^{-1}}{2}\right)^{-1}$

- **5388:** *Proposed by Jiglău Vasile, Arad, Romania*

  Let $ABCD$ be a cyclic quadrilateral, R and r its exradius and inradius respectively, and a, b, c, d its side lengths (where a and c are opposite sides.) Prove that
  $$\frac{R^2}{r^2} \geq \frac{a^2c^2}{b^2d^2} + \frac{b^2d^2}{a^2c^2}.$$

- **5389:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let ABC be a scalene triangle with semi-perimeter s and area $\mathcal{A}$. Prove that
  $$\frac{3a + 2s}{a(a - b)(a - c)} + \frac{3b + 2s}{b(b - a)(b - c)} + \frac{3c + 2s}{c(c - a)(c - b)} < \frac{3\sqrt{3}}{4\mathcal{A}}.$$

- **5390:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $A \in \mathcal{M}_2(R)$ such that $AA^T = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, where $a > b \geq 0$. Prove that $AA^T = A^T A$ if and only if $A = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$ or $A = \begin{pmatrix} \beta & \alpha \\ \alpha & \beta \end{pmatrix}$, where $\alpha = \dfrac{\pm\sqrt{a+b} \pm \sqrt{a-b}}{2}$ and $\beta = \dfrac{\pm\sqrt{a+b} \mp \sqrt{a-b}}{2}$. Here A^T denotes the *transpose* of A.

Solutions

- **5367:** *Proposed by Kenneth Korbin, New York, NY*

  Given triangle ABC with integer length sides and integer area. The vertices have coordinates $A(0,0),\ B(x,y)$ and $C(z,w)$ with $\sqrt{x^2+y^2} - \sqrt{z^2+w^2} = 1$.

  Find positive integers x, y, z and w if the perimeter is 84.

  Solution by Ed Gray, Highland Beach, FL

  Let the sides of the triangle be a, b, c where $b = \sqrt{z^2+w^2}$ and $c = \sqrt{x^2+y^2}$. We are given that

$$\begin{aligned} c - b &= 1 \\ a + b + c &= 84. \text{ So, subtracting} \\ a + 2b &= 83, \text{ or, } a = 83 - 2b. \end{aligned}$$

  By Brahmagupta's formula, the area T is given by

$$\begin{aligned} T^2 &= s(s-a)(s-b)(s-c), \text{ where } s = \frac{1}{2}(a+b+c) = 42. \text{ Then,} \\ T^2 &= 42\,(42 - (83 - 2b))\,(42 - b)\,(42 - (b+1))\,, \text{ or} \\ T^2 &= 42\,(2b - 41)\,(42 - b)(41 - b) \Longrightarrow b = 34. \text{ So} \\ T^2 &= (42)(27)(8)(7) = (14)^2 \cdot 9^2 \cdot 2^2 = (252)^2 \Longrightarrow \\ T &= 252,\ b = 34,\ c = b + 1 = 35, \text{ and } a = 15. \end{aligned}$$

  Since $b = \sqrt{z^2+w^2}$, $b^2 = 34^2 = 1156 = z^2 + w^2$ and we have $z = 30, w = 16$ since $900 + 256 = 1156$, or vice versa, $z = 16$ and $w = 30$. Similarly,

  $c = \sqrt{x^2+y^2}$, $c^2 = 35^2 = 1225 = x^2 + y^2$ and we have $x = 28, y = 21$ since $784 + 441 = 1225$, or vice versa, $x = 21$ and $y = 28$.

  In summary, $(x, y, z, w) \in \{(21, 28, 30, 16), (28, 21, 16, 30)\}$.

  Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo Sate University, San Angelo, TX; Brian D. Beasley, Presbyterian College, Clinton, SC; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania;

David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5368:** *Proposed by Ed Gray, Highland Beach, FL*

Let $abcd$ be a four digit number in base 10, none of which are zero, such that the last four digits in the square of $abcd$ are $abcd$, the number itself. Find the number $abcd$.

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

If $x = \left(a \times 10^3\right) + \left(b \times 10^2\right) + (c \times 10) + d$, with $a, b, c, d \in \{1, 2, \ldots, 9\}$,then

$$x^2 = \left(a^2 \times 10^6\right) + \left(2ab \times 10^5\right) + \left[\left(b^2 + 2ac\right) \times 10^4\right] + \left[2\left(ad + bc\right) \times 10^3\right] + \left[\left(c^2 + 2bd\right) \times 10^2\right] + (2cd \times 10) + d^2.$$

In order for the units digit of x^2 to be d, we must have $d^2 \equiv d \pmod{10}$. Since $d \in \{1, 2, \ldots, 9\}$, this restricts our choices to $d = 1$, 5, or 6

Case 1. If $d = 1$, then $d^2 = 1$ and to obtain c as the tens digit of x^2, we need $2cd \equiv c \pmod{10}$. Since $d = 1$, this reduces to $c \equiv 0 \pmod{10}$, which is impossible when $c \in \{1, 2, \ldots, 9\}$. Therefore, this case fails.

Case 2. If $d = 5$, then $d^2 = 25$ and to get c as the tens digit of x^2, we require that $2cd + 2 \equiv c \pmod{10}$. With $d = 5$, this reduces to $c \equiv 2 \pmod{10}$ and hence, $c = 2$. When $c = 2$ and $d = 5$, we have $(2cd \times 10) + d^2 = 225$. To get b as the hundreds digit of x^2, we are forced to set

$$c^2 + 2bd + 2 \equiv b \pmod{10}.$$

This reduces to $b \equiv 6 \pmod{10}$ and thus, $b = 6$. When $d = 5$, $c = 2$, and $b = 6$, we have $\left(c^2 + 2bd\right) \times 10^2 + (2cd \times 10) + d^2 = 6625$. Finally, to obtain a as the thousands digit of x^2, we are left with

$$2\left(ad + bc\right) + 6 \equiv a \pmod{10},$$

which reduces to $a \equiv 0 \pmod{10}$. Since this is impossible when $a \in \{1, 2, \ldots, 9\}$, this case also fails.

Case 3. If $d = 6$, then $d^2 = 36$ and to get c as the tens digit of x^2, we must set $2cd + 3 \equiv c \pmod{10}$. This reduces to $c \equiv 7 \pmod{10}$ and hence, $c = 7$. When $d = 6$ and $c = 7$, $(2cd \times 10) + d^2 = 876$. To get b as the hundreds digit of x^2 now requires that $c^2 + 2bd + 8 \equiv b \pmod{10}$, i.e., $b \equiv 3 \pmod{10}$. Hence, $b = 3$ and $\left(c^2 + 2bd\right) \times 10^2 + (2cd \times 10) + d^2 = 9376$. Finally, in order for the thousands digit of x^2 to be a, we need $2\left(ad + bc\right) + 9 \equiv a \pmod{10}$ or $a \equiv 9 \pmod{10}$. This yields $a = 9$ and $x = 9376$. Since $(9376)^2 = 87909376$, our solution is complete.

Solution 2 by Bruno Salguerio Fanego, Viveiro, Spain

Note that $abcd$ can be expressed as $1000a + 100b + 10c + d$, whose square $(abcd)^2$ is

$$a^2 \cdot 10^6 + 2ab \cdot 10^5 + (2ac + b2) \cdot 10^4 + (2ad + 2bc) \cdot 1000 + (2bd + c^2) \cdot 100 + 2cd \cdot 10 + d^2.$$

Moreover, $1 \le a, b, c, d \le 9$. We distinguish several cases:

If $d \le 3$, the last digit of $(abcd)^2$ is d^2, which, since its last four digits are $abcd$, must be equal to d, so $d = 1$, in which case, for $c \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, we obtain that the last two digits of $(abc1)^2 = \ldots + 2c \cdot 10 + 1$ are, respectively, $\{21, 41, 61, 81, 01, 21, 41, 61, 81\}$ and, on the other hand, since the last two digits of $(abc1)^2$ are equal to c1, they must be also equal to $\{11, 21, 31, 41, 51, 61, 71, 81, 91\}$. But none of the two possible ending digits for $(abcd)^2$ coincides with units digit of this last possible ending, and so we conclude that this case, that is, $d \leq 3$, is impossible, so $d \geq 4$. Since d^2 ends in $1, 4, 9, 6$ or $5, (abcd)^2$ ends in $1, 4, 9, 6$ or 5, so $d \in \{4, 5, 6, 9\}$ and, hence, $(abcd)^2$ ends in $6, 5, 6, 1$ respectively, so $d \in \{6, 5, 6, 1\}$ respectively, which implies that $d \in \{5, 6\}$.

When $d = 5, (abcd)^2 = \ldots + (2bd + c^2 + c) \cdot 10 + 25$ ends in 25, so $c = 2$. Then,

$$(abcd)^2 = \cdots + (2b \cdot 5 + 2 \cdot 2^2) \cdot 100 + (2 \cdot 2 \cdot 5 + 2^2) \cdot 10 + 25 = \ldots + 625,$$

which ends in 625, so $b = 6$. Hence,

$$(abcd)^2 = \ldots + (2 \cdot a \cdot 5 + 2 \cdot 6 \cdot 2) \cdot 1000 + (2 \cdot 6 \cdot 5 + 2^2) \cdot 100 + (2 \cdot 2 \cdot 5) \cdot 10 + 25,$$

which ends in 0625 and this contradicts the fact that $(abcd)^2$ must end in $abcd$ (because a cannot be equal to zero).

When $d = 6$, we obtain respectively that, for $c \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, (abcd $)^2$ ends in $56, 76, 96, 16, 36, 56, 76, 96, 16$. Thus, the only possible case is $c = 7$, being thus

$$(abcd)^2 = (ab76)^2 = (12a + 14b) \cdot 1000 + (12b + 49) \cdot 100 + 876.$$

Hence, we obtain that, when $b \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}, (abcd)^2$ ends in $976, 176, 376, 576, 776, 976, 176, 376, 576$, respectively, which implies that $b = 3$ is the only possibility.

Then, $(abcd)^2 = (a376)^2$, which ends $3376, 5376, 7376, 9376, 1376, 3376, 5376, 7376, 9376$ for a equal to $1, 2, 3, 4, 5, 6, 7, 8, 9$. This implies that $a = 9$ and since 9376^2 ends in 9376, we conclude that the only solution to the problem is the number 9376.

Solution 3 by Paul M. Harms, North Newton, KS

Let us look for the answer to the problem by checking one digit at a time. First consider a one-digit number whose square has the same units digit as the original number. The one-digit number will have to be 1, 5, or 6.

Let us now try two-digit numbers whose units digit is 1 and whose square has the same last two digits as the original number. It is easy to show that no two-digit number exists for this case.

Now consider the case where the units digit is 5. All numbers of this type have squares ending in 25. Thew number 25 is the only two-digit number whose square ends in 25.

We find 625 is the only three-digit number whose square ends in 625.

If a is any non-zero fourth digit, we find that $a625$ has a square that ends in 0625.Thus the number satisfying the problem cannot end in 5. We now consider the case where the units digit is 6. We see that $76^2 = 5776,\ 376^2 = 141376$, and $9376^2 = 87909376$. The number 9376 satisfies the problem.

Editor's comment: **Brian D. Beasley of Presbyterian College in Clinton SC, Kenneth Korbin of New York, NY, and the team of David Stone and John Hawkins of Georgia Southern University** each mentioned in their solution that

such sequences are called "automorphic numbers" and start as
$\{5, 25, 625, 0625, 90625, \ldots\}$ and $\{6, 76, 376, 9376, 09376, \ldots\}$. See: Weisstein, Eric W. "Automorphic Number" in MathWorld-A Wolfram Web Resource, <http://matheworld.wolfram.com/Automorphic Number.html>.

David Stone and John Hawkins constructed and proved the following theorem.
For any $n \geq 1$, there are exactly four n-digit integers N such that the last n digits of N^2 are the digits of N. The four numbers are 0 and 1 (considered as n-digit integers), $2^{n \cdot 4 \cdot 5^{n-1}}$ and $5^{n \cdot 2^{n-1}}$ (both being reduced mod 10^n).
They went on to say that they did not find the above theorem in the literature that they searched on automorphic numbers.

Also solved by Stephen Acampa (student at Eastern Kentucky University), Richmond, KY; Brian D. Beasley, Presbyterian College, Clinton SC; Kee-Wai Lau, Hong Kong, China; Kenneth Korbin, New York, NY; Carl Libis, Columbia Southern University, Orange Beach, AL; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Susan Popp (graduate student at Eastern Kentucky University), Richmond, KY; Erron Prickett (graduate student at Eastern Kentucky University), Richmond, KY; Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins of Georgia Southern University, Statesboro, GA; Deven Turner (student at Eastern Kentucky University), Richmond, KY, and the proposer.

- **5369:** *Proposed by Chirita Marcel, Bucuresti, Romania*

A convex quadrilateral $ABCD$ has area S and side lengths
$\overline{AB} = a, \overline{BC} = b, \overline{CD} = c, \overline{DA} = d$. Show that

$$2\left(a+b+c+d\right)^2 + a^2 + b^2 + c^2 + d^2 \geq 36\sqrt{\left(S^2 + abcd\cos^2\frac{A+C}{2}\right)}.$$

Solution by Nikos Kalapodis, Patras, Greece

Taking into account the Bretschneider's formula (see [1]) for the area of a convex quadrilateral:
$S = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\frac{A+C}{2}}$, where $s = \dfrac{a+b+c+d}{2}$,
we see that the given inequality is equivalent to

$$2(a+b+c+d)^2 + a^2 + b^2 + c^2 + d^2 \geq 36\sqrt{(s-a)(s-b)(s-c)(s-d)} \quad (*).$$

Now from the Cauchy-Schwartz inequality and the AM-GM inequality we have

$$\begin{aligned} 2(a+b+c+d)^2 + a^2 + b^2 + c^2 + d^2 &\geq 2(a+b+c+d)^2 + \frac{(a+b+c+d)^2}{4} \\ &= \frac{9}{4}(a+b+c+d)^2 \\ &= \frac{9}{4}[(s-a)+(s-b)+(s-c)+(s-d)]^2 \end{aligned}$$

$$\geq \frac{9}{4}\left[4\sqrt[4]{(s-a)(s-b)(s-c)(s-d)}\right]^2$$
$$= 36\sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

We have thus proved $(*)$ and this completes the solution.

[1] https://en.wikipedia.org/wiki/Bretschneider

Also solved by Bruno Salguerio Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania, and the proposer.

- **5370:** *Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

Let $f(x)$ and $g(x)$ be arbitrary functions defined for all $x \in \Re$. Prove that there is a function $h(x)$ such that

$$(f(x)-h(x))^{2015}\cdot(g(x)-h(x))^{2015}$$

is an odd function for all $x \in \Re$.

Solution by Moti Levy, Rehovot, Israel

If $f(x)$ is odd then $(f(x))^{2015}$ is odd, hence proving that there is a function $h(x)$ such that

$$(f(x)-h(x))(g(x)-h(x))$$

is an odd function for all $x \in R$ will suffice.
Let

$$h(x)=\frac{1}{2}\left(f(x)+f(-x)+g(x)-g(-x)\right).$$

$$\begin{aligned}&(f(x)-h(x))\,(g(x)-h(x))\\&=\left(f(x)-\frac{1}{2}\left(f(x)+f(-x)+g(x)-g(-x)\right)\right)\left(g(x)-\frac{1}{2}\left(f(x)+f(-x)+g(x)-g(-x)\right)\right)\\&=\left(\frac{f(x)-f(-x)}{2}-\frac{g(x)-g(-x)}{2}\right)\left(\frac{g(x)+g(-x)}{2}-\frac{f(x)+f(-x)}{2}\right).\end{aligned}$$

The first factor is odd function, while the second factor is even function, hence the product is an odd function, as required.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania, and the proposer.

- **5371:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers where $n \geq 4$. Prove that

$$\left(\frac{a_1}{a_n+a_2}\right)^2+\left(\frac{a_2}{a_1+a_3}\right)^2+\ldots+\left(\frac{a_n}{a_{n-1}+a_1}\right)^2\geq\frac{4}{n}$$

Solution 1 by David E. Manes, SUNY College at Oneonta, Oneonta, NY

Define vector $\vec{u}$ and $\vec{v}$ in R^n such that

$$\vec{u} = (1, 1, 1, \ldots, 1) \text{ and } \vec{v} = \left(\frac{a_1}{a_n + a_2}, \frac{a_2}{a_1 + a_3}, \cdots, \frac{a_n}{a_{n-1} + a_1}\right).$$

Then the Cauchy-Schwarz inequality implies $||\vec{u}||||\vec{v}|| \geq \vec{u}\cdot\vec{v}$. Therefore,

$$\sqrt{n}\sqrt{\left(\frac{a_1}{a_n + a_2}\right)^2 + \left(\frac{a_2}{a_1 + a_3}\right)^2 + \cdots + \left(\frac{a_n}{a_{n-1} + a_1}\right)^2} \geq \frac{a_1}{a_n + a_2} + \frac{a_2}{a_1 + a_3} + \cdots + \frac{a_n}{a_{n-1} + a_1}.$$

Squaring the inequality, we obtain

$$\left(\frac{a_1}{a_n + a_2}\right)^2 + \left(\frac{a_2}{a_1 + a_3}\right)^2 + \cdots + \left(\frac{a_n}{a_{n-1} + a_1}\right)^2 \geq \frac{\left(\frac{a_1}{a_n + a_2}, \frac{a_2}{a_1 + a_3}, \cdots, \frac{a_n}{a_{n-1} + a_1}\right)^2}{n}.$$

The result now follows provided we can show that if $n \geq 4$, then

$$J_n = \frac{a_1}{a_n + a_2} + \frac{a_2}{a_1 + a_3} + \cdots + \frac{a_n}{a_{n-1} + a_1} \geq 2.$$

To this end, let $n = 4$. Then

$$J_4 = \frac{a_1}{a_4 + a_2} + \frac{a_2}{a_1 + a_3} + \frac{a_3}{a_2 + a_4} + \frac{a_4}{a_3 + a_1} = \frac{a_1 + a_3}{a_2 + a_4} + \frac{a_2 + a_4}{a_1 + a_3} \geq 2,$$

since $x + \dfrac{1}{x} \geq 2$ for all $x > 0$. Assume inductively tht k is a positive integer, $k \geq 4$, and $J_k \geq 2$. Consider $k + 1$ positive numbers $a_1, a_2, \ldots a_k, a_{k+1}$. Since J_{k+1} is symmetric with respect to these numbers, we can assume without loss of generality that $a_j \geq a_k + 1$ for $j = 1, 2, \ldots, k$. Then

$$J_{k+1} = \frac{a_1}{a_{k+1} + a_2} + \frac{a_2}{a_1 + a_3} + \cdots + \frac{a_k}{a_{k-1} + a_{k+1}} + \frac{a_{k+1}}{a_k + a_1}.$$

Observe that

$a_{k+1} \leq a_k$ implies $a_{k+1} + a_2 \leq a_k + a_2$ implies $\dfrac{a_1}{a_{k+1} + a_2} \geq \dfrac{a_1}{a_k + a_2}$, and similarly $\dfrac{a_k}{a_{k-1} + a_{k+1}} \geq \dfrac{a_k}{a_{k-1} + a_1}$. Therefore,

$$J_{k+1} \geq J_k + \frac{a_{k+1}}{a_k + a_1} > J_k \geq 2$$

by the induction hypothesis. Hence, by induction $J_n \geq 2$ if $n \geq 4$.

Accordingly,

$$\left(\frac{a_1}{a_n + a_2}\right)^2 + \left(\frac{a_2}{a_1 + a_3}\right)^2 + \cdots + \left(\frac{a_n}{a_{n-1} + a_1}\right)^2 \geq \frac{(J_n)^2}{n} \geq \frac{4}{n}.$$

Solution 2 by Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania

Since $(a_n + a_2)^2 \leq 2\left(a_n^2 + a_2^2\right)$, it suffices to prove that

$$\frac{x_1}{x_n+x_2}+\frac{x_2}{x_1+x_3}+\cdots+\frac{x_n}{x_{n-1}+x_1}\geq\frac{8}{n}, \text{ where } x_1=a_i^2.$$

We shall prove that

$$\frac{x_1}{x_n+x_2}+\frac{x_2}{x_1+x_3}+\cdots+\frac{x_n}{x_{n-1}+x_1}\geq 2.$$

By Bergstrm's inequality we obtain

$$\frac{x_1}{x_n+x_2}+\frac{x_2}{x_1+x_3}+\cdots+\frac{x_n}{x_{n-1}+x_1}\geq\frac{(x_1+x_2+\cdots+x_n)^2}{2\left(x_1x_2+x_2x_3+\cdots+x_{n-1}x_n+x_nx_1\right)},$$

so it suffices to show that

$$(x_1+x_2+\cdots+x_n)^2\geq 4\left(x_1x_2+x_2x_3+\cdots+x_{n-1}x_n+x_nx_1\right). \qquad (1)$$

The inequality (1) is cyclic; we can assume that $x_n=\min\{x_1,x_2,\ldots,x_{n-1},x_n\}$.

• For n odd we have

$$\begin{aligned}&(x_1+x_2+\cdots+x_n)^2-4\left(x_1x_2+x_2x_3+\cdots+x_{n-1}x_n+x_nx_1\right)\\ \geq\ &(x_1-x_2+\ldots-x_{n-1}+x_n)^2+4x_1x_{n-1}-4x_1x_n\geq 0.\end{aligned}$$

• For n even we have

$$\begin{aligned}&(x_1+x_2+\cdots+x_n)^2-4\left(x_1x_2+x_2x_3+\cdots+x_{n-1}x_n+x_nx_1\right)\\ \geq\ &(x_1-x_2+\ldots+x_{n-1}-x_n)^2.\end{aligned}$$

Remark. For $n\geq 8$ we have a simple solution, i.e.,

$$\frac{x_1}{x_n+x_2}+\frac{x_2}{x_1+x_3}+\cdots+\frac{x_n}{x_{n-1}+x_1}\geq\frac{x_1}{x_1+x_2+\cdots+x_n}+\cdots+\frac{x_n}{x_1+x_2+\cdots+x_n}=1\geq\frac{8}{n}.$$

Editor′s comment: **Paolo Perfetti** mentioned in his solution that $\frac{a_1}{a_n+a_2}+\frac{a_2}{a_1+a_3}+\cdots+\frac{a_n}{a_{n-1}+a_1}\geq 2$ is known as being one of the Shapiro inequalities, and that its proof by induction can be found in <http://olympiads.mccme.ru/1ktg/2010/5/5-1en.pdf>.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Paolo Perfetti, Mathematics Department, Tor Vergata University, Rome, Italy; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania, and the proposer.

• **5372:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

(a) Let $k\geq 2$ be an integer. Calculate

$$\int_0^\infty\frac{\ln(1+x)}{x\sqrt[k]{x}}dx.$$

(b) Calculate

$$\int_0^\infty \frac{\ln(1-x+x^2)}{x\sqrt{x}}dx.$$

Solution 1 by Moti Levy, Rehovot, Israel

Reference: Emil Artin, "*The Gamma Function*", Holt, Rinehart and Winston, 1964. Page 29.

(a)

The well known Euler's reflection formula for the Gamma function is

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}, \qquad 0 < x < 1.$$

From the definition of the Beta function,

$$B(x, 1-x) = \frac{\Gamma(x)\Gamma(1-x)}{\Gamma(1)} = \int_0^1 t^{x-1}(1-t)^{-x}\,dt.$$

Since $\Gamma(1) = 1$,

$$\int_0^1 t^{x-1}(1-t)^{-x}\,dt = \frac{\pi}{\sin \pi x}, \qquad 0 < x < 1.$$

Changing the variable of integration $u = \frac{t}{1-t}$, we get

$$\int_0^\infty \frac{u^{x-1}}{1+u}du = \frac{\pi}{\sin \pi x}, \qquad 0 < x < 1.$$

By integration by parts, we get

$$\int_0^\infty \frac{u^{x-1}}{1+u}du = (1-x)\int_0^\infty \frac{\ln(1+u)}{u^{2-x}}du$$

Now set $x = 1 - \frac{1}{k}$ to obtain,

$$\frac{1}{k}\int_0^\infty \frac{\ln(1+u)}{u^{1+\frac{1}{k}}}du = \frac{\pi}{\sin \pi\left(1-\frac{1}{k}\right)} = \frac{\pi}{\sin\frac{\pi}{k}}.$$

We conclude that

$$\int_0^\infty \frac{\ln(1+x)}{x\sqrt[k]{x}}dx = \frac{k\pi}{\sin\frac{\pi}{k}}, \quad k \geq 2.$$

(b)

$1 - x + x^2 = (x+\alpha)(x+\beta)$ with $\alpha\beta = 1$ and $\alpha + \beta = -1$.

$$\int_0^\infty \frac{\ln\left(1-x+x^2\right)}{x\sqrt{x}}dx = \int_0^\infty \frac{\ln\left((x+\alpha)(x+\beta)\right)}{x\sqrt{x}}dx$$

$$= \int_0^\infty \frac{\ln(x+\alpha)}{x\sqrt{x}}dx + \int_0^\infty \frac{\ln(x+\beta)}{x\sqrt{x}}dx$$

$$= \int_0^\infty \frac{\ln\alpha + \ln\left(\frac{x}{\alpha}+1\right)}{x\sqrt{x}}dx + \int_0^\infty \frac{\ln\beta + \ln\left(\frac{x}{\beta}+1\right)}{x\sqrt{x}}dx$$

$$= \int_0^\infty \frac{\ln(\alpha\beta)}{x\sqrt{x}}dx + \frac{1}{\sqrt{\alpha}}\int_0^\infty \frac{\ln\left(\frac{x}{\alpha}+1\right)}{\frac{x}{\alpha}\sqrt{\frac{x}{\alpha}}}\frac{dx}{\alpha} + \frac{1}{\sqrt{\beta}}\int_0^\infty \frac{\ln\left(\frac{x}{\beta}+1\right)}{\frac{x}{\beta}\sqrt{\frac{x}{\beta}}}\frac{dx}{\beta}$$

Changing the variable of integration, we obtain

$$\int_0^\infty \frac{\ln\left(1-x+x^2\right)}{x\sqrt{x}}dx = \left(\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}}\right)\int_0^\infty \frac{\ln(u+1)}{u\sqrt{u}}du$$

$$\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = \frac{\sqrt{\alpha}+\sqrt{\beta}}{\sqrt{\alpha\beta}} = \sqrt{\alpha}+\sqrt{\beta} = \sqrt{\alpha+\beta+2\sqrt{\alpha\beta}} = \sqrt{-1+2} = 1.$$

We conclude that

$$\int_0^\infty \frac{\ln\left(1-x+x^2\right)}{x\sqrt{x}}dx = \int_0^\infty \frac{\ln(u+1)}{u\sqrt{u}}du = \frac{2\pi}{\sin\frac{\pi}{2}} = 2\pi.$$

Editor's comment: **Ulrich Abel of Technische Hochschule Mittelhessen in Freiberg, Germany,** wrote that "both integrals of Problem 5372 can be determined by using computer algebra. Mathematica V. 9" and he then stated:

(a) $\displaystyle\int_0^\infty \frac{\ln(1+x)}{x^a}dx = \pi\cdot\frac{\text{Cosec}(a\cdot\pi)}{1-a}$ for all constants a such that $1 < Re[a] < 2$. This is slightly more general than the proposed problem.

(b) $\displaystyle\int_0^\infty \frac{\ln(x^2-x+1)}{x^{3/2}}dx = 2\pi.$

Solution 2 by Kee-Wai Lau, Hong Kong, China

(a) Denote the integral by I. By substitution $x = y^k$, we obtain

$$I = k\int_0^\infty \frac{\ln\left(1+y^k\right)}{y^2}dy. \text{ Since } \lim_{y\to 0^+}\left(\frac{\ln(1+y^k)}{y}\right) = \lim_{y\to\infty}\left(\frac{\ln(1+y^k)}{y}\right) = 0,$$

so by integrating by parts, we obtain

$$I = -k\int_0^\infty \ln(1+y^k)d\left(\frac{1}{y}\right) = k^2\int_0^\infty \frac{y^{k-2}}{1+y^k}dy.$$

We next substitute $y = \frac{1}{z}$ to obtain $I = k^2 \int_0^\infty \frac{1}{1+z^k} dz$. It is known ([1], entry 34.24(2))
that $\int_0^\infty \frac{1}{1+z^k} dz = \frac{\pi}{k} \csc\left(\frac{\pi}{k}\right)$, and so $I = \pi k \csc\left(\frac{\pi}{k}\right)$.

(b) Denote the integral by J. By substitution $x = y^2$ we obtain

$$J = 2\int_0^\infty \frac{\ln(1-y^2+y^4)}{y^2} dy. \text{ Since}$$

$$\lim_{y \to 0^+} \left(\frac{\ln(1-y^2+y^4)}{y}\right) = \lim_{y \to \infty} \left(\frac{\ln(1-y^2+y^4)}{y}\right) = 0,$$

so by integrating by parts, we obtain

$$\begin{aligned} J &= 2\int_0^\infty \frac{\ln(1-y^2+y^4)}{y^2} dy = -2\int_0^\infty \ln(1-y^2+y^4) d\left(\frac{1}{y}\right) \\ &= 4\int_0^\infty \frac{2y^2-1}{1-y^2+y^4} dy = 8\int_0^\infty \frac{y^2}{1-y^2+y^4} dy - 4\int_0^\infty \frac{1}{1-y^2+y^4} dy. \end{aligned}$$

Substituting $y = \frac{1}{z}$, we obtain $\int_0^\infty \frac{y^2}{1-y^2+y^4} dy = \int_0^\infty \frac{1}{1-z^2+z^4} dz$, so that
$J = 4\int_0^\infty \frac{1}{1-y+y^4} dy$. It is known ([1], entry 3.242(1)) that $\int_0^\infty \frac{1}{1-y^2+y^4} dy = \frac{\pi}{2}$
and so
$J = 2\pi$.

Reference [1] I.S. Gradshteyn and I.M. Ryzhik: *Tables of Integrals, Series, and Products,* Seventh Edition, Elsevier, Inc., 2007.

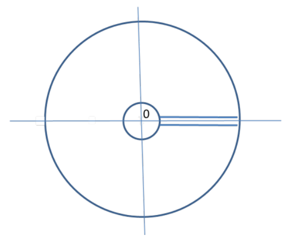
Solution 3 by Albert Stadler, Herrliberg, Switzerland

Both integrals can be evaluated by means of the following

Lemma

Let $0<a<1$. Let $0<b<2\pi$. Then $\displaystyle\int_0^\infty \frac{x^{-a}}{x-e^{ib}}dx=\frac{\pi e^{ia(\pi-b)}}{\sin(\pi a)}$.

Proof of the Lemma



Define a path C that consists of the following pieces:

$C_1: Re^{it},\ 0<t<2\pi$, run through once in the positive direction,

$C_2: t,\ \epsilon<t<R$, run through in the direction of decreasing real values,

$C_3: \epsilon e^{it},\ 0<t<2\pi$ run through once in the negative direction,

$C_4: t,\ \epsilon<t<R$, run through in the direction of increasing real values.,

Define the branch of z^{-a} such that $z^{-a}=\left(|z|\,e^{iArg(z)}\right)^{-a}$, where $0<Arg(z)<2\pi$.

Then, by Cauchy's theorem,

$$\frac{1}{2\pi i}\int_C \frac{z^{-a}}{z-e^{ib}}dz=Res\left(\frac{z^{-a}}{z-e^{ib}},\ z=e^{ib}\right)=e^{-abi}. \qquad (1)$$

The integral $\displaystyle\frac{1}{2\pi i}\int_C \frac{z^{-a}}{z-e^{ib}}dz$ splits as follows:

$$\frac{1}{2\pi i}\int_C \frac{z^{-a}}{z-e^{ib}}dz=\frac{1}{2\pi i}\int_{C_1} \frac{z^{-a}}{z-e^{ib}}dz+\frac{1}{2\pi i}\int_{C_2} \frac{z^{-a}}{z-e^{ib}}dz+\frac{1}{2\pi i}\int_{C_3} \frac{z^{-a}}{z-e^{ib}}dz+\frac{1}{2\pi i}\int_{C_4} \frac{z^{-a}}{z-e^{ib}}dz.$$

We treat each of these four integrals separately.

$$\left|\frac{1}{2\pi i}\int_{C_1} \frac{z^{-a}}{z-e^{ib}}dz\right|\le\frac{1}{2\pi}\frac{R^{-a}}{R-1}2\pi R=O\left(R^{-a}\right), as\ R\to\infty,$$

$$\left|\frac{1}{2\pi i}\int_{C_3} \frac{z^{-a}}{z-e^{ib}}dz\right|\le\frac{1}{2\pi}\frac{\epsilon^{-a}}{\epsilon-1}2\pi\epsilon=O\left(\epsilon^{1-a}\right), as\ \epsilon\to 0.$$

Therefore,

$$\frac{1}{2\pi i}\int_C \frac{z^{-a}}{z-e^{ib}}dz=\frac{1}{2\pi i}\int_0^\infty \frac{x^{-a}}{x-e^{ib}}dx-\frac{1}{2\pi i}\int_0^\infty \frac{\left(xe^{2\pi i}\right)^{-a}}{x-e^{ib}}dx=\frac{1}{2\pi i}\left(1-e^{-2\pi ia}\right)\int_0^\infty \frac{x^{-a}}{x-e^{ib}}dx. \qquad (2)$$

We combine (1) and (2) and get

$$\frac{1}{2\pi i}\left(1-e^{-2\pi i a}\right)\int_0^\infty \frac{x^{-a}}{x-e^{ib}}dx = e^{iab}$$

which is the claim of the lemma.

(a) Let $1 < a < 2$. Partial integration yields

$$\int_0^\infty \frac{\log(1+x)}{x^a}dx = \underbrace{\frac{-x^{1-a}\log(1+x)}{a-1}\Bigg|_0^\infty} + \frac{1}{a-1}\int_0^\infty \frac{x^{1-a}}{1+x}dx = \frac{1}{a-1}\int_0^\infty \frac{x^{1-a}}{1+x}dx,$$

because the first term evaluates to zero.

We set $b=\pi$ and apply the lemma to get

$$\int_0^\infty \frac{\log(1+x)}{x^a}dx = \frac{1}{a-1}\int_0^\infty \frac{x^{1-a}}{1+x}dx = \frac{1}{a-1}\cdot\frac{\pi}{\sin(\pi(a-1))} = \frac{-1}{a-1}\cdot\frac{\pi}{\sin(\pi a)}.$$

(a) is the special case $a = 1+\dfrac{1}{k}$.

(b) Let $1 < a < 2$. Partial integration yields

$$\begin{aligned}\int_0^\infty \frac{\log(1-x+x^2)}{x^a}dx &= \underbrace{\frac{-x^{1-a}\log(1-x+x^2)}{a-1}\Bigg|_0^\infty} + \frac{1}{a-1}\int_0^\infty \frac{x^{1-a}(2x-1)}{1-x+x^2}dx \\ &= \frac{1}{a-1}\int_0^\infty \frac{x^{1-a}}{x-e^{\frac{\pi i}{3}}}dx + \frac{1}{a-1}\int_0^\infty \frac{x^{1-a}}{x-e^{\frac{5\pi i}{3}}},\end{aligned}$$

because the first time evaluates to zero.

We apply the lemma to get

$$\begin{aligned}\int_0^\infty \frac{\log(1-x+x^2)}{x^a}dx &= \frac{1}{a-1}\cdot\frac{\pi e^{i(a-1)(\pi-\frac{\pi}{3})}}{\sin(\pi(a-1))} + \frac{1}{a-1}\cdot\frac{\pi e^{i(a-1)(\pi-\frac{5\pi}{3})}}{\sin(\pi(a-1))} \\ &= \frac{2\pi}{a-1}\cdot\frac{\cos\left(\frac{2\pi}{3}(a-1)\right)}{\sin(\pi(a-1))} \\ &= \frac{-2\pi}{a-1}\cdot\frac{\cos\left(\frac{2\pi}{3}(a-1)\right)}{\sin(\pi(a))}.\end{aligned}$$

In particular, if $a=\dfrac{3}{2}$ then $\displaystyle\int_0^\infty \frac{\log(1-x+x^2)}{x\sqrt{x}}dx = -4\pi\cdot\frac{\cos\left(\frac{\pi}{3}\right)}{\sin(3\pi/2)} = 2\pi.$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain, and the proposer.