Problems   Ted Eisenberg, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
April 15, 2017

- **5433:** *Proposed by Kenneth Korbin, New York, NY*

  Solve the equation: $\sqrt[4]{x+x^2} = \sqrt[4]{x} + \sqrt[4]{x-x^2}$, with $x > 0$.

- **5434:** *Proposed by Titu Zvonaru, Comnesti, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania*

  Calculate, without using a calculator or log tables, the number of digits in the base 10 expansion of 2^{96}.

- **5435:** *Proposed by Valcho Milchev, Petko Rachov Slaveikov Seconday School, Bulgaria*

  Find all positive integers a and b for which $\dfrac{a^4+3a^2+1}{ab-1}$ is a positive integer.

- **5436:** *Proposed by Arkady Alt, San Jose, CA*

  Find all values of the parameter t for which the system of inequalities

  $$\mathbf{A} = \begin{cases} \sqrt[4]{x+t} \geq 2y \\ \sqrt[4]{y+t} \geq 2z \\ \sqrt[4]{z+t} \geq 2x \end{cases}$$

  a) has solutions;
  b) has a unique solution.

- **5437:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $f : C - \{2\} \to C$ be the function defined by $f(z) = \dfrac{2-3z}{z-2}$. If $f^n(z) = (\underbrace{f \circ f \circ \ldots \circ f}_{n})(z)$, then compute $f^n(z)$ and $\lim\limits_{n \to +\infty} f^n(z)$.

- **5438:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $k \geq 0$ be an integer and let $\alpha > 0$ be a real number. Prove that

$$\frac{x^{2k}}{(1-x^2)^\alpha} + \frac{y^{2k}}{(1-y^2)^\alpha} + \frac{z^{2k}}{(1-z^2)^\alpha} \geq \frac{x^k y^k}{(1-xy)^\alpha} + \frac{y^k z^k}{(1-yz)^\alpha} + \frac{x^k z^k}{(1-xz)^\alpha},$$

for $x, y, z \in (-1, 1)$.

Solutions

- **5415:** *Proposed by Kenneth Korbin, New York, NY*

Given equilateral triangle ABC with inradius r and with cevian $\overline{CD}$. Triangle ACD has inradius x and triangle BCD has inradius y, where x, y and r are positive integers with $(x, y, r) = 1$.
Part 1: Find $x, y,$ and r if $x + y - r = 100$

Part 2: Find $x, y,$ and r if $x + y - r = 101$.

Solution by Ed Gray, Highland Beach, FL

Editor's comment: Ed's solution to this problem was 18 pages in length. Listed below is my greatly abbreviated outline of his solution method. All formulas listed below were proved and/or referenced in Ed's complete solution. He started his solution with the second part of the problem and then applied the methods constructed there to the first part of the problem. The reason for this will soon become apparent. Following is Ed's solution.

The following equations will be used in the solution.

$$\begin{aligned} x &= \frac{6r^2 - 3rp}{4r - p + \sqrt{4r^2 + p^2 - 2rp}} && (1) \\ y &= \frac{3rp}{p + 2r + \sqrt{4r^2 + p^2 - 2pr}} && (2) \\ k &= p(2r - p) && (3) \\ x + y - r &= \frac{k}{2r + \sqrt{4r^2 - k}} && (4) \end{aligned}$$

Solution to Part 2. $x + y - r = 101$.

Substituting the value $x + y - r = 101$ into (4) and solving for k we see that

$$\frac{k}{2r + \sqrt{4r^2 - k}} = 101 \Longrightarrow k = 404r - 10201,$$

and substituting this into (3) above we see that

$$404r - 10201 = 2pr - p^2 \Longrightarrow p = r - \sqrt{r^2 - 404r + 10201}.$$

Letting D equal the value under the square root we have

$$D^2 = r^2 - 404r + 10201 \Longrightarrow r^2 - 404r + 10201 - D^2 = 0.$$

Solving for r gives $r = 202 \pm \sqrt{30603 + D^2}$

Letting $b^2 = 30603 + D^2$ we have $(b - D)(b + D) = 30603 = 3^1 \cdot 101^2$.

This implies that there are three possible factorizations:

$$\begin{aligned} \text{Case I}: &\quad 1 \times 30603 \\ \text{Case II}: &\quad 3 \times 10201 \\ \text{Case III}: &\quad 101 \times 303 \end{aligned}$$

Case I: $\begin{cases} b - D &= 1 \\ b + D &= 30603 \end{cases} \Longrightarrow \begin{cases} b &= 15302 \\ D &= 15301 \end{cases}.$

So,

$$\begin{aligned} r_1 &= 202 + b = 202 + 15302 = 15504 \\ r_2 &= 202 - b = 202 - 15302 < 0 \\ p &= r - D = 155404 - 15301 = 203. \end{aligned}$$

Therefore, $r = 15504$ and $p = 203$.

For these values of r and p, we evaluate x and y by using formulas (1) and (2) above.

$$\begin{aligned} x &= \frac{6r^2 - 3rp}{4r - p + \sqrt{4r^2 + p^2 - 2rp}} \\ & \frac{6(155040)^2 - 3(15504)(203)}{4(15504) - 203 + \sqrt{4(15504)^2 + (203)^2 - 2(15504)(203)}} \\ &= 15453. \end{aligned}$$

$$\begin{aligned} y &= \frac{3pr}{p + 2r + \sqrt{4r^2 + p^2 - 2pr}} \\ &= 152. \end{aligned}$$

So for Case I, $r = 15504,\ x = 15453,\ y = 152$, and $x + y - r = 101$. Since x, y, r have no common factor, they represent a solution.

Case II: $\begin{cases} b - D &= 3 \\ b + D &= 10201 \end{cases} \Longrightarrow \begin{cases} b &= 5102 \\ D &= 5099 \end{cases}.$ So, $\begin{cases} r_1 &= 202 + 5102 = 5304 \\ r_2 &= 202 - 5102 < 0 \\ p &= r - D = 205. \end{cases}.$

Following the path in Case I, we find that $\begin{cases} r &= 5304 \\ x &= 5252 \\ y &= 153, \text{ and} \\ x + y - r &= 101. \end{cases}$

These terms have no common factor and so represent a solution.

Case III: $\begin{cases} b-D &= 101 \\ b+D &= 303 \end{cases} \Longrightarrow \begin{cases} b &= 202 \\ D &= 101 \end{cases}$. So, $\begin{cases} r_1 &= 202+202=4043 \\ r_2 &= 202-202=0, \text{ not viable} \\ p &= r-D=4043-101=303. \end{cases}$.

Given $r=404, p=303$, and calculating as before, we have for Case III,
$r=404,\ x=303,\ y=202,\ x+y-r=101$. However 101 divides all three terms, violating $(x,y,r)=1$, so we do not have a solution.

In summary, and taking into account the interchangeability of x and y, there are four solutions for Part 2 of the problem:

$$\begin{pmatrix} x \\ y \\ r \end{pmatrix} = \begin{pmatrix} 15453 \\ 152 \\ 15504 \end{pmatrix}, \begin{pmatrix} 5252 \\ 153 \\ 5304 \end{pmatrix}, \begin{pmatrix} 152 \\ 15453 \\ 15504 \end{pmatrix}, \begin{pmatrix} 153 \\ 5252 \\ 5304 \end{pmatrix}.$$

Solution to Part 1. $x+y-r=100$. In solving Part 1 of the problem we employ the same techniques that were used in Part 2. We start off by finding that if $\dfrac{k}{2r+\sqrt{4r^2-k}}=100$ then $k=400r-10,000$. Substituting this into Equation (3), gives us $400r-10,000=2pr-p^2$ and solving for p gives us $p=r-\sqrt{r^2-4004r+10,000}$. The discriminant, D is given by $D^2=r^2-400r+10000$. Writing this as a quadratic in r and solving for r gives us

$$r^2-400r+10,000-D^2=0$$

$$r=200\pm\sqrt{30,000+D^2}.$$

And as before, letting $b^2=30,000+D^2$ we obtain

$$(b-D+)(b+D)=30,000=2^4\cdot 3^1\cdot 5^4.$$

Hence there are $5\times 2\times 5=50$ factors which need to be written as the product of two factors. Since $2b$ must equal the sum of the two factors, they cannot be of opposite parity. Following is a table listing all factorizations. We eliminate those factorizations that have an odd factor by placing an asterisk in front of them.

$*1\times 30,000$	$2\times 15,000$	$*3\times 10,000$
4×7500	$*5\times 6,000$	8×3750
10×3000	12×2500	$*15\times 2000$
$*16\times 1875$	20×1500	24×1250
$*25\times 1200$	30×1000	40×750
$*48\times 625$	50×600	60×500
$*75\times 400$	$*80\times 375$	100×300
120×250	$*125\times 240$	150×200

The remaining factorizations represent potential solutions. We will do the first one in detail but the others we will only check to see if $(x,y,r)=1$.

$\begin{cases} b-D &= 2 \\ b+D &= 15000 \end{cases} \Longrightarrow \begin{cases} b &= 7501 \\ D &= 7499. \end{cases}$. So, $\begin{cases} r_1 &= 200=7501=7701 \\ r_2 &= 200-7501<0 \\ p &= r-D=7701-7499=202. \end{cases}$

Given $p=r-D=7701-7499=202$. For $r=7701, p=202$ we calculate x and y using the standard formulas.

$$x=\frac{6r^2-3rp}{4r-p+\sqrt{4r^2+p^2-2rp}} \Longrightarrow x=7650$$

$$y = \frac{3rp}{p + 2r + \sqrt{4r^2 + p^2 - 2pr}} \Longrightarrow y = 151.$$

So $x = 7650,\ y = 152,\ r = 7701.$ These have no common factor and so represent a solution.

We now move to the next case. $\begin{cases} b - D = 4 \\ b + D = 7500 \end{cases} \Longrightarrow x = 3900,\ y = 152,\ r = 3952.$ Since $(x, y, z) \neq 1$, this is not a solution.

And the next case. $\begin{cases} b - D = 6 \\ b + D = 5000 \end{cases} \Longrightarrow x = 2650,\ y = 153,\ r = 2703.$ Since $x + y - r = 100$ and $(x, y, r) = 1$ this is a solution.

And the next. $\begin{cases} b - D = 8 \\ b + D = 3750 \end{cases} \Longrightarrow x = 2025,\ y = 154,\ r = 2079.$ Since $x + y - r = 100$ and $(x, y, r) = 1$ this is a solution 10pt Working our way through the table of potential solutions we find that $\begin{cases} b - D = 24 \\ b + D = 1250 \end{cases} \Longrightarrow x = 775,\ y = 162,\ r = 837$ and since $x + y - r = 100$ and $(x, y, r) = 1$ this is a solution

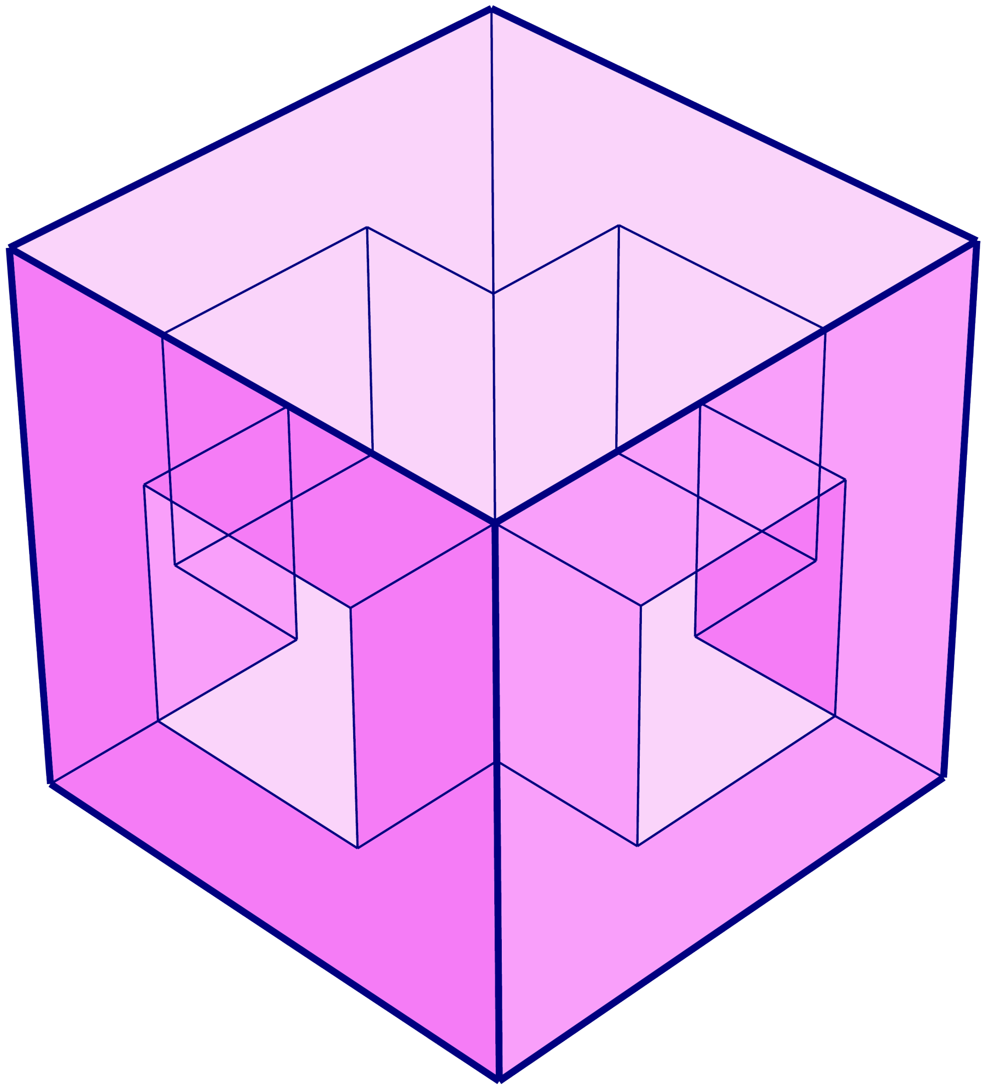
Systemically working our way the table we see that many values did not result in an answer to the problem. Summarizing Part 1 of the problem, and taking into account the interchangeability of x and y, we see that there are exactly eight solutions.

$$\begin{array}{l}
1)\ x = 7650,\ y = 151,\ r = 7701 \\
2)\ x = 2650,\ y = 153,\ r = 2703 \\
3)\ x = 2025,\ y = 154,\ r = 2079 \\
4)\ x = 775,\ y = 162,\ r = 837 \\
5)\ x = 151,\ y = 7650,\ r = 7701 \\
6)\ x = 153,\ y = 2650,\ r = 2703 \\
7)\ x = 154,\ y = 2025,\ r = 2079 \\
8)\ x = 162,\ y = 775,\ r = 837.
\end{array}$$

Also solved by Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5416:** *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

Two congruent intersecting holes, each with a square cross-section were drilled through a cube. Each of the holes goes through the opposite faces of the cube. Moreover, the edges of each hole are parallel to the appropriate edges of the original cube, and the center of each hole is at the center of the original cube. Letting the length of the original cube be a, find the length of the square cross-section of each hole that will yield the largest surface area of the solid with two intersecting holes. What is the largest surface area of the solid with two intersecting holes?



Solution by Paul M. Harms, North Newton, KS

Let the side lengths of the drilled squares be x at the surface of the original cube. The surface area of the one side of the original cube, with a square hole cut out of it, is $a^2 - x^2$. There are four of these sides on the original cube.

On a side of the original cube the shortest distance between an edge of the original cube and a parallel side of the drilled square hole is $\dfrac{a-x}{2}$.

Now consider the surface area "inside" the cube made by the part of the drilled square that starts at a side of the original cube and ends when the drilled square meets the other drilled square originating from an adjacent side of the cube. This surface area looking at one side of the cube includes four rectangles with one side length of x and "depth" length of $\dfrac{a-x}{2}$, so this surface area is $\dfrac{4x(a-x)}{2} = 2(a-x)$. There are four of these around the original cube. The surface area of each of the two sides of the original cube which have no holes is a.

In the middle of the original cube at the intersection of the two drilled square holes, there are two squares of side length x with are parallel to the sides of the original cube with no holes . The area of each square is x^2.

The total surface area of the problem is

$$4(a^2-x^2)+4\left(2x(a-x)\right)+2a^2+2x^2 = 6a^2+8ax-10x^2.$$

The maximum surface area occurs when $8a-20x=0$ or $x=\dfrac{2a}{5}$. The maximum surface area is $\dfrac{38a^2}{5}$ when a side of the drilled square holes as a length of $\dfrac{2a}{5}$.

Editor's comment: **David Stone and John Hawkins, both from Georgia Southern University, Statesboro, GA** accompanied their solution by placing the statement of the problem into a story setting. They wrote:

"An interpretation: in the ancient Martian civilization, the rulers favorite meditational spot was a levitating cube having a cubical inner sanctum formed by two horizontal square tunnels, meeting at the center of the cube, from which he could see out in all four directions. The designers were charged to construct the ship with a maximum amount of wall space for inscriptions and carved likenesses of His Highness. There are four short hallways leading from the inner room to the outside walls." They let x be the side length of the square tunnels that are drilled through the original cube and noted that each tunnel has an $x\times x$ cross section and has length a. The inner most cubical room is $x\times x\times x$. They then mentioned that "by drilling the tunnels and opening up an interior chamber, the surface area has increased from $6a^2$ to $\dfrac{38}{5}a^2$, an increase of $\dfrac{8}{5}a^2$ or 27%. So the King has his private getaway and more space for pictures and wall hangings."

Also solved by Jeremiah Bartz, University of North Dakota, Grand Forks, ND and Nicholas Newman, Francis Marion University, Florence SC; Michael N. Fried, Ben-Gurion University, Beer-Sheva, Israel; David A. Huckaby, Angelo State University, San Angelo, TX; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5417:** *Proposed by Arkady Alt, San Jose, CA*

Prove that for any positive real number x, and for any natural number $n\geq 2$,

$$\sqrt[n]{\frac{1+x+\cdots+x^n}{n+1}} \geq \sqrt[n-1]{\frac{1+x+\cdots+x^{n-1}}{n}}.$$

Solution 1 by Henry Ricardo, New York Math Circle, NY

Let $\alpha_n = (1 + x + \cdots + x^n)/(n + 1)$ and define

$$F(x) = \frac{(1 + x + x^2 + \cdots + x^{n-1})^n}{(1 + x + x^2 + \cdots + x^n)^{n-1}}.$$

Then, for $x > 0$ and $n \geq 2$, we see that

$$\sqrt[n-1]{\alpha_{n-1}} \leq \sqrt[n]{\alpha_n} \;\Leftrightarrow\; \alpha_{n-1}^n \leq \alpha_n^{n-1} \;\Leftrightarrow\; F(x) \leq \frac{n^n}{(n+1)^{n-1}} = F(1).$$

Now we show that $F(x)$ attains its absolute maximum value at $x = 1$.

For $x \neq 1$, we have

$$\begin{aligned} F'(x) &= \frac{(x^n - 1)^{n-1}(x^{n+1} - 1)^{-n}\left(-x^{2n+1} + n^2x^{n+2} + 2(1 - n^2)x^{n+1} + n^2x^n - x\right)}{x(x-1)^2} \\ &= \overbrace{\frac{(x^n - 1)^{n-1}}{(x^{n+1} - 1)^n(x-1)^2}}^{G(x)} \cdot \overbrace{\left(-x^{2n} + n^2x^{n+1} + 2(1 - n^2)x^n + n^2x^{n-1} - 1\right)}^{H(x)}. \end{aligned}$$

Noting that $G(x)$ is negative for $0 < x < 1$ and positive for $x > 1$, we examine the factor $H(x)$ to see that

$$\begin{aligned} H(x) &= -(x^n - 1)^2 + n^2x^{n-1}(x-1)^2 \\ &= -n^2(x-1)^2\left[\frac{(x^{n-1} + x^{n-2} + \cdots + x + 1)^2}{n^2} - x^{n-1}\right] \\ &= -n^2(x-1)^2\left[\left(\frac{x^{n-1} + x^{n-2} + \cdots + x + 1}{n}\right)^2 - \left(\sqrt[n]{x^{n-1} \cdot x^{n-2} \cdots x \cdot 1}\right)^2\right] \end{aligned}$$

is negative for all $x > 0$ by the AM-GM inequality.

Thus $F'(x) > 0$ for $0 < x < 1$ and $F'(x) < 0$ for $x > 1$, implying that $F(x)$ has an absolute maximum value at $x = 1$—that is, $F(x) \leq F(1)$ on $(0, \infty)$, which proves the proposed inequality.

COMMENT: This was proposed by Walther Janous as problem 1763 (1992, p. 206) in *Crux Mathematicorum*. My solution is based on the published solution of Chris Wildhagen.

Solution 2: by Moti Levy, Rehovot, Israel

If $x = 1$ then the inequality holds, since

$$\sqrt[n]{\frac{1 + x + \cdots + x^n}{n+1}} = \sqrt[n-1]{\frac{1 + x + \cdots + x^{n-1}}{n}} = 1.$$

We assume that $x > 1$.
Let us define the continuous functions $g(t)$, and $f(t)$, $t \in R$, $t > 1$, as follows,

$$g(t) := \frac{x^{t+1} - 1}{x - 1}\frac{1}{t+1}, \quad f(t) := (g(t))^{\frac{1}{t}} .$$

Clearly, $\sqrt[n]{\frac{1+x+\cdots+x^n}{n+1}} = \sqrt[n]{\frac{1}{n+1}\frac{x^{n+1}-1}{x-1}} = f\left(n\right)$. The original inequality (in terms of the function f) is

$$f\left(n\right) \geq f\left(n-1\right), \quad \text{for} \quad n \geq 2.$$

For $n=2$, $\sqrt{\frac{1+x+x^2}{3}} \geq \frac{1+x}{2}$ follows from $\frac{1+x+x^2}{3} - \left(\frac{1+x}{2}\right)^2 = \frac{1}{12}\left(x-1\right)^2 \geq 0$.
Therefore, it suffices to prove that $f\left(t\right)$ is monotone increasing function for $t \geq 1$.
We will show this by proving that the derivative of $\ln f\left(t\right)$ is postive for $t \geq 1$.
The derivative is given by

$$t^2 \frac{d}{dt}\left(\ln f\right) = -\ln g + t\frac{\frac{dg}{dt}}{g}.$$

The first step is showing $-\ln g + t\frac{\frac{dg}{dt}}{g} > 0$ for $t=1$.

$$-\ln g + t\frac{\frac{dg}{dt}}{g}\Bigg|_{t=1} = -\ln\left(\frac{1+x}{4}\right) + \frac{2x^2\ln x}{2\left(x^2-1\right)}.$$

To show that $-\ln\left(\frac{1+x}{4}\right) + \frac{2x^2\ln x}{2(x^2-1)} > 0$ for $x > 0$, we see that
$\lim_{x\to 0}\left(-\ln\left(\frac{1+x}{4}\right) + \frac{2x^2\ln x}{2(x^2-1)}\right) = \ln 4 > 0$.
Now we show that the derivative of $-\ln\left(\frac{1+x}{4}\right) + \frac{2x^2\ln x}{2(x^2-1)}$ is positive:

$$\frac{d\left(-\ln\left(\frac{1+x}{4}\right) + \frac{2x^2\ln x}{2(x^2-1)}\right)}{dx} = \frac{1}{x^2-1} - \frac{2x\ln x}{\left(x^2-1\right)^2}.$$

We use the well known inequality: $\ln x \leq \frac{x^2-1}{2x}$ for $x > 0$ to show that

$$\frac{1}{x^2-1} - \frac{2x\ln x}{\left(x^2-1\right)^2} \geq 0.$$

The second step is showing that the derivative of $-\ln g + t\frac{\frac{dg}{dt}}{g}$ is positive for $t > 0$,

$$\frac{d\left(-\ln g + t\frac{\frac{dg}{dt}}{g}\right)}{dt} = -\frac{\frac{dg}{dt}}{g} + \frac{\frac{dg}{dt}}{g} + \frac{d}{dt}\left(\frac{\frac{dg}{dt}}{g}\right) = \frac{d}{dt}\left(\frac{\frac{dg}{dt}}{g}\right).$$

After some tedious calculation we arrive at,

$$\frac{d}{dt}\left(\frac{\frac{dg}{dt}}{g}\right) = \frac{\left(x^{t+1}-1\right)^2 - x^{t+1}\ln^2 x^{t+1}}{\left(x^{t+1}-1\right)^2\left(t+1\right)^2}.$$

To show that $\left(x^{t+1}-1\right)^2 \geq x^{t+1}\ln^2 x^{t+1}$, or that $\ln x^{t+1} \leq \frac{1}{\sqrt{x^{t+1}}}\left(x^{t+1}-1\right)$, we use again the inequality $\ln y \leq \frac{y^2-1}{2y}$ for $y > 0$,

$$\ln y \leq \frac{y-1}{\sqrt{y}}\frac{y+1}{2\sqrt{y}}.$$

But $\frac{y+1}{2\sqrt{y}} \geq 1$; hence,

$$\ln y \leq \frac{y-1}{\sqrt{y}}, \quad y > 0.$$

Now set $y = x^{t+1}$ to finish the proof.

Solution 3 by Kee-Wai Lau, Hong Kong, China

Denote the inequality of the problem by $(*)$. It is easy to see that if $(*)$ holds for $x = t$ then it also holds for $x = \dfrac{1}{t}$. Hence it suffices to prove $(*)$ for $0 < x \le 1$.

Let $f(x) = (n-1)\ln\left(\displaystyle\sum_{k=0}^{n} x^k\right) - n\ln\left(\displaystyle\sum_{k=0}^{n-1} x^k\right) + \ln\left(\dfrac{n^n}{(n+1)^{n-1}}\right)$, where $0 < x \le 1$.

By taking logarithms, we see that $(*)$ is equivalent to $f(x) \ge 0$.

We have $f(1) = 0$ and for $0 < x < 1$,

$$f(x) = (n-1)\ln(1-x^{n+1}) - n\ln(1-x^n) + \ln(1-x) + \ln\left(\frac{n^n}{(n+1)^{n-1}}\right).$$

Hence to prove $(*)$, we need only prove that $f'(x) < 0$ for $0 < x < 1$.

Since $f'(x) = \dfrac{g(x)}{(x-1)(x^n-1)(x^{n+1}-1)}$, where

$g(x) = x^{2n} - n^2x^{n+1} + 2(n-1)(n+1)x^n - n^2x^{n-1} + 1$, it suffices to show

$g(x) > 0$, for $0 < x < 1$. Now

$$\begin{aligned}
g'(x) &= 2nx^{2n-1} - (n+1)n^2x^n + 2n(n-1)(n+1)x^{n-1} - (n-1)n^2x^{n-2},\\
g''(x) &= 2n(2n-1)x^{2n-2} - (n+1)n^3x^{n-1} + 2n(n+1)(n-1)^2x^{n-2} - (n-1)(n-2)n^2x^{n-3}, \text{and}\\
g'''(x) &= 4n(n-1)(2n-1)x^{2n-3} - (n-1)(*n+1)n^3xn - 2+\\
&\quad 2n(n-2)(n+1)(n-1)^2x^{n-3} - (n-1)(n-2)(n-3)n^2x^{n-4}.
\end{aligned}$$

Thus $g(1) = g'(1) = g''(1) = g'''(x) = 0$ so that 1 is a root of multiplicity 4 of the equation $g(x) = 0$. By Descartes' rule of signs, the equation $g(x) = 0$ has no other positive roots. Since $g(0) = 1 > 0$, so $g(x) > 0$ for $0 < x < 1$.

This completes the proof.

Solution 4 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

Let $f(t) = 1/x$. The inequality goes unchanged because

$$\sqrt[n]{\frac{1+\frac{1}{t}+\ldots+\frac{1}{t^n}}{t^n(n+1)}} \ge \sqrt[n-1]{\frac{1+\frac{1}{t}+\ldots+\frac{1}{t^{n-1}}}{t^{n-1}n}}$$
$$\iff \sqrt[n]{\frac{1+t+\ldots+t^n}{n+1}} \ge \sqrt[n-1]{\frac{1+t+\ldots+t^{n-1}}{n}}.$$

This means that we may assume $x \geq 1$.

Let $x = 1$. The inequality becomes

$$1 = \sqrt[n]{\frac{1}{n+1}(\underbrace{1+1+\ldots+1}_{n+1 \ times})} \geq \sqrt[n]{\frac{1}{n}(\underbrace{1+1+\ldots+1}_{n \ times})} = 1.$$

Let $x > 1$. The inequality is also

$$\sqrt[n]{\frac{1}{n+1}\frac{1-x^{n+1}}{1-x}} \geq \sqrt[n-1]{\frac{1}{n}\frac{1-x^n}{1-x}},$$

that is

$$\sqrt[n]{\frac{1}{x-1}\int_1^x t^n} \geq \sqrt[n-1]{\frac{1}{x-1}\int_1^x t^{n-1}}.$$

This is the Power–Means inequality for integrals.

Also solved by Ed Gray, Highland Beach, FL; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5418:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzaău, Romania*

Let ABC be an acute triangle with circumradius R and inradius r. If $m \geq 0$, then prove that

$$\sum_{cyclic} \frac{\cos A \cos^{m+1} B}{\cos^{m+1} C} \geq \frac{3^{m+1} R^m}{2^{m+1}(R+r)^m}.$$

Solution 1 by Nikos Kalapodis, Patras, Greece

Applying Radon's Inequality and taking into account that $\cos A + \cos B + \cos C = 1 + \dfrac{r}{R}$ and $\displaystyle\sum_{cyclic} \frac{\cos A \cos B}{\cos C} \geq \frac{3}{2}$ (see Solution 1 of Problem 5381, SSMA, April 2016) we have

$$\sum_{cyclic} \frac{\cos A \cos^{m+1} B}{\cos^{m+1} C} = \sum_{cyclic} \frac{\left(\dfrac{\cos A \cos B}{\cos C}\right)^{m+1}}{\cos^m A} \geq \frac{\left(\displaystyle\sum_{cyclic} \frac{\cos A \cos B}{\cos C}\right)^{m+1}}{\left(\displaystyle\sum_{cyclic} \cos A\right)^m} \geq$$

$$\frac{3^{m+1} R^m}{2^{m+1}(R+r)^m}.$$

Solution 2 by Arkady Alt, San Jose, CA

Firstly, we will prove that in any acute triangle the inequality

(1) $\displaystyle\sum_{cyc} \frac{\cos A \cos B}{\cos C} \geq \frac{3}{2}$, holds.

Let $\alpha := \pi - 2A, \beta := \pi - 2B, \gamma := \pi - 2C$. Then $\alpha, \beta, \gamma > 0$ (since $A, B, C < \pi/2$), $\alpha + \beta + \gamma = \pi$ and **(1)** $\Longleftrightarrow \sum_{cyc} \dfrac{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\sin\frac{\gamma}{2}} \geq \dfrac{3}{2}$.

Let a, b, c be sidelenghts of a triangle with angles α, β, γ, respectively, and s be semiperimeter of this triangle.

Then $\sin\dfrac{\alpha}{2} = \sqrt{\dfrac{1-\cos\alpha}{2}} = \sqrt{\dfrac{1}{2}\left(1 - \dfrac{b^2+c^2-a^2}{2bc}\right)} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$ and,

similarly, $\sin\dfrac{\beta}{2} = \sqrt{\dfrac{(s-c)(s-a)}{ca}}, \sin\dfrac{\gamma}{2} = \sqrt{\dfrac{(s-a)(s-b)}{ab}}$. Hence,

$$\sum_{cyc} \frac{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\sin\frac{\gamma}{2}} = \sum_{cyc} \frac{\sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{(s-c)(s-a)}{ca}}}{\sqrt{\frac{(s-a)(s-b)}{ab}}} = \sum_{cyc} \frac{s-c}{c} = \sum_{cyc} \frac{s}{c} - 3 =$$

$$\frac{1}{2}(a+b+c) \cdot \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 3 \geq \frac{1}{2} \cdot 9 - 3 = \frac{3}{2}.$$

Noting that $\cos A + \cos B + \cos C = 1 + \dfrac{r}{R}$ and using a combination of the Weighted Power Mean-Arithmetic Inequality with weights $\cos A, \cos B, \cos C > 0$ and inequality **(1)** we obtain:

$$\sum_{cyc} \frac{\cos A \cos^{m+1} B}{\cos^{m+1} C} = \sum_{cyc} \cos A \left(\frac{\cos B}{\cos C}\right)^{m+1} = \sum_{cyc} \cos A \cdot \left(\frac{\sum_{cyc} \cos A \left(\frac{\cos B}{\cos C}\right)^{m+1}}{\sum_{cyc} \cos A}\right) \geq$$

$$\sum_{cyc} \cos A \cdot \left(\frac{\sum_{cyc} \cos A \left(\frac{\cos B}{\cos C}\right)}{\sum_{cyc} \cos A}\right)^{m+1} = \sum_{cyc} \cos A \cdot \frac{\left(\sum_{cyc} \frac{\cos A \cos B}{\cos C}\right)^{m+1}}{\left(\sum_{cyc} \cos A\right)^{m+1}} =$$

$$\frac{\left(\sum_{cyc} \frac{\cos A \cos B}{\cos C}\right)^{m+1}}{\left(\sum_{cyc} \cos A\right)^{m}} \geq \frac{\left(\frac{3}{2}\right)^{m+1}}{\left(1 + \frac{r}{R}\right)^{m}} = \frac{3^{m+1} R^m}{2^{m+1}(R+r)^m}.$$

Solution 3 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

The inequality is equivalent to and Radon's inequality, and applying it we obtain

$$\sum \frac{\cos A \cos^{m+1} B}{cos^{m+1} C} = \sum \frac{\left(\frac{\cos A \cos B}{\cos C}\right)^{m+1}}{\cos^m A} \underset{Radon}{\geq} \frac{\left(\sum \frac{\cos A \cos B}{\cos C}\right)^{m+1}}{\sum \cos A)^m} \geq \frac{3^{m+1} R^m}{2^{m+1}(R+r)^m},$$

where $\sum \cos A = 1 + \dfrac{r}{R}$ and $\sum \dfrac{\cos A \cos B}{\cos C} = \sum \dfrac{\tan C}{\tan A + \tan B}$.

Denote $\tan A = x,\ \tan B = y,\ \tan C = z$. Using Nesbitt's inequality, we have

$$\sum \frac{\tan C}{\tan A+\tan B} = \sum \frac{z}{x+y} \underset{Nesbitt}{\geq} \frac{3}{2}.$$

Solution 4 by Henry Ricardo, New York Math Circle, NY.

We will use the following known results: (1) Radon's inequality: If $x_k, a_k > 0\ \forall k, p > 0$, then $\sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} \geq \left(\sum_{k=1}^{n} x_k\right)^{p+1} / \left(\sum_{k=1}^{n} a_k\right)^p$; (2) $\displaystyle\sum_{cyclic} \frac{\cos A \cos B}{\cos C} \geq 3/2$; (3) $\sum_{cyclic} \cos A = (R+r)/R$.

Now we have

$$\begin{aligned}
\sum_{cyclic} \frac{\cos A \cos^{m+1} B}{\cos^{m+1} C} &= \sum_{cyclic} \frac{\left(\frac{\cos A \cos B}{\cos C}\right)^{m+1}}{\cos^m A} \\
&\overset{(1)}{\geq} \frac{\left(\sum_{cyclic} \frac{\cos A \cos B}{\cos C}\right)^{m+1}}{\left(\sum_{cyclic} \cos A\right)^m} \\
&\overset{(2),(3)}{\geq} \frac{(3/2)^{m+1}}{((R+r)/R)^m} = \frac{3^{m+1} R^m}{2^{m+1}(R+r)^m}.
\end{aligned}$$

Comments: (a) Inequality (2) appeared as problem 4053, proposed by Šefket Arslanagić, in *Crux Mathematicorum* and reappeared in several solutions to problem 5381 in this Journal; (b) Inequality (3) appeared in Solution 1 to problem 5381 in this Journal. It is also Lemma 2.5.1 in *Inequalities: A Mathematical Olympiad Approach* by R. Manfrino et. al.; (c) The related inequality $\sum_{cyclic} \left(\frac{\cos A \cos B}{\cos C}\right)^{m+1} \geq 3/2^{m+1}$ appeared as problem 5381 by the current proposers.

Editor's comment: **Moti Levy of Rehovot Israel** stated in his solution that: "A nice article on Radon's inequality is *A generalization of Radon's Inequality* by D. M. Bătineţu-Giurgiu and Ovidiu T. Pop, in CREATIVE MATH. & INF. 19 (2010), No. 2, 116 - 121."

Also solved by Ed Gray, Highland Beach, FL; Moti Levy, Rehovot, Israel; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5419: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $a_1, a_2, \cdots, a_n$ be positive real numbers. Prove that

$$\prod_{k=1}^{n} \left(\sum_{k=1}^{n} a_k^{t_k}\right) \geq \left(\sum_{k=1}^{n} a_k^{\frac{t_{n+1}}{4}}\right)^n$$

where for all $k \geq 1$, t_k is the k^{th} tetrahedral number defined by $t_k = \dfrac{k(k+1)(k+2)}{6}$.

Counter example by Moti Levy, Rehovot, Israel

The index k appears twice in the left hand side. This seems odd. The proposer has been asked and here is his response:
"*Here, index k is used in both sum and product.*
But indices in sums and product are dummy variables and they do not need to be distinct.
Surely, it is convenient but not necessary."

Following the proposer's argument that the index k is a dummy variable, we change the first index designation from the letter k to the letter j.

Now the proposed inequality becomes:

$$\prod_{j=1}^{n}\left(\sum_{k=1}^{n} a_k^{t_k}\right) \geq \left(\sum_{k=1}^{n} a_k^{\frac{t_{n+1}}{4}}\right)^n.$$

But

$$\prod_{j=1}^{n}\left(\sum_{k=1}^{n} a_k^{t_k}\right) = \left(\sum_{k=1}^{n} a_k^{t_k}\right)^n,$$

hence the proposed inequality implies

$$\sum_{k=1}^{n} a_k^{t_k} \geq \sum_{k=1}^{n} a_k^{\frac{t_{n+1}}{4}}.$$

Let us check this inequality for the special case $n = 2$, for example:

$$\sum_{k=1}^{2} a_k^{t_k} = a_1^{t_1} + a_2^{t_2} = a_1 + a_2^4$$

$$\sum_{k=1}^{2} a_k^{\frac{t_3}{4}} = a_1^{\frac{5}{2}} + a_2^{\frac{5}{2}}$$

Now take $a_1 = 4$ and $a_2 = 1$. Since

$$4 + 1 \leq 4^{\frac{5}{2}} + 1,$$

the inequality is not true.

Editor's note : The impossibility of this problem as it originally appeared was also noted by **Albert Stadler of Herrliberg, Switzerland.** I, as editor, should have noticed this mistake, but didn't; mea culpa.

In correspondence with the proposer of the problem, José Luis Díaz-Barrero, it was acknowledged that the problem should have read as follows:

Let $a_1, a_2, \cdots, a_n$ be positive real numbers. Prove that

$$\prod_{k=1}^{n}\left(\sum_{j=1}^{n} a_j^{t_k}\right) \geq \left(\sum_{k=1}^{n} a_k^{\frac{t_{n+1}}{4}}\right)^n$$

where for all $k \geq 1$, t_k is the k^{th} tetrahedral number defined by
$t_k = \dfrac{k(k+1)(k+2)}{6}$.

However, by changing the index in this manner, as Moti Levy mentioned, "changes the meaning of the problem." Below is a proof of the problem as it was intended to be in the first place.

Solution by the proposer. We consider the function $f(x)=\ln(a_1^x+a_2^x+\cdots+a_n^x)$ that is convex in R, as can be easily proven. Applying Jensen's inequality to $f(x)$, we obtain

$$\sum_{k=1}^{n} p_k \ln\left(a_1^{x_k}+\cdots+a_n^{x_k}\right) \geq \ln\left(a_1^{\sum_{k=1}^{n} p_k x_k}+\cdots+a_n^{\sum_{k=1}^{n} p_k x_k}\right)$$

where p_k are positive numbers of sum one and $x_1, x_2, \cdots, x_n \in R$. Taking into account that $f(x)=\ln(x)$ is injective, then the preceding expression becomes

$$\ln\left(\prod_{k=1}^{n}\left(\sum_{j=1}^{n} a_j^{x_k}\right)^{p_k}\right) \geq \ln\left(a_1^{\sum_{k=1}^{n} p_k x_k}+\cdots+a_n^{\sum_{k=1}^{n} p_k x_k}\right)$$

or equivalently,

$$\prod_{k=1}^{n}\left(\sum_{j=1}^{n} a_j^{x_k}\right)^{p_k} \geq \left(a_1^{\sum_{k=1}^{n} p_k x_k}+\cdots+a_n^{\sum_{k=1}^{n} p_k x_k}\right)$$

Setting $p_k=\dfrac{1}{n}$, $1\leq k\leq n$ and $x_k=t_k$, $1\leq k\leq n$, and taking into account that $\displaystyle\sum_{k=1}^{n} t_k=\frac{n}{4}\, t_{n+1}$, as can be easily proven for instance by induction, then we have

$$\prod_{k=1}^{n}\left(\sum_{j=1}^{n} a_j^{t_k}\right)^{1/n} \geq \sum_{k=1}^{n} a_k^{\frac{t_{n+1}}{4}}$$

from which the statement follows. Equality holds when $n=1$, and we are done.

Comment: On account of the preceding for the particular case $n=2$, we have

$$\prod_{k=1}^{2}\left(\sum_{j=1}^{2} a_j^{t_k}\right) \geq \left(\sum_{k=1}^{2} a_k^{\frac{t_{n+1}}{4}}\right)^2$$

or

$$(a_1^{t_1}+a_2^{t_1})(a_1^{t_2}+a_2^{t_2}) \geq (a_1^{t_3/4}+a_2^{t_3/4})^2$$

Letting $a_1=4, a_2=1, t_1=1, t_2=4, t_3=10$ in the last expression, we obtain

$$(4^1+1)(4^4+1) \geq (4^{5/2}+1)^2 \Longleftrightarrow 1285 \geq 1089$$

5420: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $A=\begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$. Calculate

$$\lim_{n\to\infty} \frac{1}{n}\left(I_2+\frac{A^n}{n}\right)^n.$$

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA

Let

$$A = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}.$$

The characteristic polynomial of A is $\lambda^2 - 2\lambda + 1$, so $\lambda = 1$ is an eigenvalue of A with algebraic multiplicity 2. The vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

forms a basis for the eigenspace of A corresponding to $\lambda = 1$. One solution of the equation $A - I = \mathbf{v}$ is the vector

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The matrix A can therefore be written in the form

$$A = T \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} T^{-1},$$

where

$$T = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}.$$

A straightforward induction argument establishes that

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix},$$

so that

$$A^n = T \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} T^{-1} = \begin{bmatrix} 2n+1 & n \\ -4n & -2n+1 \end{bmatrix}.$$

Thus,

$$\frac{A^n}{n} = \begin{bmatrix} 2 + \dfrac{1}{n} & 1 \\ -4 & -2 + \dfrac{1}{n} \end{bmatrix},$$

and

$$I_2 + \frac{A^n}{n} = \begin{bmatrix} 3 + \dfrac{1}{n} & 1 \\ -4 & -1 + \dfrac{1}{n} \end{bmatrix} = T \begin{bmatrix} 1 + \dfrac{1}{n} & 1 \\ 0 & 1 + \dfrac{1}{n} \end{bmatrix} T^{-1}.$$

Another straightforward induction argument establishes that

$$\begin{bmatrix} 1 + \dfrac{1}{n} & 1 \\ 0 & 1 + \dfrac{1}{n} \end{bmatrix}^n = \begin{bmatrix} \left(1 + \dfrac{1}{n}\right)^n & n\left(1 + \dfrac{1}{n}\right)^{n-1} \\ 0 & \left(1 + \dfrac{1}{n}\right)^n \end{bmatrix},$$

so that

$$\begin{aligned}
\left(I_2+\frac{A^n}{n}\right)^n &= T\begin{bmatrix} \left(1+\dfrac{1}{n}\right)^n & n\left(1+\dfrac{1}{n}\right)^{n-1} \\ 0 & \left(1+\dfrac{1}{n}\right)^n \end{bmatrix} T^{-1} \\
&= \begin{bmatrix} 2n\left(1+\dfrac{1}{n}\right)^{n-1}+\left(1+\dfrac{1}{n}\right)^n & n\left(1+\dfrac{1}{n}\right)^{n-1} \\ -4n\left(1+\dfrac{1}{n}\right)^{n-1} & -2n\left(1+\dfrac{1}{n}\right)^{n-1}+\left(1+\dfrac{1}{n}\right)^n \end{bmatrix}.
\end{aligned}$$

Finally,

$$\lim_{n\to\infty}\frac{1}{n}\left(I_2+\frac{A^n}{n}\right)^n = \begin{bmatrix} 2e & e \\ -4e & -2e \end{bmatrix}.$$

Solution 2 by Henry Ricardo, New York Math Circle, NY.

To simplify the solution, we invoke a known result $(*)$ that is a consequence of the Cayley-Hamilton theorem: If $A \in M_2(C)$ and the eigenvalues λ_1, λ_2 of A are equal, then for all $n \geq 1$ we have $A^n = \lambda_1^n B + n\lambda_1^{n-1}C$, where $B = I_2$ and $C = A - \lambda_1 I_2$. (See, for example, Theorem 2.25(b) in *Essential Linear Algebra with Applications* by T. Andreescu, Birkhäuser, 2014.)

The eigenvalues of the given matrix A are both equal to 1, so we apply $(*)$ to get $A^n = nA - (n-1)I_2$. Now we use the last expression to see that $M = I_2 + A^n/n = A + I_2/n$; and, since M's eigenvalues are both equal to $1 + 1/n$, we apply $(*)$ again to determine that

$$\begin{aligned}
\frac{1}{n}\left(I_2+\frac{A^n}{n}\right)^n &= \frac{1}{n}M^n \\
&= \frac{1}{n}\left[\left(1+\frac{1}{n}\right)^n I_2 + n\left(1+\frac{1}{n}\right)^{n-1}\left(M-\left(1+\frac{1}{n}\right)I_2\right)\right] \\
&= \frac{1}{n}\left[n\left(1+\frac{1}{n}\right)^{n-1} M + \left(1+\frac{1}{n}\right)^n (1-n)I_2\right] \\
&= \frac{1}{n}\left[n\left(1+\frac{1}{n}\right)^{n-1}\left(A+\frac{I_2}{n}\right) + \left(1+\frac{1}{n}\right)^n (1-n)I_2\right] \\
&= \left(1+\frac{1}{n}\right)^n \cdot \frac{n^2A-(n^2-n-1)I_2}{n(n+1)} \\
\to e(A-I_2) &= \begin{pmatrix} 2e & e \\ -4e & -2e \end{pmatrix}.
\end{aligned}$$

Solution 3 by Albert Stadler, Herrliberg, Switzerland

Put $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 3 & -2 \\ -6 & 7 \end{pmatrix}$.

Then

$AS = SJ,\ S^{-1} = \dfrac{1}{9}\begin{pmatrix} 7 & 2 \\ 6 & 3 \end{pmatrix},\ A = SJS^{-1}, A^n = \left(SJS^{-1}\right)^n = SJ^nS^{-1},\ J^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$

$$
\begin{aligned}
\frac{1}{n}\left(I+\frac{A^n}{n}\right)^n = \frac{1}{n}\left(I+\frac{(SJS^{-1})^n}{n}\right)^n &= \frac{1}{n}\left(I+S\frac{J^n}{n}S^{-1}\right)^n \\
&= \frac{1}{n}\left(S\left(I+\frac{J^n}{n}\right)S^{-1}\right)^n \\
&= \frac{1}{n}S\left(I+\frac{J^n}{n}\right)^n S^{-1} \\
&= \frac{1}{n}S\begin{pmatrix} 1+\frac{1}{n} & 1 \\ 0 & 1+\frac{1}{n} \end{pmatrix}^n S^{-1} \\
&= \frac{\left(1+\frac{1}{n}\right)^n}{n}S\begin{pmatrix} 1 & \frac{1}{1+\frac{1}{n}} \\ 0 & 1 \end{pmatrix}^n S^{-1} \\
&= \frac{\left(1+\frac{1}{n}\right)^n}{n}S\begin{pmatrix} 1 & \frac{n}{1+\frac{1}{n}} \\ 0 & 1 \end{pmatrix}^n S^{-1} \longrightarrow \\
&= eS\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} S^{-1} \\
&= \frac{e}{9}S\begin{pmatrix} 6 & 3 \\ 0 & 0 \end{pmatrix} \\
&= e\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}, \text{ as } n \longrightarrow \infty.
\end{aligned}
$$

Solution 4 by Brian D. Beasley, Presbyterian College, Clinton, SC

Solution. Let $B_n = I_2 + (1/n)A^n$. It is straightforward to show by induction that $B_n = A + (1/n)I_2$. Using the characteristic polynomial of B_n, we have $B_n^2 = 2(1+1/n)B_n - (1+1/n)^2 I_2$. It then follows by induction on k that for each positive integer k,

$$B_n^k = k\left(1+\tfrac{1}{n}\right)^{k-1} B_n - (k-1)\left(1+\tfrac{1}{n}\right)^k I_2.$$

Thus

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}B_n^n &= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n-1}B_n-\left(\frac{n-1}{n}\right)\left(1+\frac{1}{n}\right)^n I_2\right] \\
&= eA-eI_2 \\
&= e(A-I_2) \\
&= e\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}.
\end{aligned}$$

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; David R. Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.