Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before April 15, 2011

- **5146:** *Proposed by Kenneth Korbin, New York, NY*

  Find the maximum possible value of the perimeter of an integer sided triangle with in-radius $r = \sqrt{13}$.

- **5147:** *Proposed by Kenneth Korbin, New York, NY*

  Let
  $$\begin{cases} x = 5N^2 + 14N + 23 \text{ and} \\ y = 5(N+1)^2 + 14(N+1) + 23 \end{cases}$$
  where N is a positive integer. Find integers a_i such that
  $$a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6 = 0.$$

- **5148:** *Proposed by Pedro Pantoja (student, UFRN), Natal, Brazil*

  Let a, b, c be positive real numbers such that $ab + bc + ac = 1$. Prove that
  $$\frac{a^2}{\sqrt[3]{b(b+2c)}} + \frac{b^2}{\sqrt[3]{c(c+2a)}} + \frac{c^2}{\sqrt[3]{a(a+2b)}} \geq 1.$$

- **5149:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  A regular n-gon $A_1, A_2 \cdots, A_n$ $(n \geq 3)$ has center F, the focus of the parabola $y^2 = 2px$, and no one of its vertices lies on the x axis. The rays $FA_1, FA_2, \cdots, FA_n$ cut the parabola at points $B_1, B_2, \cdots, B_n$.

  Prove that
  $$\frac{1}{n}\sum_{k=1}^{n} FB_k^2 > p^2.$$

- **5150:** *Proposed by Mohsen Soltanifar(student, University of Saskatchewan), Saskatoon, Canada*

  Let $\{A_n\}_{n=1}^{\infty}, (A_n \in M_{n\times n}(C))$ be a sequence of matrices such that $\det(A_n) \neq 0, 1$ for all $n \in N$. Calculate:
  $$\lim_{n\to\infty} \frac{n^n \ln(|\det(A_n)|)}{\ln\left(|\det\left(adj^{\circ n}\left(A_n\right)\right)|\right)},$$
  where $adj^{\circ n}$ refers to $adj \circ adj \circ \cdots \circ adj, n$ times, the n^{th} iterate of the classical adjoint.

- **5151:** *Proposed by Ovidiu Furdui, Cluj, Romania*

  Find the value of
$$\prod_{n=1}^{\infty}\left(\sqrt{\frac{\pi}{2}}\cdot\frac{(2n-1)!!\sqrt{2n+1}}{2^n n!}\right)^{(-1)^n}.$$
  More generally, if $x\neq n\pi$ is a real number, find the value of
$$\prod_{n=1}^{\infty}\left(\frac{x}{\sin x}\left(1-\frac{x^2}{\pi^2}\right)\cdots\left(1-\frac{x^2}{(n\pi)^2}\right)\right)^{(-1)^n}.$$

Solutions

- **5128:** *Proposed by Kenneth Korbin, New York, NY*

  Find all positive integers less than 1000 such that the sum of the divisors of each integer is a power of two.

  For example, the sum of the divisors of 3 is 2^2, and the sum of the divisors of 7 is 2^3.

  Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

  For $n\geq 1$, let $\sigma(n)$ denote the sum of the positive divisors of n. The problem is to find all positive integers $n<1000$ such that $\sigma(n)=2^k$ for some integer $k\geq 0$. We note first that $n=1$ is a solution since $\sigma(1)=1=2^0$. For the remainder, we will assume that $n\geq 2$. Our key result is the following:

  Lemma. If p is prime and k and e are positive integers such that $\sigma(p^e)=2^k$, then $e=1$ and $p=2^k-1$ (i.e., p is a Mersenne prime).

  Proof. First of all, $p\neq 2$ since $\sigma(2^e)=1+2+\ldots+2^e$, which is odd. Further, since p must be odd,
$$2^k=\sigma(p^e)=1+p+\ldots+p^e$$
  implies that e is also odd. It follows that
$$\begin{aligned}2^k &= (1+p)+\left(p^2+p^3\right)+\left(p^4+p^5\right)+\ldots+\left(p^{e-1}+p^e\right)\\ &= (1+p)\left(1+p^2+p^4+\ldots+p^{e-1}\right). \qquad (*)\end{aligned}$$
  Then, $1+p$ divides 2^k and $1+p>1$, which leads us to conclude that $1+p=2^m$, with $1\leq m\leq k$. Statement (*) reduces to
$$2^{k-m}=1+p^2+p^4+\ldots+p^{e-1}.$$
  If $e\geq 3$, then $m<k$ and using the same reasoning as above, we get
$$\begin{aligned}2^{k-m} &= \left(1+p^2\right)+\left(p^4+p^6\right)+\ldots+\left(p^{e-3}+p^{e-1}\right)\\ &= \left(1+p^2\right)\left(1+p^4+\ldots+p^{e-3}\right),\end{aligned}$$
  which implies that $1+p^2=2^i$, for some positive integer $i\leq k-m$. Thus,
$$2^i=1+p^2=1+(2^m-1)^2=2^{2m}-2^{m+1}+2,$$

or

$$2^{i-1} = 2^{2m-1} - 2^m + 1 = 2^m \left(2^{m-1} - 1\right) + 1.$$

This requires $i = m = 1$, which is impossible since this would entail $p = 2^m - 1 = 2 - 1 = 1$. Therefore, $e = 1$ and $2^k = \sigma(p) = p + 1$, i.e., $p = 2^k - 1$.

To return to our problem, we may write

$$n = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$$

for distinct primes $p_1, \ldots, p_m$ and positive integers $e_1, \ldots, e_m$. Since σ is multiplicative and $p_1^{e_1}, \ldots, p_m^{e_m}$ are pairwise relatively prime,

$$2^k = \sigma(n) = \sigma\left(p_1^{e_1}\right) \sigma\left(p_2^{e_2}\right) \cdots \sigma\left(p_m^{e_m}\right).$$

Further, for $i = 1, \ldots, m$, $\sigma\left(p_i^{e_i}\right) \geq p_i + 1 > 1$. Hence, there are positive integers $k_1, \ldots, k_m$ such that

$$\sigma\left(p_i^{e_i}\right) = 2^{k_i}$$

for $i = 1, \ldots, m$. By the Lemma, $e_1 = e_2 = \ldots = e_m = 1$ and

$$p_i = 2^{k_i} - 1$$

for $i = 1, \ldots, m$. Therefore, $n = p_1 p_2 \cdots p_m$, where each p_i is a distinct Mersenne prime.

To solve our problem, we need to find all Mersenne primes < 1000 and all products of distinct Mersenne primes for which the product < 1000. The Mersenne primes < 1000 are $3, 7, 31$, and 127. All solutions of $\sigma(n) = 2^k$, with $n < 1000$, are listed below.

n	$\sigma(n)$
1	2^0
3	2^2
7	2^3
$21 = 3 \cdot 7$	2^5
31	2^5
$93 = 3 \cdot 31$	2^7
127	2^7
$217 = 7 \cdot 31$	2^8
$381 = 3 \cdot 127$	2^9
$651 = 3 \cdot 7 \cdot 31$	2^{10}
$889 = 7 \cdot 127$	2^{10}

Also solved by Brian D. Beasley, Clinton, SC; Pat Costello, Richmond, KY; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Boris Rays, Brooklyn, NY; Harry Sedinger, St. Bonaventure, NY; Raúl A. Simón, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA; Tran Trong Hoang Tuan (student, Bac Lieu High School for the Gifted), Bac Lieu City, Vietnam, and the proposer.

- **5129:** *Proposed by Kenneth Korbin, New York, NY*

Given prime number c and positive integers a and b such that $a^2 + b^2 = c^2$, express in terms of a and b the lengths of the legs of the primitive Pythagorean Triangles with hypotenuses c^3 and c^5, respectively.

Solution 1 by Howard Sporn, Great Neck, NY

A Pythagorean Triple (a, b, c) can be represented by the complex number $a + bi$, with modulus c. By multiplying two Pythagorean Triples in this form, one can generate another Pythagorean Triple. For instance, the complex representation of the 3-4-5 triangle is $3 + 4i$. By multiplying the complex number by itself, (and taking the absolute value of the real and imaginary parts), one obtains the 7-24-25 triangle:

$$\begin{aligned} (3+4i)(3+4i) &= -7+24i \\ 7^2+24^2 &= 25^2 \end{aligned}$$

By cubing $a + bi$, one can obtain a Pythagorean Triple whose hypotenuse is c^3.

$$\begin{aligned} (a+bi)^3 &= (a+bi)^2(a+bi) \\ &= (a^2-b^2+2abi)(a+bi) \\ &= a^3-3ab^2+i\left(3a^2-b^3\right) \end{aligned}$$

One can verify that the modulus of this complex number is $\left(a^2+b^2\right)^3 = c^3$. Thus we obtain the Pythagorean Triple $\left(\left|a^3-3ab^2\right|, \left|3a^2b-b^3\right|, c^3\right)$.

That this Pythagorean Triangle is primitive can be seen by factoring the lengths of the legs:

$$\begin{aligned} a^3-3ab^2 &= a\left(a^2-3b^2\right), \text{ and} \\ 3a^2b-b^3 &= b\left(3a^2-b^2\right), \end{aligned}$$

generally have no factors in common.

Example: If we let $(a, b, c) = (3, 4, 5)$, we obtain the Pythagorean Triple $(117, 44, 125)$.

By a similar procedure , one can obtain a Pythagorean Triple whose hypotenuse is c^5.

$$\begin{aligned} (a+bi)^5 &= (a+bi)^3(a+bi)(a+bi) \\ &= \left[a^3-3ab^2+i\left(3a^2b-b^3\right)\right](a+bi)(a+bi) \\ &= \left[a^4-6a^2b^2+b^4+i\left(4a^3b-4ab^3\right)\right](a+bi) \\ &= a^5-10a^3b^2+5ab^4+i\left(5a^4b-10a^2b^3+b^5\right). \end{aligned}$$

Thus we obtain the Pythagorean Triple

$$\left(\left|a^5-10a^3b^2+5ab^4\right|,\ \left|5a^4b-10a^2b^3+b^5\right|,\ c^5\right).$$

Example: If we let $(a, b, c) = (3, 4, 5)$, we obtain the Pythagorean Triple $(237, 3116, 3125)$.

Solution 2 by Brian D. Beasley, Clinton, SC

Given positive integers a, b, and c with c prime and $c^2 = a^2 + b^2$, we may assume without loss of generality that $a < b < c$. Also, we note that c must be odd and that c divides neither a nor b. Using the classic identity

$$(w^2 + x^2)(y^2 + z^2) = (wy + xz)^2 + (wz - xy)^2,$$

we proceed from $c^2 = a^2 + b^2$ to obtain $c^4 = (-a^2 + b^2)^2 + (2ab)^2$. Similarly, we have

$$c^6 = (-a^3 + 3ab^2)^2 + (3a^2b - b^3)^2$$

and

$$c^{10} = (a^5 - 10a^3b^2 + 5ab^4)^2 + (-5a^4b + 10a^2b^3 - b^5)^2.$$

Thus the leg lengths for the Primitive Pythagorean Triangle (PPT) with hypotenuse c^3 are

$$m = |-a^3 + 3ab^2| \quad \text{and} \quad n = |3a^2b - b^3|,$$

while the leg lengths for the PPT with hypotenuse c^5 are

$$q = |a^5 - 10a^3b^2 + 5ab^4| \quad \text{and} \quad r = |-5a^4b + 10a^2b^3 - b^5|.$$

To show that these triangles are primitive, we first note that $(-a^2 + b^2, 2ab, c^2)$ is a PPT, since c cannot divide $2ab$. Next, we prove that (m, n, c^3) is also a PPT: If not, then c divides both $a(-a^2 + 3b^2)$ and $b(3a^2 - b^2)$, so c divides $-a^2 + 3b^2$ and $3a^2 - b^2$; thus c divides the linear combination $(-a^2 + 3b^2) + 3(3a^2 - b^2) = 8a^2$, a contradiction. Similarly, we prove that (q, r, c^5) is a PPT: If not, then c divides both $a(a^4 - 10a^2b^2 + 5b^4)$ and $b(-5a^4 + 10a^2b^2 - b^4)$, so c divides $a^4 - 10a^2b^2 + 5b^4$ and $-5a^4 + 10a^2b^2 - b^4$; thus c divides the linear combinations

$$(a^4 - 10a^2b^2 + 5b^4) + 5(-5a^4 + 10a^2b^2 - b^4) = 8a^2(-3a^2 + 5b^2)$$

and

$$5(a^4 - 10a^2b^2 + 5b^4) + (-5a^4 + 10a^2b^2 - b^4) = 8b^2(-5a^2 + 3b^2).$$

But this means that c divides the linear combination
$3(-3a^2 + 5b^2) - 5(-5a^2 + 3b^2) = 16a^2$, a contradiction.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; David E. Manes, Oneonta, NY, and the proposer.

- **5130:** *Proposed by Michael Brozinsky, Central Islip, NY*

In Cartesianland, where immortal ants live, calculus has not been discovered. A bride and groom start out from $A(-a, 0)$ and $B(b, 0)$ respectively where $a \neq b$ and $a > 0$ and $b > 0$ and walk at the rate of one unit per second to an altar located at the point P on line $L : y = mx$ such that the time that the first to arrive at P has to wait for the other to arrive is a maximum. Find, without calculus, the locus of P as m varies through all nonzero real numbers.

Solution 1 by Michael N. Fried, Kibbtuz Revivim, Israel

Let OQ be the line $y = mx$. Since it is the total time which must be a minimum, we might as well consider the minimum time from A to a point P on OQ and then from P to B. But since the speed is equal and constant for both the bride and groom the minimum time will be achieved for the path having the minimum distance. This, as is well-known, occurs when $\angle APO = \angle BPQ$. Accordingly, OP is the *external* angle bisector of angle APB, and, thus, $\dfrac{BP}{AP} = \dfrac{BO}{OA}$ = a constant ratio. So, P lies on a circle (an Apollonius circle) whose diameter is OAC, where OC is the harmonic mean between OA and OB.

Solution 2 by the proposer

Since the bride and groom go at the same rate, then for a given m, P is the point such that the maximum of $||AQ| - |BQ||$ for points Q on L occurs when Q is P. Let A' denote the reflection of A about this line.

Now since $||AQ| - |BQ|| = ||A'Q| - |BQ|| \geq |A'B|$ (from the triangle inequality) we have this maximum must be $|A'B|$ since it is attained when P is the point of intersection of the line through B and A', with L. (Note that the line through A' and B is not parallel to L because that would imply that the origin is the midpoint of AB because the line through the midpoint of AA' and the midpoint of AB is parallel to the line through A' and B.)

Let M be the midpoint of segment AA'. Now, since triangles $A'PM$ and APM are congruent, L is the angle bisector at P in triangle ABP, and since an angle bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides we have $\dfrac{AP}{BP} = \dfrac{a}{b}$ (1).

Denoting P by $P(X,Y)$ we thus have $Y \neq 0$ and thus $X \neq 0$ and so from (1)

$$\frac{\sqrt{(X+a)^2 + (mX)^2}}{\sqrt{(X-b)^2 + (mX)^2}} = \frac{a}{b},$$

and since $X \neq 0$, we have by squaring both sides and solving for X, that

$$\begin{aligned} X &= \frac{2ab}{(a-b)(m^2+1)}, \text{ and thus} \\ Y &= \frac{2mab}{(a-b)(m^2+1)} \end{aligned}$$

are parametric equations of the locus. Now replacing m by $\dfrac{Y}{X}$ and simplifying, we obtain

$$X = \frac{2abX^2}{(X^2+Y^2)(a-b)}$$

which is just the circle

$$(X^2+Y^2)(a-b) = 2abX$$

with the endpoints of the diameter deleted. The endpoints of the diameter occur when $Y = 0$; that is, at $(0,0)$, and at $\left(\dfrac{2ab}{a-b}, 0\right)$.

Note that if the line $x = 0$ were a permissible altar line, then we would add $(0,0)$ to the locus, while if the $x-$axis were a permissible altar line, then the union of the rays $(-\infty, -a] \cup [b, \infty)$ would be part of the locus, and in particular, this includes $\left(\dfrac{2ab}{a-b}, 0\right)$.

- **5131:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b, c be positive real numbers. Prove that

$$\frac{a+b+3c}{3a+3b+2c}+\frac{a+3b+c}{3a+2b+3c}+\frac{3a+b+c}{2a+3b+3c}\geq\frac{15}{8}.$$

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

The inequality is homogeneous, so we can assume without loss of generality that $a+b+c=1$, being equivalent to

$$\frac{1+2c}{3-c}+\frac{1+2b}{3-b}+\frac{1+2a}{3-a}\geq\frac{15}{8},$$

which is Jensen's inequality $f(c)+f(b)+f(a)\geq 3f\left(\frac{c+b+a}{3}\right)$ applied to the convex function $f\left(x\right)=\frac{1+2x}{3-x}$ and the numbers $c,\ b,\ a$ on the interval $(0,1)$; equality occurs if and only if $a=b=c$.

Solution 2 by Javier García Cavero (student, Mathematics Club of the Instituto de Educación Secundaria- N^o 1), Requena-Valencia, Spain

Changing the variables, that is to say, calling

$$\begin{aligned}x &= 2a+3b+3c,\\ y &= 3a+2b+3c,\ \text{and}\\ z &= 3a+3b+2c\end{aligned}$$

it is easy to see, solving the corresponding system of equations, that

$$\begin{aligned}a+b+c &= \frac{x+y+z}{8}\ \text{and that}\\ a &= \frac{-5x+3y+3z}{8}\\ b &= \frac{3x-5y+3z}{8},\ \text{and}\\ c &= \frac{3x+3y-5z}{8}.\end{aligned}$$

The numerators of the fractions will thus be:

$$a+b+3c=\frac{7x+7y-9z}{8},\quad a+3b+c=\frac{7x-9y+7z}{8},\quad 3a+b+c=\frac{-9x+7y+7z}{8}$$

Replacing everything in the initial expression:

$$\begin{aligned}&\frac{a+b+3c}{3a+3b+2c}+\frac{a+3b+c}{3a+2b+3c}+\frac{3a+b+c}{2a+3b+3c}\\ =\ &\frac{7x+7y-9z}{8z}+\frac{7x-9y+7z}{8y}+\frac{-9x+7y+7z}{8x}\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{7x}{8z}+\frac{7y}{8z}+\frac{-9}{8}\right)+\left(\frac{7x}{8y}+\frac{-9}{8}+\frac{7z}{8y}\right)+\left(\frac{-9}{8}+\frac{7y}{8x}+\frac{7z}{8x}\right)\\
&= 3\cdot\left(\frac{-9}{8}\right)+\frac{7}{8}\left(\frac{x}{z}+\frac{y}{z}+\frac{x}{y}+\frac{z}{y}+\frac{y}{x}+\frac{z}{x}\right)\\
&\quad \frac{-27}{8}+\frac{7}{8}\left(\left(\frac{x}{z}+\frac{z}{x}\right)+\left(\frac{y}{z}+\frac{z}{y}\right)+\left(\frac{x}{y}+\frac{y}{x}\right)\right)\\
&\geq \frac{-27}{8}+\frac{42}{8}\\
&= \frac{15}{8},
\end{aligned}
$$

since $r+\dfrac{1}{r}\geq 2$. Equality occurs for $x=y=z$ and, therefore, for $a=b=c$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

Since

$$
\begin{aligned}
&\frac{a+b+3c}{3a+3b+2c}+\frac{b+c+3a}{3b+3c+2a}+\frac{c+a+3b}{3c+3a+2b}-\frac{15}{8}\\
=\;&\frac{7\left(6a^3+6b^3+6c^3-a^2b-ab^2-b^2c-bc^2-c^2a-ca^2-12abc\right)}{8\left(3a+3b+2c\right)\left(3b+3c+2a\right)\left(3c+3a+2b\right)}\\
=\;&\frac{7\Big((3a+3b+2c)(a-b)^2+(3b+3c+2a)(b-c)^2+(3c+3a+2b)(c-a)^2\Big)}{8(3a+3b+2c)(3b+3c+2a)(3c+3a+2b)}\\
\geq\;& 0,
\end{aligned}
$$

the inequality of the problem follows.

Solution 4 by P. Piriyathumwong (student, Patumwan Demonstration School), Bangkok, Thailand

The given inequality is equivalent to the following:

$$
\begin{aligned}
\sum_{cyc}\left(\frac{a+b+3c}{3a+3b+2c}-\frac{5}{8}\right)\geq 0 \quad &\Leftrightarrow \quad \sum_{cyc}\left(\frac{-a-b+2c}{3a+3b+2c}\right)\geq 0\\
&\Leftrightarrow \quad \sum_{cyc}\left(\frac{(c-a)+(c-b)}{3a+3b+2c}\right)\geq 0\\
&\Leftrightarrow \quad \sum_{cyc}(a-b)\left(\frac{1}{2a+3b+3c}-\frac{1}{3a+2b+3c}\right)\geq 0\\
&\Leftrightarrow \quad \sum_{cyc}\frac{(a-b)^2}{(2a+3b+3c)(3a+2b+3c)}\geq 0,
\end{aligned}
$$

which is obviously true.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, University "Tor Vergata", Rome, Italy; Boris Rays, Brooklyn, NY; Tran Trong Hoang Tuan (student, Bac Lieu High School for the Gifted), Bac Lieu City, Vietnam, and the proposer.

- **5132:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Find all all functions $f : C \to C$ such that $f(f(z)) = z^2$ for all $z \in C$.

Solution by Kee-Wai Lau, Hong Kong, China

We show that no such functions $f(z)$ exist by considering the values of $f(1), f(-1), f(i), f(-i)$, where $i = \sqrt{-1}$.

From the given relation

$$f(f(z)) = z^2 \qquad (1)$$

we obtain $f(f(f(z))) = f(z^2)$ so that

$$(f(z))^2 = f\left(z^2\right). \qquad (2)$$

Replacing z by z^2 in (2), we get

$$f(z^4) = (f(z))^4. \qquad (3)$$

By putting $z = 1$ into (2), we obtain $f(1) = 0$ or 1. If $f(1) = 0$, then by putting $z = i$ into (3), we get $0 = f(i^4) = (f(i))^4$, so that $f(i) = 0$. Putting $z = i$ into (1) we get $f(0) = -1$ and putting $z = 0$ into (2) we obtain $(-1)^2 = -1$ which is false. It follows that

$$f(1) = 1. \qquad (4)$$

By putting $z = -1$ into (2) we get $(f(-1))^2$ so that $f(-1) = -1$ or 1.

If $f(-1) = -1$ then by (1), $-1 = f(f(-1)) = (-1)^2 = 1$, which is false.

Hence,

$$f(-1) = 1. \qquad (5)$$

By putting $z = i$ into (3), we are $(f(i))^4 = 1$, so that $f(i) = -1, 1, i, -i$.

If $f(i) = \pm 1$, then by (1), (4) and (5), $1 = f(f(i)) = i^2 = -1$, which is false.

If $f(i) = i$, then by (1), $i = f\left(f((i)\right) = -1$, which is also false. Hence,

$$f(i) = -i \qquad (6)$$

By putting $z = -i$ into (3), we have $(f(-i))^4 = 1$, so that $f(-i) = -1, 1, i, -i$.

If $f(-i) = \pm 1$, then by (1), (4), and (5) $1 = f(f(-i)) = (-i)^2 = -1$, which is false.

If $f(-i) = \pm i$, then by (1) and (6) $-i = f(f(-i)) = (-i)^2 = -1$, which is also false.

Thus $f(-i)$ can take no value, showing that no such $f(z)$ exists.

Also solved by Howard Sporn and Michael Brozinsky (jointly), of Great Neck and Central Islip, NY (respectively), and the proposer.

- **5133:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $n \geq 1$ be a natural number. Calculate

$$I_n = \int_0^1 \int_0^1 (x-y)^n dxdy.$$

Solutions 1 and 2 by Valmir Bucaj (student, Texas Lutheran University), Seguin, TX

Solution 1) We first calculate $\int_0^1 (x-y)^n dx$.

Letting $u = x - y$ we get

$$\begin{aligned}\int_0^1 (x-y)^n &= \int_{-y}^{1-y} u^n du \\ &= \frac{1}{n+1}\left[(1-y)^{n+1} + (-1)^n y^{n+1}\right].\end{aligned}$$

Now,

$$\begin{aligned}I_n &= \int_0^1 \int_0^1 (x-y)^n dxdy \\ &= \frac{1}{n+1}\int_0^1 \left[(1-y)^{n+1} + (-1)^n y^{n+1}\right] dy \\ &= \begin{cases} \dfrac{2}{(n+1)(n+2)} & : n \text{ even} \\ 0 & : n \text{ odd}\end{cases}\end{aligned}$$

Solution 2) Using the fact that

$$(x-y)^n = \sum_{k=0}^n C_n^k \left(-1\right)^k x^{n-k} y^k,$$

we get

$$\begin{aligned}I_n &= \int_0^1 \int_0^1 (x-y)^n dxdy \\ &= \int_0^1 \int_0^1 \sum_{k=0}^n C_n^k (-1)^k x^{n-k} y^k dxdy\end{aligned}$$

$$= \sum_{k=0}^{n} C_n^k (-1)^k \frac{1}{(n-k+1)(k+1)}.$$

Comment: Comparing Solution 1 with Solution 2, we obtain an interesting *side-result:* namely the identity

$$\sum_{k=0}^{n} C_n^k (-1)^k \frac{1}{(n-k+1)(k+1)} = \begin{cases} \dfrac{2}{(n+1)(n+2)} & : n \text{ even} \\ 0 & : n \text{ odd} \end{cases},$$

which one can verify directly, as well.

Solution 3 by Paul M. Harms, North Newton, KS

Let $f(x,y) = (x-y)^n$. The integration region is the square in the x,y plane with vertices at $(0,0),(1,0),(1,1)$, and $(0,1)$. The line $y=x$ divides this region into two congruent triangles. I will use the terms *lower triangle* and *upper triangle,* for these two congruent triangles.

The points (x,y) and (y,x) are symmetric with respect to the line $y=x$. Let n be an odd integer. For each point (x,y) in the lower (upper) triangle we have a point (y,x) in the upper (lower) triangle such that $f(y,x) = -f(x,y)$. Thus the value of $I_n = 0$ when n is an odd integer.

When n is an even integer, $f(y,x) = f(x,y)$ and the value of the original double integral should equal $2\int_0^1 \int_y^1 (x-y)^n dxdy$ where the region of the integration is the lower triangle. The first integration of the last double integral yields

$$\frac{(x-y)^{n+1}}{n+1}\Bigg|_y^1 = \frac{(1-y)^{n+1}}{n+1}.$$

The second integration of the double integral then yields the expression

$$\frac{-2(1-y)^{n+2}}{(n+1)(n+2)}\Bigg|_0^1 = \frac{2}{(n+1)(n+2)} = I_n$$

when n is an even integer.

Also solved by Brian D. Beasley, Clinton, SC; Michael C. Faleski, University Center, MI; G. C. Greubel, Newport News, VA; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy; James Reid (student, Angelo State University), San Angelo, TX; Raúl A. Simón, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.