Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before April 15, 2013

- **5242:** *Proposed by Kenneth Korbin, New York, NY*

  Let N be any positive integer, and let $x = N(N+1)$. Find the value of

  $$\sum_{K=0}^{x/2} \binom{x-K}{K} x^K.$$

- **5243:** *Proposed by Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

  If a, b, c are consecutive Pythagorean numbers, then solve in the integers the equation:

  $$\frac{x^2 + bx}{a^y - 1} = c.$$

  (A consecutive Pythagorean triple is a Pythagorean triple that is composed of consecutive integers.)

- **5244:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  Let T_a and S_b denote the a^{th} triangular and the b^{th} square number, respectively. Find explicit instances of such numbers to prove that every Fibonacci number F_n occurs among the values $\gcd(T_a, S_b)$.

- **5245:** *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany*

  Determine all functions $f : \Re \to \Re - \{-2, -\frac{1}{2}, -1, 0, \frac{1}{2}, 2\}$, which satisfy the relation

  $$f(x) + f\left(\frac{-x-5}{2x+1}\right) + f\left(\frac{4x+5}{-2x+2}\right) = ax + b$$

  where $a, b, \in \Re$.

- **5246:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

Let $a_1, a_2, \ldots a_n, (n \geq 3)$ be distinct complex numbers. Compute the sum

$$\sum_{k=1}^{n} s_k \prod_{j \neq k} \frac{(-1)^n}{a_j - a_k},$$

where $s_k = \left(\sum_{i=1}^{n} a_i\right) - a_k,\ 1 \leq k \leq n.$

- **5247:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Calculate

$$\lim_{n\to\infty} \frac{1}{n} \sqrt[n]{\int_0^1 \ln(1+e^x)\ln(1+e^{2x})\cdots\ln(1+e^{nx})\ dx}.$$

Solutions

- **5224:** *Proposed by Kenneth Korbin, New York, NY*

  Let $T_1 = T_2 = 1, T_3 = 2$, and $\mathrm{T}_N = T_{N-1} + T_{N-2} + T_{N-3}$. Find the value of

$$\sum_{N=1}^{\infty} \frac{T_N}{\pi^N}.$$

Solution 1 by Arkady Alt, San Jose, CA

Noting that $\{T_n\}_{n\geq 1}$ is an increasing sequence of positive integers we obtain:

$$\begin{aligned}
\frac{T_{n+1}}{T_n} &= 1 + \frac{T_{n-1}}{T_n} + \frac{T_{n-2}}{T_n} \\
&= 1 + \frac{T_{n-1}}{T_n} + \frac{T_{n-2}}{T_{n-1}} \cdot \frac{T_{n-1}}{T_n} \\
&< 1 + 1 + 1 \cdot 1 = 3,\ n \in N.
\end{aligned}$$

Hence,

$$\frac{T_{n+1}}{T_n} < 3 \iff \frac{T_{n+1}}{3^{n+1}} < \frac{T_n}{3^n},\ n \in N \Longrightarrow \frac{T_n}{3^n} < \frac{T_1}{3^1} \iff T_n < 3^{n-1},\ n \in N.$$

and therefore, by the comparison test for series, $\sum_{i=1}^{n} T_i x^{i-1}$ is convergent for any $x \in \left(0, \frac{1}{3}\right)$ because for such x it is bounded by $\sum_{n=1}^{\infty} (3x)^{n-1} = \frac{1}{1-3x}$.

Since

$$\left(1 - x - x^2 - x^3\right) \sum_{n=1}^{\infty} T_n x^{n-1} = T_1 + x(T_2 - T_1) + x^2(T_3 - T_2 - T_1)$$

$$
\begin{aligned}
&+ \quad \sum_{n=1}^{\infty} x^{n+2}\left(T_{n+3}-T_{n+2}-T_{n+2}-T_n\right) \\
&= \quad T_1 + x(1-1) + x^2(2-1-1) + \sum_{n=1}^{\infty} x^{n+2}\cdot 0 = 1
\end{aligned}
$$

then

$$
\sum_{n=1}^{\infty} T_n x^{n-1} \frac{1}{1-x-x^2-x^3} \iff \sum_{n=1}^{\infty} T_n x^n = \frac{x}{1-x-x^2-x^3}
$$

and therefore, for $x = \dfrac{1}{\pi} < 3$, we obtain

$$
\sum_{n=1}^{\infty} \frac{T_n}{\pi^n} = \frac{\frac{1}{\pi}}{1-\frac{1}{\pi}-\frac{1}{\pi^2}-\frac{1}{\pi^3}} = \frac{\pi^2}{\pi^3-\pi^2-\pi-1}.
$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We first claim that $1 \le T_n \le 2^{n-1}$ for $n \ge 1$. Indeed this is true for $n = 1, 2,$ and 3 and

$1 \le T_n = T_{n-1}+T_{n-2}+T_{n-3} \le 2^{n-2}+2^{n-3}+2^{n-4} < 2^{n-2}+2^{n-3}+2^{n-3} = 2^{n-1}$, as claimed.

So, $S = \displaystyle\sum_{n=1}^{\infty} \frac{T_n}{\pi^n}$ is convergent and

$$
\begin{aligned}
S &= \sum_{n=1}^{\infty} \frac{T_n}{\pi^n} = \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{2}{\pi^3} + \sum_{n=1}^{\infty} \frac{T_{n-1}+T_{n-2}+T_{n-3}}{\pi^n} \\
&= \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{2}{\pi^3} + \frac{1}{\pi}\sum_{n=3}^{\infty} \frac{T_n}{\pi^n} + \frac{1}{\pi^2}\sum_{n=2}^{\infty} \frac{T_n}{\pi^n} + \frac{1}{\pi^3}\sum_{n=1}^{\infty} \frac{T_n}{\pi^n} \\
&= \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{2}{\pi^3} + \frac{1}{\pi}\left(S - \frac{1}{\pi} - \frac{1}{\pi^2}\right) + \frac{1}{\pi^2}\left(S - \frac{1}{\pi}\right) + \frac{1}{\pi^3}S \\
&= \frac{1}{\pi} + S\left(\frac{1}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi^3}\right). \quad \text{So,} \\
S &= \frac{\pi^2}{\pi^3-\pi^2-\pi-1}
\end{aligned}
$$

Solution 3 by Adrian Naco, Polytechnic University, Tirana, Albania

Let us pose, $a_n = \dfrac{T_n}{\pi^n}$, $T_0 = 0$. We prove by induction that, $\;T_n \le T_{n+1} \le 2T_n$.

$$
T_n \le T_{n+1} = T_n + T_{n-1} + T_{n-2} \le 2T_{n-1} + 2T_{n-2} + 2T_{n-3} = 2T_n.
$$

Thus, it implies that,

$$
\forall n \in N: \qquad \frac{1}{\pi}a_n \le a_{n+1} = \frac{T_{n+1}}{\pi^{n+1}} = \frac{1}{\pi}\cdot\frac{T_{n+1}}{T_n}\cdot\frac{T_n}{\pi^n} \le \frac{2}{\pi}a_n,
$$

and by induction it results that

$$\left(\frac{1}{\pi}\right)^n = \left(\frac{1}{\pi}\right)^n a_1 \leq a_{n+1} \leq \left(\frac{1}{\pi}\right)^n a_1 = \left(\frac{2}{\pi}\right)^n.$$

Thus, the given series converges, and

$$\frac{1}{\pi-1} = \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\right)^n \leq \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{T_n}{\pi^n} \leq \sum_{n=1}^{\infty}\left(\frac{2}{\pi}\right)^n = \frac{1}{\pi-2}.$$

Considering the given difference equation for T_n we transform it to a difference equation for a_n

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \quad \Leftrightarrow \quad \frac{T_n}{\pi^n} = \frac{1}{\pi}\cdot\frac{T_{n-1}}{\pi^{n-1}} + \frac{1}{\pi^2}\cdot\frac{T_{n-2}}{\pi^{n-2}} + \frac{1}{\pi^3}\cdot\frac{T_{n-3}}{\pi^{n-3}}$$

$$\Leftrightarrow \quad a_n = \frac{1}{\pi}\cdot a_{n-1} + \frac{1}{\pi^2}\cdot a_{n-2} + \frac{1}{\pi^3}\cdot a_{n-3}.$$

The respective characteristic equation is the following one, the left side of which is a nonnegative polynomial,

$$p(\lambda) = 0 \quad \Leftrightarrow \quad \lambda^3 - \frac{1}{\pi}\cdot\lambda^2 - \frac{1}{\pi^2}\cdot\lambda - \frac{1}{\pi^3} = 0.$$

Studying its derivative, $p'(\lambda) = 3(\lambda + \frac{1}{3})(\lambda - 1)$, we come to the conclusion that the characteristic polynomial has a unique positive real root, $\alpha \in (0;1)$, and two complex conjugate roots, $\beta, \gamma \in C$.

Recall the Theorem for the dominance of the unique positive root of a nonnegative polynomial that states:

Theorem. *If λ_0 is a positive root of a nonnegative polynomial $p(x)$, then λ_0 is a dominant root, in the sense that any other root $\lambda \in C$ satisfies the relation $|\lambda| \leq \lambda_0$.*
Thus, $0 < |\beta| = |\gamma| < \alpha < 1$.

The general structure of the term a_n is,

$$\forall n = 0,1,2,: \quad a_n = c_1\cdot\alpha^n + c_2\cdot\beta^n + c_3\cdot\gamma^n, \quad \textit{where} \quad c_1, c_2, c_3 \in C.$$

To define the constants we consider the initial conditions,

$$\begin{aligned} a_0 = \ 0 \ &= c_1\cdot\alpha^0 + c_2\cdot\beta^0 + c_3\cdot\gamma^0 \\ a_1 = \frac{1}{\pi} &= c_1\cdot\alpha^1 + c_2\cdot\beta^1 + c_3\cdot\gamma^1 \\ a_2 = \frac{1}{\pi^2} &= c_1\cdot\alpha^2 + c_2\cdot\beta^2 + c_3\cdot\gamma^2 \end{aligned}$$

And these imply:

$$c_1 = \frac{(\beta-\gamma)(\beta+\gamma-\frac{1}{\pi})}{\pi(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)},\ c_2 = \frac{(\gamma-\alpha)(\gamma+\alpha-\frac{1}{\pi})}{\pi(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)},\ c_3 = \frac{(\alpha-\beta)(\alpha+\beta-\frac{1}{\pi})}{\pi(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}$$

Since, $\alpha, \beta, \gamma \in \{z \in C : |z| < 1\} \quad \Rightarrow \quad \lim_{n\to\infty} \alpha^n = \lim_{n\to\infty} \beta^n = \lim_{n\to\infty} \gamma^n = 0$

Doing some simple operations and based on Vieta's formulas

$$\alpha\beta\gamma = \frac{1}{\pi^3}, \qquad \alpha\beta + \beta\gamma + \gamma\alpha = -\frac{1}{\pi^2}, \qquad \alpha + \beta + \gamma = \frac{1}{\pi^3}$$

implies that

$$\begin{aligned}
\Rightarrow \quad \sum_{n=0}^{\infty} a_n &= \sum_{n=0}^{\infty} (c_1 \cdot \alpha^n + c_2 \cdot \beta^n + c_3 \cdot \gamma^n) = c_1 \cdot \sum_{n=0}^{\infty} \alpha^n + c_2 \cdot \sum_{n=0}^{\infty} \beta^n + c_3 \cdot \sum_{n=0}^{\infty} \gamma^n \\
&= \frac{c_1}{1-\alpha} + \frac{c_2}{1-\beta} + \frac{c_3}{1-\gamma} = \frac{c_1}{1-\alpha} + \frac{c_2}{1-\beta} + \frac{c_3}{1-\gamma} \\
&= \frac{1}{\pi(1-\alpha)(1-\beta)(1-\gamma)} = \frac{1}{\pi\left(1 - \frac{1}{\pi} - \frac{1}{\pi^2} - \frac{1}{\pi^3}\right)} \\
&= \frac{\pi^2}{\pi^3 - \pi^2 - \pi - 1}
\end{aligned}$$

since $p(1) = (1-\alpha)(1-\beta)(1-\gamma) = 1 - \frac{1}{\pi} - \frac{1}{\pi^2} - \frac{1}{\pi^3}$ is the value of the characteristic polynomial for $\lambda = 1$.

Comment: Let us prove that $a_n = (c_1 \cdot \alpha^n + c_2 \cdot \beta^n + c_3 \cdot \gamma^n) \in R$, even if c_1, c_2, c_3 are complex constants.

The first term $c_1\alpha^n$ is a real number since $c_1 \in R$ and $\alpha \in R$. Indeed,

$$c_1 = \frac{(\beta-\gamma)(\beta+\gamma-\frac{1}{\pi})}{\pi(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)} = \frac{-\alpha}{\pi(\alpha-\beta)(\gamma-\alpha)} = \frac{-\alpha}{\pi[\alpha(\beta+\gamma) - \alpha^2 - \beta\gamma]} \in R$$

since $\frac{\alpha}{\pi} \in R$, $(\beta+\gamma) = 2Re\beta \in R$ and $\beta\gamma = |\beta|^2 \in R$.

To prove that the summation of the other two terms in the expression for a_n is a real number, we need to prove by induction in n that $\forall n \in N$,$\frac{(\beta^n - \gamma^n)}{(\beta-\gamma)} \in R$.

Indeed, supposing that the given expression is a real number $\forall\, k < n$. Then

$$\begin{aligned}
\frac{(\beta^n - \gamma^n)}{(\beta-\gamma)} &= \frac{(\beta^{n-1} - \gamma^{n-1})(\beta+\gamma) - \beta\gamma(\beta^{n-2} - \gamma^{n-2})}{(\beta-\gamma)} \\
&= (\beta+\gamma)\frac{(\beta^{n-1} - \gamma^{n-1})}{(\beta-\gamma)} - \beta\gamma\frac{(\beta^{n-2} - \gamma^{n-2})}{(\beta-\gamma)} \in R \text{ since} \\
(\beta+\gamma) &= 2Re\beta \in R\ , \quad \beta\gamma = |\beta|^2 \in R.
\end{aligned}$$

Thus,

$$c_2 \cdot \beta^n + c_3 \cdot \gamma^n = \frac{(\gamma-\alpha)(\gamma+\alpha-\frac{1}{\pi})}{\pi(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)} \cdot \beta^n + \frac{(\alpha-\beta)(\alpha+\beta-\frac{1}{\pi})}{\pi(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)} \cdot \gamma^n$$

$$= \frac{(\gamma-\alpha)(-\beta)\beta^n + (\alpha-\beta)(-\gamma)\gamma^n}{\pi(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}$$

$$= \frac{\alpha(\beta^{n+1}-\gamma^{n+1}) - \beta\gamma(\beta^{n-1}-\gamma^{n-1})}{\pi(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}$$

$$= \frac{1}{\pi[\alpha(\beta+\gamma)-\alpha^2-\beta\gamma)]}\left[\alpha\frac{(\beta^{n+1}-\gamma^{n+1})}{(\beta-\gamma)} - \beta\gamma\frac{(\beta^{n-1}-\gamma^{n-1})}{(\beta-\gamma)}\right] \in R$$

$\forall n \in N,\ \dfrac{(\beta^n-\gamma^n)}{(\beta-\gamma)} \in R\ ,\quad \alpha \in R,\quad (\beta+\gamma) = 2Re\beta \in R\ ,\quad \beta\gamma = |\beta|^2 \in R.$

Editor's Comment: **David Stone and John Hawkins of Georgia Southern University, Statesboro, GA** noted in their solution that the π in the statement of the problem is simply a stand in. They found the characteristic equation for the linear recurrence to be $p(x) = x^3 - x^2 - x - 1$. Letting $z, \overline{z}$, and r be the roots of the characteristic polynomial they observed that $\sum_{n=0}^{\infty} \frac{T_n}{\pi^n} = \sum_{n=0}^{\infty} \frac{k_1 z^n + k_2 (\overline{z})^n + k_3 r^n}{\pi^n}$ is the sum of three geometric series, each of which must necessarily converge. They then found the values of $z, \overline{z}$, and r.

$$\begin{aligned} p(x) &= x^3 - x^2 - x - 1, \text{ and also} \\ &= (x-z)(x-\overline{z})(x-r) \\ &= x^3 - (z+\overline{z}+r)x^2 - (z\overline{z}+zr+\overline{z}r)x - z\overline{z}r, \\ &\quad \text{and by equating coefficients} \\ z+\overline{z} &= 1-r \text{ and} \\ |z|^2 &= z\overline{z} = \frac{1}{r}. \end{aligned}$$

Using a calculator they approximated $r \approx 1.87$ so $|z| = |\overline{z}| \approx 0.54$. They went on to say that they could have solved the characteristic equation with Cardan's formula, but all they needed to know about the roots is that each, in absolute value, is smaller than π, which they just saw; so that the three geometric series in $\sum_{n=0}^{\infty} \frac{T_n}{\pi^n}$ converge. By Cardan's formula, the root r equals $\frac{1}{3} - \frac{C}{3} - \frac{4}{3C}$ where $C = \sqrt[3]{3\sqrt{33}-19}$. They calculated $r \approx 1.839286755$.

They then noted that if t is *any* real constant larger than r, the same calculations hold, thus showing

$$\sum_{n=0}^{\infty} \frac{T_n}{t_n} = \frac{t^2}{p(t)} = \frac{t^2}{t^3 - t^2 - t - 1}.$$

For instance, $\sum_{n=0}^{\infty} \frac{T_n}{2^n} = \frac{2^2}{p(t)} = \frac{4}{1}.$

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Bruno Salgueiro Fanego Viveiro, Spain; Michael N. Fried, Ben-Gurion University, Beer

Sheva, Israel; Noel Evens, Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo TX; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Enkel Hysnelaj, University of Technology, Sydney, Australia together with Elton Bojaxhiu, Kriftel, Germany; Anastasios Kotronis, Athens, Greece; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins of Georgia Southern University, Statesboro, GA and the proposer.

- **5225:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  Find infinitely many integer squares x that are each the sum of a square and a cube and a fourth power of positive integers $a, b, c.$ That is, $x = a^2 + b^3 + c^4$.

Solution 1 by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX

By observation, we conclude that for $n \geq 1$,

$$\begin{aligned}(2n^3)^4 + (2n^2)^3 + 1^2 &= 16n^{12} + 8n^6 + 1 \\ &= (4n^6 + 1)^2.\end{aligned}$$

Also, it can be observed for $n \geq 1$,

$$\begin{aligned}1^4 + (2n^2)^3 + (4n^6)^2 &= 1 + 8n^6 + 16n^{12} \\ &= (4n^6 + 1)^2.\end{aligned}$$

Thus, for $n \geq 1$, $x^2 = (4n^6 + 1)^2$ generates infinitely many integer squares such that $x^2 = a^2 + b^3 + c^4$ where a, b, c are positive integers

Solution 2 by Ángel Plaza, University of Las Palmas de gran, Canaria, Spain

Since $(a + c^2)^2 = a^2 + c^4 + 2ac^2$ it is enough to consider $b = 2a = c$ to obtain infinitely many integer squares $x = (a + c^2)^2 = a^2 + c^4 + 2ac^2 = a^2 + b^3 + c^4$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

Let m and n be any positive integers. Using the identity

$$\left(4m^3 + 4n^3 + n\right)^2 = \left(4m^3 - 4n^3 + n\right)^2 + (4mn)^3 + (2n)^4,$$

we find infinitely many such x.

Solution 4 by David E. Manes, SUNY College at Oneonta, Oneonta, NY

For each positive integer n, let $a = 2^{3n-2} + 3,\ b = 2^n$, and $c = 2.$ Then

$$\begin{aligned}a^2 + b^3 + c^4 &= \left(2^{3n-2} + 3\right)^2 + 2^{3n} + 2^4 \\ &= 2^{6n-4} + 10 \cdot 2^{3n-2} + 25 \\ &= \left(2^{3n-2} + 5\right)^2 = x.\end{aligned}$$

Note that if $b = 2^n, c = 2$ and $x = y^2$, then $y^2 = a^2 + 2^{3n} + 2^4$. Therefore,

$$y^2 - a^2 = 2^{3n} + 2^4 \text{ or } (y+a)(y-a) = 2(2^{3n-1} + 2^3).$$

Let

$$\begin{cases} y + a = 2^{3n-1} + 2^3 & \text{and} \\ y - a = 2 & . \end{cases}$$

The simultaneous solution for this system of equations is $y = 2^{3n-2} + 5$ and $a = 2^{3n-2} + 3$.

Accordingly, the infinitely many integer squares $x = a^2 + b^3 + c^4$ are $x = \left(2^{3n-2} + 5\right)^2$ for each positive integer n.

Solution 5 by Ken Korbin, New York, NY

There are infinitely many pairs of positive integers b and c such that $b + c$ is odd. If $a = \dfrac{b^3 + c^4 - 1}{2}$ then $a^2 + b^3 + c^4 = (a+1)^2 = x$. Examples:

a	b	c	$x = (a+1)^2$
316	2	5	$(317)^2$
70	5	2	$(71)^2$
128	1	4	$(129)^2$
72	4	3	$(73)^2$

If a, b, and c are positive integers such that $a^2 + b^3 + c^4 = (a+1)^2$ and if k is a positive integer then

$$\begin{aligned} a^2 \cdot k^{12} + b^3 \cdot k^{12} + c^4 \cdot k^{12} &= (a+1)^2 \cdot k^{12} \\ &= \left(a \cdot k^6\right)^2 + \left(b \cdot k^4\right)^3 + \left(c \cdot k^3\right)^4 \\ &= \left((a+1) \cdot k^6\right)^2 = x. \end{aligned}$$

Solution 6 by Brian D. Beasley, Presbyterian College, Clinton, SC

In order to have $x = k^2 = a^2 + b^3 + c^4$ for positive integers k, a, b, and c, we need $b^3 + c^4$ to be expressible as the difference of two squares. As Burton notes (*Elementary Number Theory* , 7th ed., Theorem 13-4, p. 269), a positive integer n has such an expression if and only if n is not congruent to 2 modulo 4. Thus as long as $b^3 + c^4$ is not congruent to 2 modulo 4, we may solve for k and a.

In particular, when $b^3 + c^4$ is odd, we may take $a = (b^3 + c^4 - 1)/2$ and $k = a + 1$, as seen in the following two cases:

One infinite set of solutions occurs when $c = 1$ and $b = 2n$ for any positive integer n, which makes $b^3 + c^4 = 8n^3 + 1$ odd. We then take $a = 4n^3$ to produce $k = 4n^3 + 1$ and hence $x = (4n^3 + 1)^2 = 16n^6 + 8n^3 + 1$.

Another infinite set of solutions occurs when $b = 1$ and $c = 2n$ for any positive integer n, which makes $b^3 + c^4 = 16n^4 + 1$ odd. We then take $a = 8n^4$ to produce $k = 8n^4 + 1$ and hence $x = (8n^4 + 1)^2 = 64n^8 + 16n^4 + 1$.

Also solved by Farideh Firoozbakht and Jahangeer Kholdi University of Isfahan, Khansar, Iran; Enkel Hysnelaj, University of Technology, Sydney, Australia together with Elton Bojaxhiu, Kriftel, Germany; Paul M. Harms, North Newton, KS; Charles McCracken, Dayton, OH; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Southern Georgia University, Statesboro, GA, and the proposer.

- **5226:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

If a and b, $a < b$ are real-valued positive numbers, then calculate:

$$\int_a^b \frac{\sqrt[n]{x-a}\left(1+\sqrt[n]{b-x}\right)}{\sqrt[n]{x-a}+2\sqrt[n]{-x^2+(a+b)x-ab}+\sqrt[n]{b-x}}\,dx,$$

where n is a positive integer greater than one, $(n > 1)$.

Solution 1 by Adrian Naco, Polytechnic University, Tirana, Albania

Let

$$\begin{aligned}
I_1 &= \int_a^b \frac{\sqrt[n]{x-a}(1+\sqrt[n]{b-x})}{\sqrt[n]{x-a}+2\sqrt[n]{x-a}\sqrt[n]{b-x}+\sqrt[n]{b-x}}dx \text{ and} \\
I_2 &= \int_a^b \frac{\sqrt[n]{b-x}(1+\sqrt[n]{x-a})}{\sqrt[n]{x-a}+2\sqrt[n]{x-a}\sqrt[n]{b-x}+\sqrt[n]{b-x}}dx.
\end{aligned}$$

Setting $y = b + a - x$, we have

$$\begin{aligned}
I_1 &= \int_a^b \frac{\sqrt[n]{x-a}(1+\sqrt[n]{b-x})}{\sqrt[n]{x-a}+2\sqrt[n]{x-a}\sqrt[n]{b-x}+\sqrt[n]{b-x}}dx \\
&= \int_b^a \frac{\sqrt[n]{b-y}(1+\sqrt[n]{y-a})}{\sqrt[n]{y-a}+2\sqrt[n]{y-a}\sqrt[n]{b-y}+\sqrt[n]{b-y}}d(b+a-y) \\
&= \int_a^b \frac{\sqrt[n]{b-y}(1+\sqrt[n]{y-a})}{\sqrt[n]{y-a}+2\sqrt[n]{y-a}\sqrt[n]{b-y}+\sqrt[n]{b-y}}dy = I_2
\end{aligned}$$

So,

$$I_1 + I_2 = \int_a^b dx = b - a, \text{ and therefore,}$$

$$\int_a^b \frac{\sqrt[n]{x-a}(1+\sqrt[n]{b-x})}{\sqrt[n]{x-a}+2\sqrt[n]{x-a}\sqrt[n]{b-x}+\sqrt[n]{b-x}}dx = \frac{b-a}{2}$$

Solution 2 by Anastasios Kotronis, Athens, Greece

$$\int_a^b \frac{\sqrt[n]{x-a}(1+\sqrt[n]{b-x})}{\sqrt[n]{x-a}+2\sqrt[n]{-x^2+(a+b)x-ab}+\sqrt[n]{b-x}}\,dx; \text{ letting } x = y+\frac{a+b}{2}, \text{we obtain}$$

$$\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \frac{\sqrt[n]{y+\frac{b-a}{2}}\left(1+\sqrt[n]{\frac{b-a}{2}-y}\right)}{\sqrt[n]{y+\frac{b-a}{2}}+2\sqrt[n]{\left(y+\frac{b-a}{2}\right)\left(\frac{b-a}{2}-y\right)}+\sqrt[n]{\frac{b-a}{2}-y}}-\frac{1}{2}+\frac{1}{2}\,dy$$

$$= \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} g(y)+\frac{1}{2}\,dy$$

$$= \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} g(y)\,dy+\frac{b-a}{2}.$$

Now it is easy to see that $g(y)$ is odd so the given integral equals $\dfrac{b-a}{2}$.

Solution 3 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

Answer: $\dfrac{b-a}{2}$

Proof: The integral is actually

$$\int_a^b \frac{\sqrt[n]{x-a}(1+\sqrt[n]{b-x})}{\left(\sqrt[n]{x-a}+\sqrt[n]{b-x}\right)^2}dx = \int_a^b \frac{1}{1+\sqrt[n]{\frac{b-x}{x-a}}}dx$$

Setting $t=(b-x)/(x-a)$ we get

$$(b-a)\int_0^\infty \frac{1}{(1+t)^2}\frac{1}{1+t^{1/n}}dt$$

The further change $t=y^n$ yields

$$(b-a)\int_0^\infty \frac{1}{(1+y^n)^2}\frac{1}{1+y}ny^{n-1}dy$$

Integrating by parts

$$(b-a)\frac{1}{1+y}\frac{1}{1+y^n}\Big|_\infty^0-\int_0^\infty \frac{b-a}{(1+y)^2}\frac{1}{1+y^n}dy = b-a-\int_0^\infty \frac{b-a}{(1+y)^2}\frac{1}{1+y^n}dy.$$

To compute the last integral we set $y=1/z$ and obtain

$$\int_0^\infty \frac{1}{(1+y)^2}\frac{1}{1+y^n}dy = \int_0^\infty \frac{z^2}{(1+z)^2}\frac{z^n}{1+z^n}\frac{1}{z^2}dz = \int_0^\infty \frac{1}{(1+z)^2}\frac{z^n}{1+z^n}dz =$$

$$= \int_0^\infty \frac{1}{(1+z)^2}dz-\int_0^\infty \frac{1}{(1+z)^2}\frac{1}{1+z^n}dz$$

that is,

$$\int_0^\infty \frac{1}{(1+y)^2}\frac{1}{1+y^n}dy = \frac{1}{2}\int_0^\infty \frac{1}{(1+z)^2}dz = \frac{1}{2}.$$

The final result is $\dfrac{1}{2}(b-a.)$

Also solved by Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5227:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

Compute

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{(n+1)+\sqrt{nk}}{n+\sqrt{nk}}\right).$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

Since $\ln(1+x) = x + O\left(x^2\right)$ as $x \longrightarrow 0$, so

$$\sum_{k=1}^{n}\ln\left(1+\frac{1}{n+\sqrt{nk}}\right) = \sum_{k=1}^{n}\frac{1}{n+\sqrt{nk}} + O\left(\frac{1}{n}\right).$$

Hence,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(1+\frac{1}{n+\sqrt{nk}}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\frac{1}{\left(1+\sqrt{\frac{k}{n}}\right)} = \int_0^1 \frac{dx}{1+\sqrt{x}}.$$

By the substitution $x = y^2$, we easily evaluate the last integral to be $2(1-\ln 2)$.

Now by exponentiation, we find the limit of the problem to be $\dfrac{e^2}{4}$.

Solution 2 by Arkady Alt, San Jose, CA

First note that for any positive real x we have

$$e^x\left(1-\frac{x^2}{2}\right) < 1+x < e^x. \qquad (1)$$

Indeed, for any positive x we can obtain from the Taylor representation of e^x that:

$$\begin{aligned} 1+x < e^x &= 1+x+\frac{x^2}{2!}+\sum_{n=1}^{\infty}\frac{x^{n+2}}{(n+2)!} \\ &= 1+x+\frac{x^2}{2}\left(1+\sum_{n=1}^{\infty}\frac{2x^n}{(n+2)!}\right) \end{aligned}$$

$$< \quad 1+x+\frac{x^2}{2}\left(1+\sum_{n=1}^{\infty}\frac{x^n}{n!}\right)$$

$$= \quad 1+x+\frac{x^2e^x}{2} \text{ and then we have}$$

$$e^x < 1+x+\frac{x^2e^x}{2} \iff e^x\left(1-\frac{x^2}{2}\right) < 1+x.$$

Applying inequality (1) to $x=\dfrac{1}{n+\sqrt{nk}}, k=1,2,...,n$ we obtain

$$e^{a_{kn}}b_{kn} < 1+\frac{1}{n+\sqrt{nk}} < e^{a_{kn}}, k=1,2,...,n, \qquad (2)$$

where $a_{kn}=\dfrac{1}{n+\sqrt{nk}}$ and $b_{kn}=1-\dfrac{1}{2\left(n+\sqrt{nk}\right)^2}$.

Let $S_n=\sum_{k=1}^{n}a_{kn}$. Hence,

$$e^{S_n}\prod_{k=1}^{n}b_{kn} < \prod_{k=1}^{n}\left(\frac{(n+1)+\sqrt{nk}}{n+\sqrt{nk}}\right) < e^{S_n}.$$

Note that $\lim\limits_{n\to\infty}\prod\limits_{k=1}^{n}b_{kn}=1$. Indeed, since $n<n+\sqrt{nk}<2n, k=1,2,...,n$ then

$$1-\frac{1}{2n^2} < 1-\frac{1}{2\left(n+\sqrt{nk}\right)^2} < 1-\frac{1}{8n^2}, k=1,2,...,n$$

and we obtain

$$\left(1-\frac{1}{2n^2}\right)^n < \prod_{k=1}^{n}b_{kn} < \left(1-\frac{1}{8n^2}\right)^n < 1.$$

Since

$$\lim_{n\to\infty}\left(1-\frac{1}{2n^2}\right)^{n^2}=\frac{1}{\sqrt{e}} \text{ then}$$

$$\lim_{n\to\infty}\left(1-\frac{1}{2n^2}\right)^{n}=\lim_{n\to\infty}\sqrt[n]{\left(1-\frac{1}{2n^2}\right)^{n^2}}=1.$$

Since

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\sqrt{\frac{k}{n}}}$$

$$= \int_0^1\frac{1}{1+\sqrt{x}}dx=\left[x=t^2; dx=2tdt\right]$$

$$= \quad 2\int_0^1 \frac{t}{1+t}dt = 2\left(t - \ln\left(1+t\right)\right)|_0^1 = 2\left(1 - \ln 2\right), \text{ then}$$

$$\lim_{n\to\infty} e^{S_n} = \lim_{n\to\infty} e^{S_n} \prod_{k=1}^{n} b_{kn} = e^{2(1-\ln 2)} = \frac{e^2}{4}.$$

By the Squeeze Principle we see that

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(\frac{(n+1)+\sqrt{nk}}{n+\sqrt{nk}}\right) = \frac{e^2}{4}.$$

Solution 3: by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

The proposed limit may be written as $L = \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{\frac{1}{n}}{1+\sqrt{\frac{k}{n}}}\right)$. So,

$\ln L = \lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(1 + \frac{\frac{1}{n}}{1+\sqrt{\frac{k}{n}}}\right)$. Now we expand each of the logs according to its power series and write this as a double sum. Then we change order of summation and sum up by columns. This is allowed because both directions provide convergent sums. So

$$\ln\left(1 + \frac{\frac{1}{n}}{1+\sqrt{\frac{1}{n}}}\right) = \frac{\frac{1}{n}}{1+\sqrt{\frac{1}{n}}} - \frac{\left(\frac{\frac{1}{n}}{1+\sqrt{\frac{1}{n}}}\right)^2}{2} + \frac{\left(\frac{\frac{1}{n}}{1+\sqrt{\frac{1}{n}}}\right)^3}{3} + \cdots.$$

$$\ln\left(1 + \frac{\frac{1}{n}}{1+\sqrt{\frac{2}{n}}}\right) = \frac{\frac{1}{n}}{1+\sqrt{\frac{2}{n}}} - \frac{\left(\frac{\frac{1}{n}}{1+\sqrt{\frac{2}{n}}}\right)^2}{2} + \frac{\left(\frac{\frac{1}{n}}{1+\sqrt{\frac{2}{n}}}\right)^3}{3} + \cdots.$$

$$\ln\left(1 + \frac{\frac{1}{n}}{1+\sqrt{\frac{3}{n}}}\right) = \frac{\frac{1}{n}}{1+\sqrt{\frac{3}{n}}} - \frac{\left(\frac{\frac{1}{n}}{1+\sqrt{\frac{3}{n}}}\right)^2}{2} + \frac{\left(\frac{\frac{1}{n}}{1+\sqrt{\frac{3}{n}}}\right)^3}{3} + \cdots.$$

Note that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\frac{1}{n}}{1+\sqrt{\frac{k}{n}}} = \int_0^1 \frac{1}{1+\sqrt{x}}dx = \ln\left(\frac{e^2}{4}\right),$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\left(\frac{\frac{1}{n}}{1+\sqrt{\frac{k}{n}}}\right)^m}{m} = 0, \text{ for } m > 1.$$

From where $\ln L = \ln\left(\frac{e^2}{4}\right)$, and therefore $L = \frac{e^2}{4}$.

Also solved by Bruno Salgueiro Fanego Viveiro, Spain; Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel,

German; Anastasios Kotronis, Athens, Greece; Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland;

- **5228:** *Proposed by Mohsen Soltanifar, University of Saskatchewan, Saskatoon, Canada*

Given a random variable X with non-negative integer values. Assume the n^{th} moment of X is given by

$$E\left(X^n\right) = \sum_{k=1}^{\infty} f_n(k) P\left(X \geq k\right) \qquad n = 1, 2, 3, \cdots,$$

where f_n is a non-negative function defined on N. Find a closed formula for f_n.

Solution by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany.

From the first principle we have

$$E(X^n) = \sum_{k=1}^{\infty} k^n P(X = k)$$

Doing easy manipulations we have

$$\begin{aligned}
E(X^n) &= \sum_{k=1}^{\infty} f_n(k) P(X \geq k) \\
&= f_n(1)P(X \geq 1) + f_n(2)P(X \geq 2) + ... + f_n(k)P(X \geq k) + ... \\
&= f_n(1)(P(X = 1) + P(X = 2) + ...) + f_n(2)(P(X = 2) + P(X = 3) + ...) + ... \\
& \quad + f_n(k)(P(X = k) + P(X = k+1) + ...) + ... \\
&= f_n(1)P(X = 1) + (f_n(1) + f_n(2))P(X = 2) + ... \\
& \quad + (f_n(1) + f_n(2) + ... + f_n(k))P(X = k) + ... \\
&= \sum_{k=1}^{\infty} \sum_{i=1}^{k} f_n(i) P(X = k)
\end{aligned}$$

Comparing this with the expression we have from the first principle we have

$$\sum_{i=1}^{k} f_n(i) = k^n$$

for any non-negative integers k and n.

Finally, using the above result implies

$$f_n(k) = \sum_{i=1}^{k} f_n(i) - \sum_{i=1}^{k-1} f_n(i) = k^n - (k-1)^n$$

and this is the end of the proof.

Also solved by Kee-Wai Lau, Hong Kong, China, and the proposer.

- **5229:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $\beta > 0$ be a real number and let $(x_n)_{n\in N}$ be the sequence defined by the recurrence relation

$$x_1 = a > 0,\ x_{n+1} = x_n + \frac{n^{2\beta}}{x_1 + x_2 \cdots + x_n}.$$

1) Prove that $\lim\limits_{n\to\infty} x_n = \infty$.

2) Calculate $\lim\limits_{n\to\infty} \dfrac{x_n}{n^\beta}$.

Solution 1 by Kee-Wai Lau, Hong Kong, China

1) By induction, we have $x_n > 0$ for positive integers n. Hence x_n is strictly increasing. Suppose, on the contrary that, $\lim\limits_{x\to\infty} s_n = L$, where $0 < L < \infty$.

Since $0 < x_1 + x_2 + \ldots + x_n < nL$, so, $x_{n+1} > x_n + \dfrac{n^{2\beta-1}}{L}$.

Hence for any positive integer N, we have $\sum\limits_{n=1}^{N} x_{n+1} > \sum\limits_{n=1}^{N} x_n + \dfrac{1}{L}\sum\limits_{n=1}^{N} n^{2\beta-1}$, so that

$L > x_{N+1} > a + \dfrac{1}{L}\sum\limits_{n=1}^{N} n^{2\beta-1}$. Since $\sum\limits_{n=1}^{N} n^{2\beta-1} \longrightarrow \infty$ as $N \longrightarrow \infty$, this is a contradiction. It follows that $\lim\limits_{n\to\infty} x_n = \infty$.

2) To find the leading behavior of x_n as $n \to \infty$, we try

$$x_n \sim kn^\alpha \qquad (1)$$

for some positive constants k and α. We then have $x_1 + x_2 + \ldots + x_n \sim \dfrac{kn^{\alpha+1}}{\alpha+1}$.

Hence $x_{n+1} - x_n \sim \dfrac{(\alpha+1)n^{2\beta-\alpha-1}}{k}$. If $\alpha > 2\beta$, then x_{n+1} is bounded, which is not true.

If $\alpha = 2\beta$, then $x_{n+1} \sim \dfrac{(\alpha+1)\ln n}{k}$, which is inconsistent with (1) . So

$0 < \alpha < 2\beta$, and we we have

$$x_{n+1} \sim \frac{(\alpha+1)n^{2\beta-\alpha}}{k(2\beta-\alpha)}.$$

By (1) and (2), we see that $\alpha = 2\beta - \alpha$ and $k = \dfrac{\alpha+1}{k(2\beta-\alpha)}$. Hence $\alpha = \beta$ and $k = \sqrt{\dfrac{\beta+1}{\beta}}$. It follows that $\lim\limits_{n\to\infty} \dfrac{s_n}{n^\beta} = \sqrt{\dfrac{\beta+1}{\beta}}$.

Solution 2 by proposer

(1) It is easy to see that $x_n > 0$, for all $n \in N$. Also, $x_{n+1} - x_n = \frac{n^{2\beta}}{x_1+x_2+\cdots+x_n} > 0$, and hence the sequence is strictly increasing. By way of contradiction, we assume that $\lim_{n\to\infty} x_n = l$. We have, since (x_n) increases, that $l \neq 0$ and $x_n < l$ for all $n \in N$. Iterating the recurrence relation we get that

$$x_{n+1} \;=\; x_1 + \frac{1}{x_1} + \frac{2^{2\beta}}{x_1+x_2} + \cdots + \frac{n^{2\beta}}{x_1+x_2+\cdots+x_n} > x_1 + \frac{1}{l} + \frac{2^{2\beta}}{2l} + \cdots + \frac{n^{2\beta}}{nl}$$

$$= \quad x_1 + \frac{1}{l}\left(1 + 2^{2\beta-1} + \cdots + n^{2\beta-1}\right).$$

Passing to the limit in the preceding inequality we get that $l \geq \infty$, which is a contradiction.
2) The limit equals $\sqrt{(\beta+1)/\beta}$. We apply Cesaro-Stolz Lemma and we have that

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{x_n}{n^\beta} = \lim_{n\to\infty} \frac{x_{n+1}-x_n}{(n+1)^\beta - n^\beta} = \lim_{n\to\infty} \frac{\frac{n^{2\beta}}{x_1+x_2+\cdots+x_n}}{(n+1)^\beta - n^\beta} \\
&= \lim_{n\to\infty}\left(\frac{n^{\beta+1}}{x_1+x_2+\cdots+x_n}\cdot\frac{n^{\beta-1}}{(n+1)^\beta-n^\beta}\right) \\
&= \frac{1}{\beta}\cdot\lim_{n\to\infty}\left(\frac{n^{\beta+1}}{x_1+x_2+\cdots+x_n}\right) \\
&\quad Cesaro-Stolz\ again = \frac{1}{\beta}\cdot\lim_{n\to\infty}\frac{(n+1)^{\beta+1}-n^{\beta+1}}{x_{n+1}} \\
&= \frac{1}{\beta}\lim_{n\to\infty}\left(\frac{(n+1)^\beta}{x_{n+1}}\cdot\frac{(n+1)^{\beta+1}-n^{\beta+1}}{(n+1)^\beta}\right) \\
&= \frac{(\beta+1)}{\beta\cdot L}.
\end{aligned}$$

Thus, $L = \sqrt{(\beta+1)/\beta}$ and the problem is solved.

Solution 3: by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

1) Since $x_1 = 1 > 0$ it is easy to see that sequence $\{x_n\}_{n\in N}$ is increasing and also that

$$\begin{aligned}
x_{n+1} &= x_1 + \frac{1}{x_1} + \frac{2^{2\beta}}{x_1+x_2} + \cdots + \frac{n^{2\beta}}{x_1+x_2+\cdots+x_n} \\
&> x_1 + \frac{1}{x_n} + \frac{2^{2\beta}}{2x_n} + \cdots + \frac{n^{2\beta}}{nx_n} \\
&= x_1 + \frac{1}{x_n}H_n
\end{aligned}$$

where, $H_n = 1 + 2^{2\beta-1} + \cdots + n^{2\beta-1}$. Since $\{x_n\}_{n\in N}$ is increasing, then either $\{x_n\}_{n\in N}$ is convergent if bounded, or $\lim_{n\to\infty} x_n = \infty$.
Now, since $\lim_{n\to\infty} H_n = \infty$, the hypothesis of $\{x_n\}_{n\in N}$ convergent gives a contradiction with the fact that $x_1 + \frac{1}{x_n}H_n < x_{n+1}$. Therefore $\lim_{n\to\infty} x_n = \infty$.

2. Note that since $x_{n+1} = x_1 + \dfrac{1}{x_1} + \dfrac{2^{2\beta}}{x_1+x_2} + \cdots + \dfrac{n^{2\beta}}{x_1+x_2+\cdots+x_n}$, then, by Stolz-Cezaro criteria

$$L = \lim_{n\to\infty}\frac{x_{n+1}}{(n+1)^\beta} = \lim_{n\to\infty}\frac{\frac{n^{2\beta}}{x_1+x_2+\cdots+x_n}}{(n+1)^\beta - n^\beta} = \lim_{n\to\infty}\frac{\frac{1}{\beta}n^{\beta+1}}{x_1+x_2+\cdots+x_n}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{1}{\beta} \frac{n^{\beta+1} - (n-1)^{\beta+1}}{x_n} = \lim_{n\to\infty} \frac{1}{\beta} \cdot \frac{(\beta+1)n^{\beta}}{x_n} \\
&= \frac{\beta+1}{\beta} \cdot \frac{1}{L},
\end{aligned}
$$

from where $L = \sqrt{\frac{\beta+1}{\beta}}$.

Also solved by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

Notes and Comments

From Charles McCracken of Dayton, OH:

In their solution to Problem 5213 David Stone and John Hawkins note that n^4 is always the sum of two triangular numbers. But n^2 is also the sum of two (consecutive) triangular numbers:

$$
\begin{aligned}
T_n + T_{n+1} &= \frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2} \\
&= \frac{n^2 + n + n^2 + 3n + 2}{2} = \frac{2n^2 + 4n + 2}{2} \\
&= n^2 + 2n + 1 = (n+1)^2.
\end{aligned}
$$

Thus, adding the triangular numbers in sequential pairs generates all the squares; which generates all the fourth powers.

Mea Culpa

The names of **Brian D. Beasley of Presbyterian College in Clinton, SC and of Arkady Alt of San Jose, CA** were inadvertently left off the list of having solved problem 5218. Arkady also solved 5220 and 5221, and I missed listing his name for those too. To Brian and Arkardy, mea culpa, sorry.

Additionally, **David Stone and John Hawkins of Georgia Southern University in Statesboro, GA** should receive credit for having solved 5215. I am happy to report that this time the "senior moment" is theirs and not mine; they forgot to send me their solution!