Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
March 15, 2015

- **5331:** *Proposed by Kenneth Korbin, New York, NY*

  Given equilateral $\triangle ABC$ with cevian $\overline{CD}$. Triangle ACD has inradius $3N+3$ and $\triangle BCD$ has inradius N^2+3N where N is a positive integer.

  Find lengths $\overline{AD}$ and $\overline{BD}$.

- **5332:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  Inspired by the prime number 1000000000000066600000000000001, known as *Belphegor's prime* where there are thirteen consecutive zeros to the left and right of 666, we consider the numbers $100\ldots0201500\ldots01$ where there are $k-$zeros left and right of 2015. For $k<28$ only $k=9$ and $k=27$ yield prime numbers.

  (a) Prove that the sequence $120151,\ 10201501,\ 1002015001, \ldots$ has an infinite subsequence of all composite numbers.

  (b) Find the next prime in both the sequences $100\ldots066600\ldots01$ and $100\ldots0201500\ldots01$, after the ones noted above.

- **5333:** *Proposed by Paolo Perfetti, Department of Mathematics, Tor Vergata Roma University, Rome, Italy*

  Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\left(\ln\left(1+\tan x+\tan^2 x\right)\right)^2}{1+\sin x\cos x}dx.$

- **5334:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $x_{ij},\ (1\le i\le m, 1\le j\le n)$ be nonnegative real numbers. Prove that

$$\prod_{j=1}^{n}\left(1-\prod_{i=1}^{m}\frac{\sqrt{x_{ij}}}{1+\sqrt{x_{ij}}}\right)+\prod_{i=1}^{m}\left(1-\prod_{j=1}^{n}\frac{1}{1+\sqrt{x_{ij}}}\right)\ge 1.$$

- **5335:** *Proposed by Arkady Alt, San Jose, CA*

Prove that for any real $p > 1$ and $x > 1$ that

$$\frac{\ln x}{\ln(x+p)} \le \left(\frac{\ln(x+p-1)}{\ln(x+p)}\right)^p.$$

- **5336:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Caculate:

$$\sum_{k=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}-\ln\left(k+\frac{1}{2}\right)-\gamma\right).$$

Solutions

- **5313:** *Proposed by Kenneth Korbin, New York, NY*

  Find the sides of two different isosceles triangles if they both have perimeter 256 and area 1008.

Solution by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Let $s = \dfrac{2a+b}{2}$ be the semiperimeter of the triangle. By Heron's formula for the area we also have: $A = 1008 = \sqrt{s(s-a)^2(s-b)}$. Solving the system we obtain $(a,b) = (65, 126)$ and $(a,b) = \left(\dfrac{255-\sqrt{253}}{2}, 1+\sqrt{253}\right)$.

Also solved by Corneliu Manescu-Avram, Transportation High School, Ploiesti, Romania; Brian D. Beasley, Presbyterian College, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX; Jerry Chu (student at Saint George's School), Spokane, WA; Bruno Salgueiro Fanego Viveiro, Spain; Ed Gray, Highland Beach, FL; G.C. Greubel, Newport, News, VA; Paul M. Harms, North Newton, KS; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY at Oneonta, Oneonta, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5314:** *Proposed by Roger Izard, Dallas TX*

  A biker and a hiker like to workout together by going back and forth on a road which is ten miles long. One day, at 8 AM, at the starting end of the road, they went out together. The biker soon got far past the hiker, reached the end of the road, reversed his direction, and soon passed by the hiker at 9:06 AM. Then, the biker got down to the beginning part of the road, reversed his direction, and got back to the hiker at 9:24 AM. The biker and the hiker were, then, going in the same direction. Calculate in miles per hour the speeds of the hiker and the biker.

Solution 1 by Jerry Chu (student at Saint George's School), Spokane, WA

Let the speed of the biker be x (mph) and let the speed of the hiker be y (mph).

Because it takes $\frac{11}{10}$ hours (8 am to 9:06 am) to meet we have

$$biker - - - - - - - - - - - - - > - - -\downarrow$$
$$hiker - - - - - - - - - - >< - - - - - -^{\downarrow}$$

$\frac{11(x+y)}{10} = 20$, together they made 20 miles.

And because it takes $\frac{7}{5}$ hours for them to meet again, we have

$$biker - - - - - - - - - - - - - > - - -\downarrow$$
$$\downarrow - - - - - - - - - - < - - - - - - -^{\downarrow}$$
$$^{\downarrow} - - - - - - - - - >$$
$$hiker - - - - - - - - >$$

The difference in the distances they traveled is $\frac{7(x-y)}{5} = 20$. Solving the system of equations

$$\begin{aligned} 11(x+y) &= 200 \\ 7(x-y) &= 100, \end{aligned}$$

we obtain $x = \frac{1250}{77}$mph and $y = \frac{150}{77}$ mph, for the biker and hiker respectively.

Solution 2 by Michael Thew (student at Saint George's School), Spokane WA

We are given that the entire length of the road is 20 miles. At their first meeting, the biker has already hit the ten mile mark and started his way back to the starting line. He passes the hiker (who is still traveling away from the starting line) after a total time of 1.1 hours has elapsed. Let the distance from the 10 mile mark to this meeting point be x. Therefore, the biker has traveled $10 + x$, and the hiker has traveled $10 - x$. Letting h and b be the speed in mph of the hiker and the biker respectively, we have, by the $distance = (rate)(time)$ equation, that $10 - x{=}(h)(1.1)$ and $10 + x = (b)(1.1)$. If we add these two equations and cancel the $x's$, we obtain: $20 = (1.1)(h + b)$.
The two continue moving until they end up meeting again after a total of 1.4 hours has elapsed (from the beginning). Therefore, the biker has finished the 10 mile return to the starting line and has reversed his direction again. The hiker was still traveling in the same direction (away from the starting line). Labeling y as the distance from the starting line to this second meeting point, we obtain $y = h(1.4)$ and $20 + y = b(1.4)$ Subtracting the first equation from the second equation and canceling the $y's$ gives : $20 = 1.4(b - h)$.
Once knowing that

$$20 = 1.1(b + h)20 = 1.4(b - h)$$

we solve for b and h obtaining that $b = 16.234$ mph and $h = 1.948$ mph.

Also solved by Adnan Ali (student at A.E.C.S-4), Mumbai, India; Harold Don Allen, Brossard, Quebec, Canada; Brian D. Beasley, Presbyterian College, Clinton, SC; Michael Brozinsky, Central Islip, NY; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY at Oneonta, Oneonta, NY; Guy Preskill, Butler University, Indianapolis, IN; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Titu Zvonaru, Comănesti, Romania (jointly with) Neculai Stanciu, "George Emil Palade School," Buzău, Romania, and the proposer.

- **5315:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

The hexagonal numbers have the form $H_n = 2n^2 - n,\ n = 1, 2, 3, \ldots$. Prove that infinitely many hexagonal numbers are the sum of two hexagonal numbers.

Solution 1 by Susan Abernathy, Dionne Bailey, Elsie Campbell, Charles Diminnie, and Jesse Taylor (jointly), Angelo State University, San Angelo, TX

Suppose that

$$H_{n+j} = H_n + H_k$$

with $j \geq 1$. Then, we have

$$\begin{aligned} 2\left(n+j\right)^2 - \left(n+j\right) &= 2n^2 - n + 2k^2 - k, \\ 4nj &= 2k^2 - k - 2j^2 + j \\ &= \left[2k + \left(2j-1\right)\right]\left(k-j\right). \end{aligned}$$

One possibility is to let $k - j = 4j$ or $k = 5j$. Then,

$$n = 2\left(5j\right) + 2j - 1 = 12j - 1$$

and

$$n + j = 13j - 1.$$

To check whether these assignments are feasible, note that

$$\begin{aligned} H_{13j-1} &= 2\left(13j-1\right)^2 - \left(13j-1\right) \\ &= 338j^2 - 65j + 3, \\ H_{12j-1} &= 2\left(12j-1\right)^2 - \left(12j-1\right) \\ &= 288j^2 - 60j + 3, \end{aligned}$$

and

$$H_{5j} = 2\left(5j\right)^2 - 5j = 50j^2 - 5j.$$

It is now clear that

$$H_{5j} + H_{12j-1} = 338j^2 - 65j + 3 = H_{13j-1}$$

for all $j \geq 1$. The first five solutions of this type are shown in the following table:

j	$5j$	$12j-1$	$13j-1$	H_{5j}	H_{12j-1}	H_{13j-1}
1	5	11	12	45	231	276
2	10	23	25	190	1035	1225
3	15	35	38	435	2415	2850
4	20	47	51	780	4371	5151
5	25	59	64	1225	6903	8128

Solution 2 by Jerry Chu (student at Saint George's School), Spokane, WA

The difference between two consecutive hexagonal numbers is

$$H(n+1) - H(n) = \big(2(n+1)^2 - (n+1)\,\big) - \big(2n^2 - n\big) = 4n + 1.$$

This is to say that any hexagonal number of the form $4n+1$ is the difference between $H(n)$ and $H(n+1)$. So we look for hexagonal numbers of the form $4m+k$, where $k = 0, 1, 2, 3$

$$H(4m+k)\ (k = 0,1,2,3) = 32m^2 + 16mk + 2k^2 - 4m - k = 2k^2 - k \pmod 4).$$

Only $k = 1$ satisfies this equation. Therefore, hexagonal numbers of the form $H(4m+1) = 32m^2 + 12m + 1$, can be expressed as the difference between $H(8m^2+3m)$ and $H(8m^2+3m+1)$. So there are an infinite number of hexagonal numbers of the form $H(8m^2+3m+1)$ that can be expressed as the sum of two hexagonal numbers.

Comment by Editor: **William J. O'Donnell of Centennial, CO** *mentioned in his solution that:* It can further be shown that infinitely many hexagonal numbers are the sum and difference of two hexagonal numbers, specifically,

$$\begin{aligned} H_{128n^2+12n+1} &= H_{16n+1} + H_{128n^2+12n} \\ &= H_{8192n^4+1536n^3+168n^2+9n+1} - H_{8192n^4+1536n^3+168n^2+9n}, \text{ for } n \geq 1. \end{aligned}$$

For more detail, see O'Donnell, W.J., Two theorems concerning hexagonal numbers *Fibonacci Quarterly* 1979, 17(1), 77-79. Similar results have also been published for triangular, pentagonal, and octagonal numbers. See:

−Hansen, R.T. Arithmetic of pentagonal numbers. *Fibonacci Quarterly,* 1970, 8, 83-87.

− O'Donnell, W.J. A theorem concerning octagonal numbers. *Journal of Recreational Mathematics,* 1979-80, 12(4), 271-272.

−Sierpinski W. Un Théorème sur les nombres triangulaires *Elemente der Mathematik,* 1968, 23, 31-32.

Also solved by Adnan Ali (student at A.E.C.S-4), Mumbai, India; Arkady Alt, San Jose, CA; Corneliu Manescu-Avram,Transportation High School, Ploiesti, Romania; Brian D. Beasley, Presbyterian College, Clinton, SC; Bruno Salgueiro Fanego, Viveiro, Spain; Moti Levy, Rehovot, Israel; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Jahangeer

Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY at Oneonta, Oneonta, NY; William J. O'Donnell, Centennial, CO; Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Bob Sealy, Sackville, New Brunswick, Canada; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Titu Zvonaru, Comănesti, Romania (jointly with) Neculai Stanciu, "George Emil Palade School," Buzău, Romania, and the proposer.

- **5316:** *Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

Let $\{u_n\}_{n\geq 0}$ be a sequence defined recursively by

$$u_{n+1}=\sqrt{\frac{u_n^2+u_{n-1}^2}{2}}.$$

Determine $\lim\limits_{n\to\infty} u_n$ in terms of u_0, u_1.

Solution 1 by Moti Levy, Rehovot, Israel

Let $v_n=u_n^2$, then the sequence $\{v_n\}_{n\geq 0}$ follows the linear recurrence

$$2v_{n+1}=v_n+v_{n-1}.$$

The closed form of the sequence $\{v_n\}_{n\geq 0}$ is $v_n=\alpha+\beta\left(-\frac{1}{2}\right)^n$, with initial conditions:

$$u_0^2=\alpha+\beta,$$
$$u_1^2=\alpha-\frac{\beta}{2}.$$
$$u_n^2=\frac{1}{3}u_0^2+\frac{2}{3}u_1^2+\left(\frac{2}{3}u_0^2-\frac{2}{3}u_1^2\right)\left(-\frac{1}{2}\right)^n.$$
$$\lim_{n\to\infty} u_n=\sqrt{\frac{1}{3}u_0^2+\frac{2}{3}u_1^2}.$$

Solution 2 by Jerry Chu (student at Saint George's School), Spokane, WA

First some observations:

$$u_2 = \sqrt{\frac{u_0^2+u_1^2}{2}}$$

$$u_3 = \sqrt{\frac{u_0^2+3u_1^2}{4}}$$

$$u_4 = \sqrt{\frac{3u_0^2+5u_1^2}{8}}.$$

This suggests that the general of the sequence is

$$u_n=\sqrt{\frac{\left(2^{n-2}\pm\frac{1}{2}\right)u_0^2+\left(2^{n-1}\mp\frac{1}{2}\right)u_1^2}{3\cdot 2^{n-2}}},$$

which is true by induction. So

$$\lim_{n\to\infty} u_n = \sqrt{\frac{1}{3}u_0^2 + \frac{2}{3}u_1^2}$$

Solution 3 by Henry Ricardo, New York Math Circle, NY

In [1], the authors provide three ways of determining convergence and limiting values for linear mean recurrences. In particular, they prove that given a sequence $\{x_n\}$ such that $x_n := (x_{n-1} + x_{n-2} + \cdots + x_{n-m})/m$ for $n \geq m+1$, where $x_1, x_2, \ldots, x_m$ are given real numbers, we can conclude that

$$\lim_{n\to\infty} x_n = \frac{2}{m(m+1)} \sum_{n=1}^{m} n x_n.$$

The substitution $U_k = u_k^2$ converts our given relation to the linear mean recurrence

$$U_n = \frac{U_{n-1} + U_{n-2}}{2} \qquad \text{for } n = 2, 3, \ldots.$$

Then the paper cited above provides the formula

$$\lim_{n\to\infty} U_n = \frac{2}{2(3)} \sum_{n=0}^{1} (n+1)U_n = \frac{U_0 + 2U_1}{3},$$

giving us $u_n^2 \to (u_0^2 + 2u_1^2)/3$, or $u_n \to \sqrt{(u_0^2 + 2u_1^2)/3}$.

Reference

[1] D. Borwein, J. M. Borwein, B. Sims, *On the Solution of Linear Mean Recurrences*, Amer. Math. Monthly **121** 2014, pp. 486-498.

Editor's comment: Henry Ricardo submitted three solutions to problem 5316. Taken together, the above solutions represent the different ways for determining the convergence of linearly stated recursion sequences, as pointed out in the reference Henry cited.

Also solved by Susan Abernathy, Dionne Bailey, Elsie Campbell, Charles Diminnie, and Jesse Taylor (jointly), Angelo State University, San Angelo, TX; Arkady Alt, San Jose, CA; Corneliu Manescu-Avram,Transportation High School, Ploiesti, Romania; Bruno Salgueiro Fanego Viveiro, Spain; Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News, VA; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; Adrian Naco, Polytechnic University, Tirana, Albania; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănesti, Romania (jointly with) Neculai Stanciu, "Geroge Emil Palade School," Buzău, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5317:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $a_k, b_k > 0,\ 1 \le k \le n$, be real numbers such that $a_1 + a_2 + \ldots + a_n = 1$. Prove that

$$\frac{1}{n^3}\left(\sum_{k=1}^{n} b_k\right)^5 \le \sum_{k=1}^{n} \frac{b_k^5}{a_k}.$$

Solution 1 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

By the Hölder inequality,

$$\left(\sum_{k=1}^{n} 1\right)^{p-q-1} \sum_{k=1}^{n} \frac{b_k^p}{a_k^q}\left(\sum_{k=1}^{n} a_k\right)^q \ge \left(\sum_{k=1}^{n} 1^{\frac{p-q-1}{p}} \frac{b_k^{\frac{p}{p}}}{a_k^{\frac{q}{p}}} a_k^{\frac{q}{p}}\right)^p = \left(\sum_{k=1}^{n} b_k\right)^p.$$

The choice $p = 5$ and $q = 1$ gives the result.

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

We will need two preliminary results:

1. Let $f(x) = x^{\frac{5}{2}}$ on $(0, \infty)$. Since $f''(x) = \dfrac{15}{4} x^{\frac{1}{2}} > 0$ for $x > 0$, it follows that $f(x)$ is strictly convex on $(0, \infty)$. Then, Jensen's Theorem implies that

$$f\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right) \le \frac{1}{n}\sum_{k=1}^{n} f(b_k),$$

i.e.,

$$\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right)^{\frac{5}{2}} \le \frac{1}{n}\sum_{k=1}^{n} b_k^{\frac{5}{2}}. \tag{1}$$

Further, equality is attained in (1) if and only if $b_1 = b_2 = \ldots = b_n$.

2. If we apply the Cauchy - Schwarz Inequality to the vectors

$$\overrightarrow{X} = \left(\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_n}\right) \quad \text{and} \quad \overrightarrow{Y} = \left(\sqrt{\frac{b_1^5}{a_1}}, \sqrt{\frac{b_2^5}{a_2}}, \cdots, \sqrt{\frac{b_n^5}{a_n}}\right),$$

we get

$$\begin{aligned}\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{b_k^5}{a_k}\right) &= \left\|\overrightarrow{X}\right\|^2 \left\|\overrightarrow{Y}\right\|^2 \\ &\ge \left(\overrightarrow{X} \cdot \overrightarrow{Y}\right)^2 \\ &= \left(\sum_{k=1}^{n} b_k^{\frac{5}{2}}\right)^2. \end{aligned} \tag{2}$$

Since $\sum_{k=1}^{n} a_k = 1$, we use (1) and (2) to obtain

$$\begin{aligned}
\sum_{k=1}^{n} \frac{b_k^5}{a_k} &= \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{b_k^5}{a_k}\right) \\
&\geq \left(\sum_{k=1}^{n} b_k^{\frac{5}{2}}\right)^2 \\
&= n^2 \left(\frac{1}{n}\sum_{k=1}^{n} b_k^{\frac{5}{2}}\right)^2 \\
&\geq n^2 \left[\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right)^{\frac{5}{2}}\right]^2 \\
&= n^2 \cdot \frac{1}{n^5}\left(\sum_{k=1}^{n} b_k\right)^5 \\
&= \frac{1}{n^3}\left(\sum_{k=1}^{n} b_k\right)^5.
\end{aligned}$$

Also, by the criterion for equality in (1), equality results above if and only if $b_1 = b_2 = \ldots = b_n$.

Solution 3 by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

The proposed inequality is homogeneous in b_k, so we may assume in addition that also $b_1 + b_2 + \cdots + b_n = 1$, and the inequality is equivalent to prove that $\frac{1}{n^3} \leq \sum_{k=1}^{n} \frac{b_k^5}{a_k}$. We use the Cauchy-Schwarz inequality in Engel's form which states that

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \cdots + \frac{x_n^2}{y_n} \geq \frac{(x_1 + x_2 + \cdots + x_n)^2}{y_1 + y_2 + \cdots + y_n},$$

for all real numbers x_i and positive real numbers y_i.

$$\begin{aligned}
\sum_{k=1}^{n} \frac{b_k^5}{a_k} &= \sum_{k=1}^{n} \frac{\left(b_k^{5/2}\right)^2}{a_k} \text{ (which by Engel's inequality)} \\
&\geq \frac{\left(\sum_{k=1}^{n} b_k^{5/2}\right)^2}{\sum_{k=1}^{n} a_k} = \left(\sum_{k=1}^{n} b_k^{5/2}\right)^2.
\end{aligned}$$

Now, by the power-mean arithmetic-mean inequality

$$\frac{\sum_{k=1}^{n} b_k^{5/2}}{n} \geq \left(\frac{\sum_{k=1}^{n} b_k}{n}\right)^{5/2} = \frac{1}{n^{5/2}}, \quad \text{and so, } \sum_{k=1}^{n} b_k^{5/2} \geq \frac{1}{n^{3/2}}.$$

Therefore, $\displaystyle\sum_{k=1}^{n} \frac{b_k^5}{a_k} \geq \left(\sum_{k=1}^{n} b_k^{5/2}\right)^2 \geq \frac{1}{n^3}$ and we are done.

Solution 4 by Adrian Naco, Polytechnic University,Tirana, Albania

Let's prove a more general inequality, that is,

$$\frac{1}{n^s}\left(\sum_{k=1}^{n} b_k\right)^{s+2} \leq \sum_{k=1}^{n} \frac{b_k^{s+2}}{a_k}, \quad (1)$$

Based on the well known Chebyschev inequality we get that

$$\sum_{k=1}^{n} \frac{b_k^{s+2}}{a_k} \geq \frac{1}{n}\left(\sum_{k=1}^{n} \frac{1}{a_k}\right)\left(\sum_{k=1}^{n} b_k^{s+2}\right) \quad (2)$$

Furthermore if we apply the same inequality s times recursively then,

$$\sum_{k=1}^{n} \frac{b_k^{s+2}}{a_k} \geq \frac{1}{n}\left(\sum_{k=1}^{n} \frac{1}{a_k}\right)\left(\sum_{k=1}^{n} b_k^{s+2}\right) \geq \frac{1}{n}\left(\sum_{k=1}^{n} \frac{1}{a_k}\right)\frac{1}{n}\left(\sum_{k=1}^{n} b_k\right)\left(\sum_{k=1}^{n} b_k^{s+1}\right)$$

$$\geq \frac{1}{n}\left(\sum_{k=1}^{n} \frac{1}{a_k}\right)\left(\frac{1}{n}\right)^2\left(\sum_{k=1}^{n} b_k\right)^2\left(\sum_{k=1}^{n} b_k^{s}\right) \geq \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\geq \frac{1}{n}\left(\sum_{k=1}^{n} \frac{1}{a_k}\right)\left(\frac{1}{n}\right)^{s+1}\left(\sum_{k=1}^{n} b_k\right)^{s+1}\left(\sum_{k=1}^{n} b_k\right)$$

$$= \left(\frac{1}{n}\right)^{s+2}\left(\sum_{k=1}^{n} \frac{1}{a_k}\right)\left(\sum_{k=1}^{n} b_k\right)^{s+2} \geq \left(\frac{1}{n}\right)^{s+2} n^2\left(\sum_{k=1}^{n} b_k\right)^{s+2}$$

$$= \left(\frac{1}{n}\right)^{s}\left(\sum_{k=1}^{n} b_k\right)^{s+2}$$

since,

$$\left(\sum_{k=1}^{n}\frac{1}{a_k}\right) \geq \frac{n}{\sqrt[n]{\prod_{k=1}^{n} a_k}} \geq \frac{n^2}{\left(\sum_{k=1}^{n} a_k\right)} = n^2$$

Finally, the given inequality to prove is a special case of the general inequality (1) taken for $s = 5$.

Solution 5 by Adnan Ali (Student at A.E.C.S-4), Mumbai, India

From Holder's Inequality, one notices that

$$n^3\left(\sum_{k=1}^{n}\frac{b_k^5}{a_k}\right) = \left(\sum_{k=1}^{n}\frac{b_k^5}{a_k}\right)\left(\sum_{k=1}^{n} a_k\right)\left(\underbrace{1+\cdots+1}_{n\text{ times}}\right)\left(\underbrace{1+\cdots+1}_{n\text{ times}}\right)\left(\underbrace{1+\cdots+1}_{n\text{ times}}\right) \geq \left(\sum_{k=1}^{n} b_k\right)^5$$

whence the result immediately follows and so, $\dfrac{1}{n^3}\left(\sum_{k=1}^{n} b_k\right)^5 \leq \left(\sum_{k=1}^{n}\dfrac{b_k^5}{a_k}\right)$.

Solution 6 by Nicusor Zlota. "Traian Vula" Technical College, Focsani, Romania

We shall prove the following more general inequality.

If $a_k b_k > 0,\ 1 \leq k \leq n$ and $a, b, \in \Re$, such that $a - b \geq 1$, then

$$\sum_{k=1}^{n}\frac{b_k^q}{a_k^b} \geq n\frac{\left(\dfrac{1}{n}\sum_{k=1}^{n} b_k\right)^a}{\left(\dfrac{1}{n}\sum_{k=1}^{n} a_k\right)^b} \qquad (*)$$

Proof: Using the Radon and Jensen inequalities, we have

$$\sum_{k=1}^{n}\frac{b_k^q}{a_k^b} = \sum_{k=1}^{n}\frac{\left(b_k^{\frac{a}{b+1}}\right)^{b+1}}{a_k^b} \underset{Radon}{\geq} \sum_{k=1}^{n}\frac{\left(b_k^{\frac{a}{b+1}}\right)^{b+1}}{a_k^b} \underset{Jensen}{\geq} \frac{\left[n\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right)^{\dfrac{a}{b+1}}\right]^{b+1}}{\left(\sum_{k=1}^{n} a_k\right)^b} = n\frac{\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right)^a}{\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^b}.$$

If in $(*)$, $a = 5, b = 1$ and $\sum_{k=1}^{n} a_k = 1$, then we obtain the inequality of the problem.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Moti Levy, Rehovot, Israel; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; Henry

Ricardo, New York Math Circle, NY; Titu Zvonaru, Comănesti, Romania (jointly with) Neculai Stanciu, "George Emil Palade School," Buzău, Romania, and the proposer.

- **5318:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Prove that $(1+x)^x \leq 1+x^2$ for $0 \leq x \leq 1$.

Solution 1 by Arkady Alt, San Jose, California, USA.

The inequality $(1+x)^x \leq 1+x^2, 0 \leq x \leq 1$ immediately follows from the Bernoulli inequality:

$$(1+t)^\alpha \geq 1+\alpha t, t>-1, a \geq 1. \qquad \textbf{(1)}$$

Indeed, for $0 < x \leq 1 \iff \dfrac{1}{x} \geq 1,$ and by **(1)** we have

$$\left(1+x^2\right)^{\frac{1}{x}} \geq 1+x^2 \cdot \frac{1}{x} = 1+x \iff 1+x^2 \geq (1+x)^x.$$

For $x=0$ the original inequality is obvious.

Another way to prove the inequality $(1+x)^x \leq 1+x^2$ is based on using the Weighted AM-GM Inequality: $u^p v^q \leq pu+qv$ where $u,v,p,q \geq 0$ and $p+q=1$.

Indeed, for $u=1+x, v=1,\ p=x,\ q=1-x$ we have

$$(1+x)^x \cdot 1^{1-x} \leq (1+x)\,x+1\cdot(1-x) \iff (1+x)^x \leq 1+x^2.$$

$(1+x)^a \leq 1+ax,$ where $x>-1$ and $0 \leq a \leq 1. \qquad$ **(1)**

Applying inequality **(1)** to $a=x$ we obtain $(1+x)^x \leq 1+x^2$.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We have equalityfor $x=0$ and $x=1$.

We assume that $0<x<1$. We expand $(1+x)^x$ into a binomial series and get

$$\begin{aligned}(1+x)^x &= \sum_{n=0}^{\infty}\binom{x}{n}x^n = 1+x^2+\sum_{j=1}^{\infty}\left(\frac{x(x-1)\cdots(x-2j+1)}{(2j)!}x^{2j}+\frac{x(x-1)\cdots(x-2j)}{(2j+1)!}x^{2j+1}\right)\\ &= 1+x^2-\sum_{j=1}^{\infty}\underbrace{\frac{x(1-x)\cdots(2j-1-x)}{(2j)!}x^{2j}}_{>0}\underbrace{\left(1-\frac{2j-x}{2j+1}x\right)}_{>0} < 1+x^2.\end{aligned}$$

Solution 3 by Michael Brozinsky, Central Islip, NY

Consider $g(u)$=(1+u)$^{\frac{1}{u}}$ on $(0,1)$. It is decreasing since

$$g'(u) = (1+u)^{\frac{1}{u}} \cdot \frac{\left(\dfrac{u}{1+u}-\ln(1+u)\right)}{u^2}$$

.

Note that $(1+u)^{\frac{1}{u}}$ and u^2 are positive and $f(u) = \dfrac{u}{1+u} - \ln(1+u)$ is negative on $(0,1)$ since and $f(0) = 0$ and $\dfrac{d}{du}f(u) = \dfrac{u}{(1+u)^2}$. Hence since $x^2 < x$ if $0 < x < 1$ we have $g(x^2) > g(x)$ i.e.

$$(1+x^2)^{\frac{1}{x^2}} > (1+x)^{\frac{1}{x}} \qquad (*).$$

Now if $0 < a < b$ then $\dfrac{b}{a} > 1$ and so if $t > 0$ then $\left(\dfrac{b}{a}\right)^t > 1$ and so $\mathrm{b}^t > a^t$. $\quad$ (**)

The desired inequality then follows from (**) upon setting $b = (1+x^2)^{\frac{1}{x^2}}$, $a = (1+x)^{\frac{1}{x}}$ and $t = x^2$, i.e., raising both sides of (*) to the x^2 power.

Solution 4 by Ed Gray, Highland Beach, FL

Consider the the general binomial theorem.

$$(a+b)^n = a^n + na^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \binom{n}{4}a^{n-4}b^4 + \cdots.$$

Since $a = 1$, we suppress its appearance after the first term. Letting $b = x = n$, and substituting in the above we obtain

$$(1+x)^x = 1 + x^2 + \frac{x(x-1)x^2}{2!} + \frac{x(x-1)(x-2)x^3}{3!} + \frac{x(x-1)(x-2)(x-3)x^4}{4!} + \cdots.$$

Clearly, for $x = 0$ and $x = 1$ the expression on the left hand side of the equality sign is the same as the expression on the right hand side of the equality sign.

If $0 < x < 1$ we note that starting with the third term, which is negative, we have an alternating decreasing series that approaches zero. Since each term is numerically less than its precedent and the third term is negative, it is clear that the series must be less than the sum of the first two terms, $1 + x^2$, therefore, for $0 < x < 1$, it must be that $(1+x)^x < 1 + x^2$.

Solution 5 by Kee-Wai Lau, Hong Kong, China

For $0 \le x \le 1$, let $f(x) = x\ln(1+x) - \ln(1+x^2)$. We have

$$f'(x) = \ln(1+x) + \frac{x(x^2-2x-1)}{(1+x)(1+x^2)} \text{ and } f''(x) = \frac{x(x^2+2x+3)\left(x^2+2x-1\right)}{(1+x)^2(x^2+1)^2}.$$

Hence for $0 < x < \sqrt{2} - 1$, we have $f''(x) < 0$. Since $f'(0) = 0$, so $f'(x) < 0$ for $0 < x \le \sqrt{2} - 1$. Since $f(0) = 0$, so $f(x) < 0$ for $0 < x < \sqrt{2} - 1$ as well.

Now for $\sqrt{2} - 1 < x \le 1$, we have $f''(x) > 0$ so that $f(x)$ is convex. Since $f(\sqrt{2} - 1) < 0$ and $f(1) = 0$, so $f(x) \le 0$ for $\sqrt{2} - 1 < x \le 1$. It follows that $f(x) \le 0$ for $0 \le x \le 1$.

The inequality of the problem follows by exponentiation.

Also solved by Adnan Ali (Student at A.E.C.S-4), Mumbai, India; Corneliu Manescu-Avram,Transportation High School, Ploiesti, Romania; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York, NY; Moti Levy, Rehovot, Israel; Adrian Naco, Polytechnic University, Tirana, Albania; Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

Mea Culpa

Adnan Ali (Student at A.E.C.S-4 in Mumbai, India) submitted solutions to problems 5307 and 5309. Unfortunately these solutions were unintentionally not acknowledged in the previous issue of the column. **And similarly for Hatef I. Arshagi of Guilford Technical Community College in Jamestown, NC** for his solutions to problems 5307, 5309 and 5312. Once again I plead mea culpa to them both, and also to **William J. O'Donnell, Centennial, CO** for not acknowledging his solution to 5213.