Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before March 15, 2013

- **5236:** *Proposed by Kenneth Korbin, New York, NY*

  Given positive numbers (a, b, c, x, y, z) such that

$$\begin{aligned} x^2 + xy + y^2 &= a, \\ y^2 + yz + z^2 &= b, \\ z^2 + zx + x^2 &= c. \end{aligned}$$

  Express the value of the sum $x + y + z$ in terms of a, b, and c.

- **5237:** *Proposed by Michael Brozinsky, Central Islip, NY*

  Let $0 < R < 1$ and $0 < S < 1$, and define

$$\begin{aligned} a &= \sqrt{-2\sqrt{1-S^2}\sqrt{1-R^2}+2+2RS}, \\ b &= \sqrt{-R-S+1+RS}, \text{ and} \\ c &= \sqrt{R+S+1+RS}. \end{aligned}$$

  Determine whether there is tuple (R, S) such that a, b, and c are sides of a triangle.

- **5238:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  It is fairly well-known that $(1111\ldots1)_9$, a number written in base 9 with an arbitrary number of digits 1, always evaluates decimally to a triangular number. Find another base b and a single digit d in that base, such that $(ddd\ldots d)_b$, using k digits d, has the same property, $\forall k \geq 1$.

- **5239:** *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany*

  Determine all functions $f : \Re - \{-3, -1, 0, 1, 3\} \to \Re$, which satisfy the relation

$$f(x) + f\left(\frac{13+3x}{1-x}\right) = ax + b,$$

  where a and b are given arbitrary real numbers.

- **5240:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

  Let x be a positive real number. Prove that

  $$\frac{x[x]}{(x+\{x\})^2}+\frac{x\{x\}}{(x+[x])^2}>\frac{1}{8},$$

  where $[x]$ and $\{x\}$ represent the integral and fractional part of x, respectively.

- **5241:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $\alpha \geq 0$ be a real number. Calculate

  $$\lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{x^n+\alpha}\,dx\right)^n.$$

Solutions

- **5218:** *Proposed by Kenneth Korbin, New York, NY*

  Find positive integers x and y such that,

  $$2x-y-\sqrt{3x^2-3xy+y^2}=2013$$

  with $(x,y)=1$.

  Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

  If we re-write the equation in the form $2x-y-2013=\sqrt{3x^2-3xy+y^2}$ (1) and then square both sides and simplify, we get successively

  $$x^2-8052x+(2013)^2+4026y-xy=0, \text{ and}$$

  $$(x-4026)^2-(x-4026)\,y=3\,(2013)^2.$$

  To simplify further, substitute $w=x-4026$ to obtain

  $$\begin{aligned} w^2-wy &= 3\,(2013)^2 && (2)\\ \text{or } w\,(w-y) &= 3\,(2013)^2. && (3) \end{aligned}$$

  Since w and $w-y$ are integers, the problem can be solved by considering all factorizations of

  $$3\,(2013)^2=3^3 11^2 61^2 \qquad (4)$$

  into a product of two integers. Also, since $y>0$, we have $w-y<w$ in each instance.

  Before proceeding, we note that (2) implies that

  $$y=\frac{w^2-3\,(2013)^2}{w}$$

and we get

$$\begin{aligned} 2x - y - 2013 &= 2(w + 4026) - \frac{w^2 - 3(2013)^2}{w} - 2013 \\ &= \frac{w^2 + 6093w + 3(2013)^2}{w}. \end{aligned}$$

Since

$$w^2 + 6093w + 3(2013)^2 = \left(w + \frac{6039}{2}\right)^2 + \frac{3}{4}(2013)^2 > 0$$

for all w, we end up with $2x - y - 2013 < 0$ when $w < 0$. Hence, when $w - y < w < 0$, (1) implies that we will get extraneous solutions.

Next, suppose that $(w, w - y) > 1$. Then, there is a prime p which is a divisor of both w and $w - y$. Conditions (3) and (4) tell us that $p = 3$, 11, or 61 and hence, p divides 2013. First of all, p divides both w and $w - y$ implies that p divides $w - (w - y) = y$. Also, since p divides both w and 2013, it follows that p divides $w + 2(2013) = x$. As a result, when $(w, w - y) > 1$, we have $(x, y) > 1$ as well. Therefore, we may restrict our work to the case where $(w, w - y) = 1$.

Finally then, we need only consider (3) and (4) with $0 < w - y < w$ and $(w, w - y) = 1$. The full set of solutions is given in the following table.

$w - y$	w	$x = w + 4026$	$y = w - (w - y)$
1	$3^3 11^2 61^2$	$12,160,533$	$12,156,506$
3^3	$11^2 61^2$	$454,267$	$450,214$
11^2	$3^3 61^2$	$104,493$	$100,346$
$3^3 11^2$	61^2	$7,747$	454

With a good software package, it's possible to check that all of these are solutions of (1) with $(x, y) = 1$.

Solution 2 by Adrian Naco, Polytechnic University, Tirana, Albania

The left side of the equation can be transformed to

$$2x - y - \sqrt{(2x-y)^2 - x(x-y)} = 2013 \quad \Rightarrow \quad x(x-y) > 0 \quad \Rightarrow \quad 0 < y < x. \qquad (1)$$

(since x and y are positive integers). Further more,

$$\sqrt{3x^2 - 3xy - y^2} = 2x - y - 2013 \quad \Rightarrow \quad 2x - y - 2013 \geq 0 \quad \Leftrightarrow \quad 2x - y \geq 2013. \quad (2)$$

Solving the equation we have that

$$\begin{aligned} 3x^2 - 3xy - y^2 = (2x - y - 2013)^2 \quad &\Rightarrow \quad x^2 - xy - 2 \cdot 2013y - 4 \cdot 2013x + 2013^2 = 0 \\ &\Rightarrow \quad y = x - 2 \cdot 2013 - \frac{3 \cdot 2013^2}{x - 2 \cdot 2013}. \qquad (3) \end{aligned}$$

where

$$\frac{3\cdot 2013^2}{x-2\cdot 2013}=r\in Z \quad\Rightarrow\quad x=2\cdot 2013+\frac{3\cdot 2013^2}{r}, \qquad (4)$$

and since

$$\frac{3\cdot 2013^2}{r}=s\in Z \quad\Rightarrow\quad rs=3\cdot 2013^2=3^3\cdot 11^2\cdot 61^2. \qquad (5)$$

Considering (3), (4), (5) we have that,

$$\begin{aligned} x &= 2\cdot 2013+s \\ y &= s-r \\ rs &= 3\cdot 2013^2=3^3\cdot 11^2\cdot 61^2 \quad \text{where} \quad r,s\in Z. \end{aligned}$$

The general structure of r and s is

$$r \;=\; 3^{\alpha_1}11^{\alpha_2}61^{\alpha_3} \quad \text{and s} = 3^{\beta_1}11^{\beta_2}61^{\beta_3} \quad \text{where}$$

$$\alpha_1+\beta_1=3,\quad \alpha_2+\beta_2=2,\quad \alpha_3+\beta_3=2.$$

From (1) and (2)

$$\begin{aligned} x>y>0 \quad &\Rightarrow \quad s>r>-2\cdot 2013 \\ 2x-y-2013>0 \quad &\Rightarrow \quad s+r>-3\cdot 2013 \\ &\Rightarrow \quad s+\frac{3\cdot 2013^2}{s}>-3\cdot 2013 \\ &\Rightarrow \quad \frac{s^2+3\cdot 2013s+3\cdot 2013^2}{s}>0 \\ &\Rightarrow \quad s>0 \quad\Rightarrow\quad r>0. \end{aligned}$$

Furthermore, if $(r,s)=p$ then $p|2013$ and consequently $p|x$ and $p|y$. Since $(x,y)=1$ then $p=1$, resulting that there are only eight possible combinations for r and s (since for each combination we have $\alpha_i=0$ or $\beta_i=0$, $\forall i\in\{1,2,3\}$)

$$r=3^0 11^0 61^0 \quad \text{and} \quad s=3^3 11^2 61^2$$

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and since $s > r$, there are only four possible combinations, each of them generates a solution for the given equation. More concretely the four solutions are

$$r = 3^0 11^0 61^0, \quad s = 3^3 11^2 61^2 \quad \Rightarrow \quad x = 12160533, \quad y = 12156506$$

$$r = 3^3 11^0 61^0, \quad s = 3^0 11^2 61^2 \quad \Rightarrow \quad x = 454267, \quad y = 450214$$

$$r = 3^0 11^2 61^0, \quad s = 3^3 11^0 61^2 \quad \Rightarrow \quad x = 104493, \quad y = 100346$$

$$r = 3^3 11^2 61^0, \quad s = 3^0 11^0 61^2 \quad \Rightarrow \quad x = 7747, \quad y = 454.$$

Comment by David Stone and John Hawkins of Georgia Southern University, Statesboro, GA. The above four points (x, y) are called *visible* points (i.e., the view from the origin is not blocked by any other lattice point.)

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney Australia and Elton Bojaxhiu, Kriftel, Germany; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Albert Stadler Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5219: *Proposed by David Manes and Albert Stadler, SUNY College at Oneonta, Oneonta, NY and Herrliberg, Switzerland (respectively)*

Let k and n be natural numbers. Prove that:

$$\sum_{j=1}^{n} \cos^k\left(\frac{(2j-1)\pi}{2n+1}\right) = \begin{cases} \dfrac{2n+1}{2^{k+1}}\dbinom{k}{k/2} - \dfrac{1}{2}, & k \text{ even} \\ \\ \dfrac{1}{2}, & k \text{ odd.} \end{cases}$$

Solution by Kee-Wai Lau, Hong Kong, China

Since the stated result is not true for $(k,n)=(3,1),(6,1)$, we modify it to

$$\sum_{j=1}^{n}\cos^k\left(\frac{(2j-1)\pi}{2n+1}\right)=\begin{cases}\dfrac{2n+1}{2^{k+1}}\dbinom{k}{k/2}-\dfrac{1}{2}, & k=2,4,6,\ldots,4n\\[2ex] \dfrac{1}{2}, & k=1,3,5,\ldots,2n-1.\end{cases}$$

Let $i=\sqrt{-1}$ and $\theta=\theta(j,n)=\dfrac{(2j-1)\pi}{2n+1}$. By the binomial theorem we have

$$\begin{aligned}\sum_{j=1}^{n}\cos^k\theta &= \frac{1}{2}\sum_{j=-n}^{n}\cos^k\theta+\frac{(-1)^{k-1}}{2}\\ &= \frac{1}{2^{k+1}}\sum_{j=-n}^{n}\left(e^{i\theta}+e^{-i\theta}\right)+\frac{(-1)^{k-1}}{2}\\ &= \frac{1}{2^{k+1}}\sum_{j=-n}^{n}\sum_{t=0}^{k}\binom{k}{t}e^{i(k-2t)\theta}+\frac{(-1)^{k-1}}{2}\\ &= \frac{1}{2^{k+1}}\sum_{t=0}^{k}\binom{k}{t}\sum_{j=-n}^{n}e^{i(k-2t)\theta}+\frac{(-1)^{k-1}}{2}.\end{aligned}$$

For $k=2,4,6,\ldots,4n$ and $t=0,1,2,\ldots,k$, $\dfrac{2(k-2t)}{2n+1}$ is not an integer unless $t=\dfrac{k}{2}$. So for $t=\dfrac{k}{2}$, we have $\displaystyle\sum_{j=-n}^{n}e^{i(k-2)\theta}=2n+1$ and for $t=0,1,2,\ldots,\dfrac{k-2}{2},\ \dfrac{k+2}{2},\ \ldots,k$, we have $\displaystyle\sum_{j=-n}^{n}e^{i(k-2t)\theta}=\frac{1-e^{2(k-2t)\pi i}}{1-e^{(2(k-2t)\pi i)/(2n+1)}}=0.$

This proves the first part of the modified statement of the problem.

For $k=1,3,5,\ldots,2n-1$ and $t=0,1,2,\ldots,k$, $\dfrac{2(k-2t)}{2n+1}$ is not an integer and so $\displaystyle\sum_{j=-n}^{n}e^{i(k-2t)\theta}=0$, and this proves the second part of the modified statement of the problem.

Editor's note: David Manes and Anastasios Kotronis, noted the error in the statement of the problem, but the problem had already been posted. Each went on to correct the mistake and each made reference to a general technique for solving such problems that is discussed in a paper by Mircea Merca (of the University of Craiova in Romania) entitled: "A Note on Cosine Power Sums" that appeared in the Journal of Integer Sequences, Vo. 15(2012); Article 12.5.3. Other solvers of 5219 parenthetically referenced the need to modify of the original statement.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Anastasios Kotronis, Athens, Greece; Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy, and the proposers.

5220: *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

The pentagonal numbers begin $1, 5, 12, 22\ldots$ and are generally defined by $P_n = \dfrac{n(3n-1)}{2}$, $\forall n \geq 1$. The triangular numbers begin $1, 3, 6, 10, \ldots$ and are generally defined by $T_n = \dfrac{n(n+1)}{2}, \forall n \geq 1$. Find the greatest common divisor, $\gcd(T_n, P_n)$.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

Case 1: If n is even, $n = 2k$ for some $k \geq 1$, so, using properties of the gcd, the Euclidean algorithm, and the fact that $2k+1$ is odd $\Longrightarrow \gcd(2k+1, 4) = 1$, we obtain

$$\begin{aligned}
\gcd(P_n, T_n) &= \gcd\left(k\left(6k-1\right), k\left(2k+1\right)\right)\\
&= k\gcd\left(6k-1, 2k+1\right)\\
&= k\gcd\left(2k+1, -4\right)\\
&= k\gcd\left(2k+1, 4\right)\\
&= k = \frac{n}{2}.
\end{aligned}$$

Case 2: If n is odd, then $\dfrac{n}{4}$ gives a remainder of 1 or 3; so $n \equiv 1 \ (mod\ 4)$ or $n \equiv 3 \ (mod\ 4)$. We have two cases to consider.

Case 2.1: $n = 4k+1$ for some $k \geq 0$; then

$$\begin{aligned}
\gcd(P_n, T_n) &= \gcd\left(n(2k+1), n\left(6k+1\right)\right)\\
&= n\gcd\left(2k+1, 6k+1\right)\\
&= n\gcd\left(-2, 2k+1\right)\\
&= n\gcd\left(2, 2k+1\right)\\
&= n.
\end{aligned}$$

Case 2.2 $n = 4k+3$ for some $k \geq 0$; then

$$\gcd(P_n, T_n) = \gcd\left(n(6k+4), n\left(2k+2\right)\right)$$

$$
\begin{aligned}
&= 2n \gcd(3k+2, k+1) \\
&= 2n \gcd(k+1, -1) \\
&= 2n \gcd(k+1, 1) \\
&= 2n.
\end{aligned}
$$

Hence,

$$\gcd(P_n, T_n) = \begin{cases} \dfrac{n}{2} & n \text{ } even \\ n & n \equiv 1 \ (mod\ 4) \\ 2n & n \equiv 3 \ (mod\ 4) \end{cases}$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

If n is even then,

$$(P_n, T_n) = \frac{n}{2}(3n-1, n+1) = \frac{n}{2}(3n-1-3(n+1), n+1) = \frac{n}{2}(-4, n+1) = \frac{n}{2}.$$

If $n \equiv 1 \ (mod\ 4)$ then,

$$(P_n, T_n) = n\left(\frac{3n-1}{2}, \frac{n+1}{2}\right) = n\left(\frac{3n-1}{2} - 3 \cdot \frac{n+1}{2}, \frac{n+1}{2}\right) = n\left(-2, \frac{n+1}{2}\right) = n.$$

If $n \equiv 3 \ (mod\ 4)$ then

$$(P_n, T_n) = n\left(\frac{3n-1}{2}, \frac{n+1}{2}\right) = n\left(\frac{3n-1}{2} - 3 \cdot \frac{n+1}{2}, \frac{n+1}{2}\right) = n\left(-2, \frac{n+1}{2}\right) = 2n.$$

These three lines can be summarized in one formula by, e.g.,

$$(P_n, T_n) = \frac{n}{2}\left(2\sin^2\frac{\pi n}{2} - \sin\frac{\pi n}{2} + 1\right).$$

Solution 3 by Brian D. Beasley, Presbyterian College, Clinton, SC

Editor's comment: Brian generalized the problem for the nth r-gonal number.

Given integers $n \geq 1$ and $r \geq 3$, the nth r-gonal number is defined by

$$p_n^r = \tfrac{1}{2}n[(r-2)n - (r-4)].$$

Find the following greatest common divisors for $a)$ $\gcd(p_n^r, p_n^{r+1})$ $b)$ $\gcd(p_n^r, p_n^{r+2})$, r even, and c) $\gcd(p_n^r, p_n^{r+2})$, r odd.

a) We show that $\gcd(p_n^r, p_n^{r+1}) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$.

If $n = 2m$ for some positive integer m, then $p_n^r = m(2mr-4m-r+4)$ and $p_n^{r+1} = m(2mr-2m-r+3)$. Since $2mr-2m-r+3 = (2mr-4m-r+4)+(2m-1)$, $2mr-4m-r+4 = (2m-1)(r-2)+2$, and $\gcd(2m-1, 2) = 1$, we also have $\gcd(2mr-2m-r+3, 2mr-4m-r+4) = 1$. Hence $\gcd(p_n^r, p_n^{r+1}) = m = n/2$.

If $n = 2m+1$ for some nonnegative integer m, then $p_n^r = (2m+1)(mr-2m+1)$ and $p_n^{r+1} = (2m+1)(mr-m+1)$. Since $mr-m+1 = mr-2m+1+(m)$ and $mr-2m+1 = m(r-2)+1$, we have $\gcd(mr-m+1, mr-2m+1) = 1$. Hence $\gcd(p_n^r, p_n^{r+1}) = 2m+1 = n$.

b) We show that for even r, $\gcd(p_n^r, p_n^{r+2}) = n$.

Write $r = 2m$ for some positive integer m. Then $p_n^r = n(mn-m-n+2)$ and $p_n^{r+2} = n(mn-m+1)$. Since $mn-m+1 = (mn-m-n+2)+(n-1)$ and $mn-m-n+2 = (n-1)(m-1)+1$, we have $\gcd(mn-m+1, mn-m-n+2) = 1$. Hence $\gcd(p_n^r, p_n^{r+2}) = n$.

c) We show that for odd r, $\gcd(p_n^r, p_n^{r+2}) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ n & \text{if } n \equiv 1 \pmod 4 \\ 2n & \text{if } n \equiv 3 \pmod 4 \end{cases}$.

Write $r = 2m+1$ for some nonnegative integer m. Then $p_n^r = n(2mn-n-2m+3)/2$ and $p_n^{r+2} = n(2mn+n-2m+1)/2$. Since $2mn+n-2m+1 = (2mn-n-2m+3)+(2n-2)$, $2mn-n-2m+3 = (2n-2)(m-1)+(n+1)$, and $2n-2 = (n+1)(2)-(4)$, we have three cases:

If n is even, then $\gcd(n+1, 4) = 1$, so $\gcd(p_n^r, p_n^{r+2}) = (n/2)(1) = n/2$.

If $n \equiv 1 \pmod 4$, then $\gcd(n+1, 4) = 2$, so $\gcd(p_n^r, p_n^{r+2}) = (n/2)(2) = n$.

If $n \equiv 3 \pmod 4$, then $\gcd(n+1, 4) = 4$, so $\gcd(p_n^r, p_n^{r+2}) = (n/2)(4) = 2n$.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney Australia and Elton Bojaxhiu, Kriftel, Germany; Kee-Wai Lau, Hong Kong, China; David Manes, SUNY College at Oneonta, Oneonta, NY; Melfried Olson, University of Hawaii, Honolulu, HI; Boris Rays, Brooklyn, NY; Neculai Stanciu "George Emil Palade" Secondary School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins of Georgia Southern University, Statesboro, GA, and the proposer.

5221: *Proposed by Michael Brozinsky, Central Islip, NY*

If x, y and z are positive numbers find the maximum of

$$\frac{\sqrt{(x+y+z)\cdot x\cdot y\cdot z}}{(x+y)^2+(y+z)^2+(x+z)^2}.$$

Solution 1 by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany

Normalising the expression, the problem will be equivalent to finding the maximum of

$$\frac{\sqrt{xyz}}{(x+y)^2+(y+z)^2+(x+z)^2}$$

subject to $x+y+z=1$.
Using the AM-GM Inequality we have

$$\sqrt[3]{xyz}\le\frac{x+y+z}{3}=\frac{1}{3}\Rightarrow\sqrt{xyz}\le\left(\frac{1}{3}\right)^{\frac{3}{2}}$$

and

$$(x+y)^2+(y+z)^2+(x+z)^2\ge\frac{1}{3}((x+y)+(y+z)+(x+z))^2=\frac{4}{3}$$

Applying these two results we have

$$\frac{\sqrt{xyz}}{(x+y)^2+(y+z)^2+(x+z)^2}\le\frac{\left(\frac{1}{3}\right)^{\frac{3}{2}}}{\frac{4}{3}}=\frac{1}{4\sqrt{3}}.$$

So the maximum value of the required expression is $\frac{1}{4\sqrt{3}}$, and this is achieved when $x=y=z$.

Solution 2 by Kee-Wai Lau, Hong Kong, China

Denote the expression of the problem by f. We show that the maximum of f is $\frac{\sqrt{3}}{12}$.

Since f equals the constant $\frac{\sqrt{3}}{12}$ whenever $x=y=z>0$, so it suffices to show that for $x,y,z>0$, we have

$$f\le\frac{\sqrt{3}}{12}. \qquad (1)$$

From $f=\frac{\sqrt{(x+y+z)\cdot xyz}}{(x-y)^2+(y-z)^2+(x-z)^2+4(xy+yz+zx)}\le\frac{\sqrt{(x+y+z)\cdot xyz}}{4(xy+yz+zx)}$, we see

that (1) will follow from $\frac{(x+y+z)xyz}{(xy+yz+zx)^2}\le\frac{1}{3}$, or equivalently

$$(xy+yz+zx)^2-3xyz(x+y+z)\ge 0. \qquad (2)$$

But (2) in fact holds because its left side equals

$$\frac{x^2(y-z)^2+y^2(z-x)^2+z^2(x-y)^2}{2}.$$

This completes the solution.

Solution 3 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

Since $x, y, z > 0$, the Arithmetic - Geometric Mean Inequality implies that

$$xyz \le \left(\frac{x+y+z}{3}\right)^3 = \frac{(x+y+z)^3}{27},$$

with equality if and only if $x = y = z$. Hence,

$$\sqrt{(x+y+z)\cdot xyz} \le \sqrt{\frac{(x+y+z)^4}{27}} = \frac{\sqrt{3}}{9}(x+y+z)^2, \qquad (1)$$

with equality if and only if $x = y = z$.

Next, we use the strict convexity of $f(t) = t^2$ and Jensen's Theorem to get

$$\begin{aligned}(x+y)^2+(y+z)^2+(x+z)^2 &\ge 3\left[\frac{(x+y)+(y+z)+(x+z)}{3}\right]^2 \\ &= \frac{4}{3}(x+y+z)^2. \qquad (2)\end{aligned}$$

Here, equality results if and only if $x+y = y+z = x+z$, i.e., if and only if $x = y = z$.

Therefore, by (1) and (2),

$$\frac{\sqrt{(x+y+z)\cdot xyz}}{(x+y)^2+(y+z)^2+(x+z)^2} \le \frac{\sqrt{3}}{9}\cdot\frac{3}{4}\cdot\frac{(x+y+z)^2}{(x+y+z)^2} = \frac{\sqrt{3}}{12},$$

with equality if and only if $x = y = z$. It follows that the maximum value of

$$\frac{\sqrt{(x+y+z)\cdot xyz}}{(x+y)^2+(y+z)^2+(x+z)^2}$$

is $\dfrac{\sqrt{3}}{12}$ and this is attained precisely when $x = y = z$.

Solution 4 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

We prove that the maximum is $\sqrt{3}/12$. To this end

$$\frac{\sqrt{(x+y+z)xyz}}{(x+y)^2+(y+z)^2+(z+x)^2} = \frac{\sqrt{(x+y+z)xyz}}{(x+y+z)^2+(x^2+y^2+z^2)} \le \frac{\sqrt{3}}{12}$$

and this is implied by

$$\frac{\sqrt{(x+y+z)}\dfrac{(x+y+z)^{3/2}}{3^{3/2}}}{(x+y+z)^2+\dfrac{(x+y+z)^2}{3}}\le\frac{\sqrt{3}}{12}$$

which is actually an identity and this completes the proof.

Also solved by Bruno Salgueiro Fanego (two solutions), Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Adrian Naco, Polytechnic University,Tirana, Albania; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5222: *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

Calculate without the aid of a computer the following sum

$$\sum_{n=0}^{\infty}(-1)^n\,(n+1)(n+3)\left(\frac{1}{1+2\sqrt{2}i}\right)^n,\quad \text{where } i=\sqrt{-1}.$$

Solution by David E. Manes, SUNY College at Oneonta, Oneonta, NY

The sum of the series is $\dfrac{164+103\sqrt{2}i}{108}$.

Consider the complex function $f(z)=\dfrac{1}{1+z}$ that is represented by the power series

$$f(z)=\frac{1}{1+z}=\sum_{n=0}^{\infty}(-1)^n z^n$$

on the interior of the unit circle $|z|<1$. Since $\left|\dfrac{1}{1+2\sqrt{2}i}\right|=\dfrac{1}{3}$, the power series and all of its derivatives converge absolutely for $z=\dfrac{1}{1+2\sqrt{2}i}$. For the first derivative

$$f'(z)=\frac{-1}{(1+z)^2}=\sum_{n=1}^{\infty}(-1)^n n z^{n-1}=\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)\,z^n.$$

Therefore,

$$\frac{1}{(1+z)^2}=\sum_{n=0}^{\infty}(-1)^n(n+1)z^n.$$

Differentiating again, one obtains

$$\frac{-2}{(1+z)^3}=\sum_{n=1}^{\infty}(-1)^n(n+1)z^{n-1}=\sum_{n=0}^{\infty}(-1)^{n+1}(n+2)(n+1)z^n.$$

Therefore,

$$\frac{2}{(1+z)^3} = \sum_{n=0}^{\infty}(-1)^n(n^2+3n+2)z^n.$$

Let $z = \dfrac{1}{1+2\sqrt{2}i}$. Then

$$\begin{aligned}
\frac{1}{1+z} &= \frac{1}{1+\frac{1}{1+2\sqrt{2}i}} = \frac{1+2\sqrt{2}i}{2(1+\sqrt{2}i)} = \frac{(1+2\sqrt{2}i)(1-\sqrt{2}i)}{2(1+\sqrt{2}i)(1-\sqrt{2}i)} = \frac{5+\sqrt{2}i}{6}. \\
\frac{1}{(1+z)^2} &= \left(\frac{1}{1+z}\right)^2 = \frac{1}{36}(5+\sqrt{2}i)^2 = \frac{23+10\sqrt{2}i}{36}, \\
\frac{2}{(1+z)^3} &= \left(\frac{23+10\sqrt{2}i}{36}\right)\left(\frac{5+\sqrt{2}i}{3}\right).
\end{aligned}$$

Consequently, if $z = \dfrac{1}{1+2\sqrt{2}i}$, then

$$\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n(n+1)(n+3)\left(\frac{1}{1+2\sqrt{2}i}\right)^n &= \sum_{n=0}^{\infty}(-1)^n(n^2+3n+2)z^n + \sum_{n=0}^{\infty}(-1)^n(n+1)z^n \\
&= \frac{2}{(1+z)^3} + \frac{1}{(1+z)^2} \\
&= \left(\frac{23+10\sqrt{2}i}{36}\right)\left(\frac{5+\sqrt{2}i}{3}\right) + \left(\frac{23+10\sqrt{2}i}{36}\right) \\
&= \left(\frac{23+10\sqrt{2}i}{36}\right)\left(1+\frac{5+\sqrt{2}i}{3}\right) \\
&= \left(\frac{23+10\sqrt{2}i}{36}\right)\left(\frac{8+\sqrt{2}i}{3}\right) \\
&= \left(\frac{164+103\sqrt{2}i}{108}\right),
\end{aligned}$$

as claimed.

Also solved by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany; Anastasios Kotronis, Athens, Greece; Kee-Wai Lau, Hong Kong, China; Adrian Naco, Polytechnic University,Tirana, Albania; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland;

David Stone and John Hawkins of Georgia Southern University, Statesboro, GA, and the proposer.

5223: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

a) Find the value of

$$\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\cdots\right).$$

b) More generally, if $x \in (-1,1]$ is a real number, calculate

$$\sum_{n=0}^{\infty}(-1)^n\left(\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{n+2}+\frac{x^{n+3}}{n+3}-\cdots\right).$$

Solution by Albert Stadler, Herrliberg, Switzerland

We have

$$\begin{aligned}\sum_{j=0}^{k-1}(-1)^j\frac{x^{n+1+j}}{n+1+j} &= \sum_{j=0}^{k-1}(-1)^j\int_0^x t^{n+j}dt\\ &= \int_0^x t^n\frac{1-(-t)^k}{1+t}dt\\ &= \int_0^x \frac{t^n}{1+t}dt+O\left(\int_0^x t^{n+k}dt\right)\\ &= \int_0^x \frac{t^n}{1+t}dt+O\left(\frac{1}{n+k+1}\right).\end{aligned}$$

We let k tend to infinity and get

$$\sum_{j=0}^{\infty}(-1)^j\frac{x^{n+1+j}}{n+1+j}=\int_0^x\frac{t^n}{1+t}dt.$$

Then

$$\begin{aligned}\sum_{j=0}^{k-1}(-1)^n\int_0^x\frac{t^n}{1+t}dt &= \int_0^x\frac{1}{1+t}\cdot\frac{1-(-1)^k}{1+t}dt\\ &= \int_0^x\frac{1}{(1+t)^2}dt+O\left(\int_0^x t^k dt\right)\\ &= \left[\frac{-1}{1+t}\right]_0^x+O\left(\frac{1}{k+1}\right).\end{aligned}$$

So,

$$\sum_{n=0}^{\infty}(-1)^n\left(\sum_{j=0}^{\infty}(-1)^j\,\frac{x^{n+1+j}}{n+1+j}\right)=\frac{x}{1+x}.$$

Letting $x=1$ implies that the sum of the first series is $\frac{1}{2}$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Anastasios Kotronis, Athens, Greece; Kee-Wai Lau, Hong Kong, China; Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy, and the proposer.