Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
March 15, 2018

- **5475:** *Proposed by Kenneth Korbin, New York, NY*

  Given positive integers a, b, c and d such that $\begin{cases} a+b = 14\sqrt{ab-48}, \\ b+c = 14\sqrt{bc-48}, \\ c+d = 14\sqrt{cd-48}, \end{cases}$

  with $a < b < c < d$. Express the values of b, c, and d in terms of a.

- **5476:** *Proposed by Ed Gray, Highland Beach, FL*

  Find all triangles with integer area and perimeter that are numerically equal.

- **5477:** *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu-Sevrin, Meredinti, Romania*

  Compute:

  $$L = \lim_{n\to\infty}\left(\ln n + \lim_{x\to 0}\frac{1-\sqrt{1+x^2}\,\sqrt[3]{1+x^2}\cdot\ldots\cdot\sqrt[n]{1+x^2}}{x^2}\right).$$

- **5478:** *Proposed by D. M. Btinetu-Giurgiu, "Matei Basarab" National Collge, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzu, Romania*

  Compute:

  $$\int_0^{\pi/2}\cos^2 x\left(\sin x\sin^2\left(\frac{\pi}{2}\cos x\right)+\cos x\sin^2\left(\frac{\pi}{2}\sin x\right)\right)dx.$$

- **5479:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $f:[0,1]\to\Re$ be a continuous convex function. Prove that

  $$\frac{2}{5}\int_0^{1/3}f(t)dt+\frac{3}{10}\int_0^{2/3}f(t)dt\ge\frac{5}{8}\int_0^{8/15}f(t)dt.$$

- **5480:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $n \geq 1$ be a nonnegative integer. Prove that in $C[0, 2\pi]$

$$span\{1, \sin x, \sin(2x), \ldots, \sin(nx)\} = span\{1, \sin x, \sin^2 x, \ldots, \sin^n x\}$$

if and only if $n = 1$.

We mention that $span\{v_1, v_2, \ldots, v_k\} = \sum_{j=1}^{k} a_j v_j,\ a_j \in \Re, j = 1, \ldots, k$, denotes the set of all linear combinations with $v_1, v_2, \ldots, v_k$.

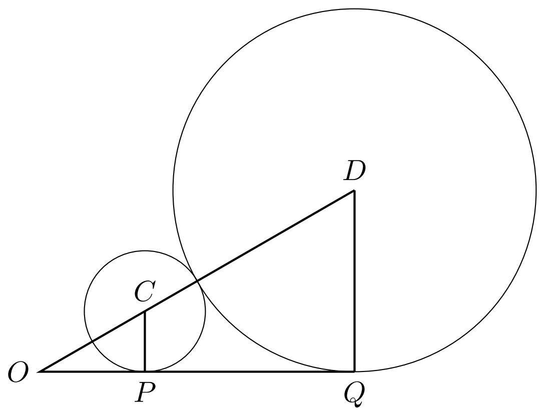
Solutions

- **5457:** *Proposed by Kenneth Korbin, New York, NY*

Given angle A with $\sin A = \dfrac{12}{13}$. A circle with radius 1 and a circle with radius x are each tangent to both sides of the angle. The circles are also tangent to each other. Find x.

Solution by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

The angle bisector passes through the centers C and D of the two circles, and the radii from the centers to the points of tangency P and Q of the circles with a side of the angle make right angles $\angle CPO$ and $\angle DQO$. Thus we have a pair of similar right triangles as follows.



Here $\angle DOQ = A/2$ and $|CD| = 1 + x$.

Suppose $x > 1$. Then $|CP| = 1$ and $|DQ| = x$. We have

$$\sin(A/2) = \frac{|CP|}{|CO|} = \frac{1}{|CO|}$$

so $|CO|\sin(A/2) = 1$. And

$$\sin(A/2) = \frac{|DQ|}{|DO|} = \frac{|DQ|}{|DC| + |CO|} = \frac{x}{1 + x + |CO|}$$

so

$$\sin(A/2) + x\sin(A/2) + |CO|\sin(A/2) = x.$$

Thus

$$x = \frac{1 + \sin(A/2)}{1 - \sin(A/2)}.$$

Now $\sin A = 12/13$ so $\cos A = \pm\sqrt{1 - \sin^2 A} = \pm 5/13$, and thus

$$\sin(A/2) = \sqrt{\frac{1 - \cos(A)}{2}} = 2\sqrt{13}/13 \text{ or } 3\sqrt{13}/13.$$

Therefore

$$x = \frac{13 + 2\sqrt{13}}{13 - 2\sqrt{13}} \approx 3.491 \text{ or } x = \frac{13 + 3\sqrt{13}}{13 - 3\sqrt{13}} \approx 10.908.$$

If $x < 1$ then scale the plane by $1/x$ and appeal to the last paragraph. This gives two more values of x:

$$x = \frac{13 - 2\sqrt{13}}{13 + 2\sqrt{13}} \approx 0.286 \text{ or } x = \frac{13 - 3\sqrt{13}}{13 + 3\sqrt{13}} \approx 0.092.$$

Editor's Comment : **David Stone and John Hawkins of Georgia Southern University, Statesboro, GA** generalized the problem. First, they proved the following lemma:

Let A be an angle, $0 < A < \pi$. Suppose a circle C_1 of radius $r = 1$ is inscribed in A and a *larger* circle C_2 of radius $R = x$ is also inscribed in A, with C_2 tangent to C_1. Then

$$x = \frac{1 + \sin\alpha}{1 - \sin\alpha}, \ \alpha = \frac{1}{2}A.$$

They proved this lemma by showing that it held when angle A is acute and also obtuse. Then they magnified the entire figure by a factor of r, so that the smaller circle C_1 has a radius of r and the larger circle C_2 has a radius of $R = rx$, and this allowed them to generalize the lemma: Let A be an angle, $0 < A < \pi$. Suppose that two circles, circle C_1 of radius r and C_2 of radius R are also inscribed in A, with C_2 tangent to C_1. Then

$$R = \frac{1 + \sin\alpha}{1 - \sin\alpha} r, \ \alpha = \frac{1}{2}A.$$

They went on to say that "with the result stated in this way, we see the co-dependency between r and R – if we know one we know the other." Applying the lemma they went on to solve the problem.

In conclusion they stated the following:

We can apply this result in several interesting ways. For example, as a Corollary, if $A = 60^\circ$, then $\sin\alpha = \sin 30^\circ = \frac{1}{2}$. Let circle C_0 of radius 1 be inscribed in A. Then we have a larger inscribed circle C_1 tangent to C_0 which has radius

$$R_1 = \frac{1 + \sin\alpha}{1 - \sin\alpha} \cdot 1 = \frac{1 + 1/2}{1 - 1/2} = 3.$$

And in continuing on in this manner we have a larger inscribed circle C_2 tangent to C_1 which has radius

$$R_2 = \frac{1+\sin\alpha}{1-\sin\alpha}\cdot 3 = 3^2.$$

There is an infinite sequence of expanding inscribed pairwise tangent circles having radii $R_n = 3^n,\ n \geq 0$.

Not to be outdone, we have a smaller inscribed circle C_{-1}, tangent to C_0 which has radius $R_{-1} = \dfrac{1-\sin\alpha}{1+\sin\alpha}\cdot 1 = \dfrac{1}{3}$

Continuing, there is an infinite sequence of shrinking inscribed pairwise tangent circles of radii $R_{-n} = \dfrac{1}{3^n}, n \geq 0$.

We could carry out this construction for any angle A, (but the numbers won't work out so nicely.)

In summary they stated: Given values for R and r, we can solve the Corollary equation for α and then find A. That is, given tangent circles of radii r and R, with $r < R$, we can compute the angle A which will "circumscribe" the circles: $A = 2\sin^{-1}\left(\dfrac{R-r}{R+r}\right)$.

Also solved by Arkady Alt, San Jose, CA; Charles Burnette, Academia Sinica, Taipei, Taiwan; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; David A. Huckaby, Angelo State University, San Angelo TX; Kee-Wai, Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Vijaya Prasad, Nalluri, India; Trey Smith, Angelo State University, San Angelo, TX; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5458:** *Proposed by Michal Kremzer, Gliwice, Silesia, Poland*

Find two pairs of integers (a, b) from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that for all positive integers n, the number

$$c = 537aaa\underbrace{b\dots b}18403$$

is composite, where there are $2n$ numbers b between a and 1 in the string above.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

Note that

$$\begin{aligned} c &= 18403 + b\cdot 10^5\cdot \underbrace{1...1}_{2n \text{ times}} + a\cdot 10^{2n+5}\cdot 111 + 537\cdot 10^{2n+8} \\ &= 18403 + b00000\cdot \underbrace{1...1}_{2n \text{ times}} + 10^{2n+5}\cdot 537aaa. \end{aligned}$$

Thus, if $(a, b) \in \{(2, 7), (9, 7)\}$, since $18403 \equiv_7 0,\ 700000 \equiv_7 0,\ 537222 \equiv_7 0$, and $537999 \equiv_7 0$, where $\equiv_7$ denotes congruence modulo 7, then

$$c \equiv_7 0 + 0\cdot 1\ \underbrace{1...1}_{2n \text{ times}} + 10^{2n+5}\cdot 0 \equiv_7 0,$$

so c is divisible by 7 and, hence composite.

Solution 2 by Ed Gray, Highland Beach, FL

The two pairs which guarantee that $c = 537aaa\underbrace{bbbbbb\ldots bb}_{2n \text{ times}}18403$ is always composite are: $a = 2, b = 7$ and $a = 9, b = 7$. We will show that with these integers, c is always divisible by 7.

A test for divisibility by 7 is as follows: double the last digit and subtract it from the remaining truncated number. If the result is divisible by 7, then so was the original number. As a simple example, consider the number 826. Double the last digit which gives 12, and subtract it from the leading truncated number, which is 82. Then $82 - 12 = 70$, which is divisible by 7, so 826 is divisible by 7.

Now consider our number. It's last digit is 3, and we double it to get 6. Subtracting 6 from the "truncated" number, we have $537aaa\underbrace{bbbbbb\ldots bb}_{2n \text{ times}}1834$.

We note that 1834 is divisible by 7; that if we let $b = 7$, every b will be divisible by 7. It remains to find 2 values for a such that $537aaa$ is divisible by 7. If $a = 2$, we have the number $537222 = 7 \cdot 76746$, and if $a = 9$, we have the number $537999 = 7 \cdot 76857$. This concludes the proof.

Solution 3 by David E. Manes, Oneonta, NY

Two pairs of integers (a, b) that satisfy the problem are $(2, 7)$ and $(9, b)$ where b is any nonnegative integer. For the pair $(2, 7)$, the integer c is always divisible by the prime 7 and for the pair $(9, b)$, c is always divisible by 11.

Given: N is a positive integer and $N = a_n 10^n + a_{n-1}10^{n-1} + \cdots + a_1 10 + a_0$. Then

$$N \equiv (100a_2 + 10a_1 + a_0) - (100a_5 + 10a_4 + a_3) + (100a_8 + 10a_7 + a_6) - \cdots \pmod 7.$$

For this case, N is divisible by 7 if and only if $N \equiv 0 \pmod 7$. Moreover,

$$N \equiv (-1)^n a_n + (-1)^{n-1} a_{n-1} + \cdots - a_1 + a_0 \pmod{11}$$

and N is divisible by 11 if and only if $N \equiv 0 \pmod{11}$. Let n be a positive integer and define

$$C_n = 537aaab\ldots b18403$$

where the number of digits b is $2n$. If $a = 2$ and $b = 7$, then $C_1 = 5372227718403$, $C_2 = 537222777718403$ and $C_3 = 53722277777718403$. Therefore, modulo 7,

$$\begin{aligned}
C_1 &\equiv 403 - 718 + 227 - 372 + 5 \equiv 4 - 4 + 3 - 1 + 5 \equiv 0 \pmod 7,\\
C_2 &\equiv 403 - 718 + 777 - 222 + 537 \equiv 4 - 4 + 0 - 5 + 5 \equiv 0 \pmod 7,\\
C_3 &\equiv 403 - 718 + 777 - 277 + 722 - 53 \equiv 4 - 4 + 0 - 4 + 1 - 4 \equiv 0 \pmod 7.
\end{aligned}$$

Thus, C_1, C_2 and C_3 are all divisible by 7 and hence, each one is composite. Furthermore, $C_{3n+1} \equiv C_1 \pmod 7$, $C_{3n+2} \equiv C_2 \pmod 7$ and $C_{3n} \equiv C_3 \pmod 7$ for all positive integers n. Hence, if $a = 2$ and $b = 7$, then C_n is always composite since all of these integers are divisible by 7.

If $a = 9$ and b is any nonnegative integer, then the number of digits in C_n is always odd and

$$\begin{aligned} C_n &\equiv 5 - 3 + 7 - a + a - a + b - b + \cdots + b - b + 1 - 8 + 4 - 0 + 3 \\ &\equiv 9 - a \pmod{11}. \end{aligned}$$

Therefore, for all positive integers n, the prime 11 is a divisor of C_n if and only if $a = 9$ and the value of b is superfluous. Hence, C_n is always composite.

Solution 4 Anthony J. Bevelacqu, University of North Dakota, Grand Forks, ND

We have

$$c = 18403 + 10^5 \cdot b \cdot (\overbrace{1 \cdots 1}^{2n}) + 10^{5+2n} \cdot (a \cdot 111 + 10^3 \cdot 537).$$

Note that $c > 18403 = 7 \cdot 11 \cdot 239$.

Since $10 \equiv -1 \bmod 11$ we have $(\overbrace{1 \cdots 1}^{2n}) \equiv 0 \bmod 11$ for any n and so $c \equiv -(a+2) \bmod 11$. Thus c will be divisible by 11 when $a = 9$ for *any* choice of the digit b and for any non-negative n.

Now if $b = 7$ we have $c \equiv 10^{5+2n}(6a + 2) \bmod 7$. Thus c will be divisible by 7 when $a = 2$ for any number of digits $b = 7$.

Therefore c will be composite when $(a, b) = (9, b)$ for any choice of the digit b and when $(a, b) = (2, 7)$.

Editor's comments : Most of the other solvers of this problem noticed that an even number of b digits forces the number formed by them alone, to be divisible by 11. Hence, they found the value $a = 9$ makes the number $537aaa18403$ divisible by 11, and so $(9, \text{any digit})$ solves the problem. The solutions listed above pick up another ordered pair. But then **The Honor Students at Ashland University in Ashland, Ohio** upped the ante by finding additional ordered pairs to $(9, \text{ any other digit})$. Using MAPLE they found 6 pairs of values for (a, b) that satisfy the problem. They checked these values for all positive integers $n \leq 25$. Letting $c = 537aaa\underbrace{b \ldots b}18403$ they found that:

(a, b)	c is divisible by
$(2, 7)$	7
$(4, 1)$	29
$(4, 5)$	13
$(6, 5)$	17
$(6, 9)$	59
$(7, 5)$	89

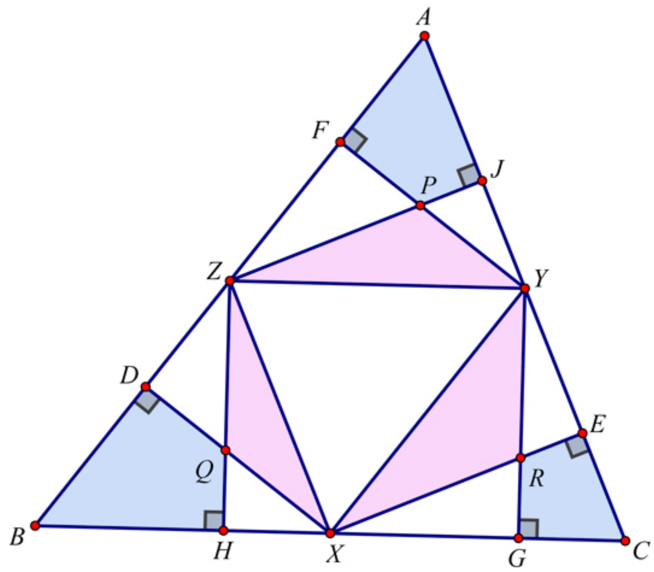
David Stone and John Hawkins of Georgia Southern University, Statesboro, GA found all of the solutions in the above table and an additional one $(4, 8)$, which is divisible by 13. They also found that if $b = 0$ were allowed, then $(2, 0)$ is divisible by 7, for all $n \geq 0$. With respect to the pair $(4, 8)$ they stated that it seemingly has a unique property. If $c_n = 537aaab....b18403$ as defined in the problem, then no single prime divides all c_n, but 7 divides all c_{3k}, 3 divides all c_{3k+1}, and and 13 divides all c_{3k+2}.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Pat Costello, Eastern Kentucky University, Richmond, KY; Kee-Wai Lau, Hong

Kong, China; Zachary Morgan, student at Eastern Kentucky University, Richmond, KY; Nathan Russell, Eastern Kentucky University, Richmond, KY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5459:** *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

Triangle ABC is an arbitrary acute triangle. Points X, Y, and Z are midpoints of three sides of $\triangle ABC$. Line segments XD and XE are perpendiculars drawn from point X to two of the sides of $\triangle ABC$. Line segments YF and YG are perpendiculars drawn from point Y to two of the sides of $\triangle ABC$. Line segments ZJ and ZH are perpendiculars drawn from point Z to two of the sides of $\triangle ABC$. Moreover,
$P = ZJ \cap FY$, $Q = ZH \cap DX$, and $R = YG \cap XE$. Three of the triangles, and three of the quadrilaterals in the figure are shaded. If the sum of the areas of the three shaded triangles is 5, find the sum of the areas of the three shaded quadrilaterals.



Solution 1 by David A. Huckaby, Angelo State University, San Angelo, TX

Let a be the area of triangle ABC. Since Y and Z are the midpoints of AC and AB, respectively, $\triangle AYZ$ is similar to $\triangle ACB$ with a scale factor of 2, so that the area of $\triangle AYZ$ is $\frac{1}{4}a$. Similarly, the areas of $\triangle BXZ$ and $\triangle CXY$ are each $\frac{1}{4}a$, and therefore the area of $\triangle XYZ$ is also $\frac{1}{4}a$.

The area of rectangle $GYZH$ is $\frac{1}{2}a$, since it has the same height and base as $\triangle XYZ$. Similarly, the areas of rectangles $FYXD$ and $EXZJ$ are each $\frac{1}{2}a$.

Consider the sum of the areas of these three rectangles:

$$\begin{aligned}\text{area of three rectangles} &= \text{area of six outer white triangles}\\ &\quad +2(\text{area of three pink triangles}) + 3(\text{area of } \triangle XYZ),\end{aligned}$$

that is,

$$3(\frac{1}{2}a) = \text{area of six outer white triangles} + 2(5) + 3(\frac{1}{4}a),$$

so that the sum of the areas of the six outer white triangles is $\frac{3}{4}a - 10$.

Now consider the sum of the areas of triangles AYZ, BXZ, and CXY:

$$\begin{aligned}\text{area of triangles } AYZ,\ BXZ, \text{ and } CXY &= \text{area of six outer white triangles}\\ &\quad +\text{area of three pink triangles} + \text{area of three blue quadrilaterals},\end{aligned}$$

that is,

$$3(\frac{1}{4}a) = \left[\frac{3}{4}a - 10\right] + 5 + \text{area of three blue quadrilaterals},$$

so that the sum of the areas of the three blue quadrilaterals is 5.

Solution 2 by Andrea Fanchini, Cantú, Italy

We use barycentric coordinates and the usual Conway's notations with reference to the triangle ABC. Then we have $X(0:1:1)$, $Y(1:0:1)$, $Z(1:1:0)$.

• *Coordinates of points D, E, F, G, H, J.*

Line segments XD and XE perpendiculars drawn from point X to two of the sides of $\triangle ABC$ are

$$XAB_{\infty\perp} : (c^2+S_A)x - S_By + S_Bz = 0, \quad XAC_{\infty\perp} : (b^2+S_A)x + S_Cy - S_Cz = 0$$

therefore the points D, E have coordinates

$$D = XAB_{\infty\perp} \cap AB = (S_B : c^2+S_A : 0), \quad E = XAC_{\infty\perp} \cap AC = (S_C : 0 : b^2+S_A)$$

then cyclically

$$G = (0 : S_C : a^2+S_B), \quad J = (b^2+S_C : 0 : S_A), \quad F = (c^2+S_B : S_A : 0), \quad H = (0 : a^2+S_C : S_B)$$

• *Coordinates of point P, Q, R.*

Coordinates of point P are

$$P = ZAC_{\infty\perp} \cap YAB_{\infty\perp} = (2S^2 - a^2S_A : S_AS_C : S_AS_B)$$

then cyclically

$$Q = (S_BS_C : 2S^2 - b^2S_B : S_AS_B), \qquad R = (S_BS_C : S_AS_C : 2S^2 - c^2S_C)$$

• *Areas of the three shaded triangles.*

Areas of the three shaded triangles are

$$[PZY] = \frac{S_BS_C}{4S^2}[ABC], \qquad [QZX] = \frac{S_AS_C}{4S^2}[ABC], \qquad [RXY] = \frac{S_AS_B}{4S^2}[ABC]$$

If the sum of the areas of the three shaded triangles is 5, we have

$$[PZY] + [QZX] + [RXY] = \frac{[ABC]}{4}, \quad \Rightarrow \quad [ABC] = 20$$

• *Areas of the three shaded quadrilaterals.*

Area of the quadrilateral $[AFPJ]$ is given from $[AFJ] + [PFJ]$ so

$$[AFJ] = \frac{S_A^2}{4b^2c^2}[ABC], \qquad [PFJ] = \frac{S_A^2S_BS_C}{4b^2c^2S^2}[ABC], \quad \Rightarrow \quad [AFPJ] = \frac{S_A^2(S^2+S_BS_C)}{4b^2c^2S^2}[ABC]$$

then cyclically

$$[BDQH] = \frac{S_B^2(S^2+S_AS_C)}{4a^2c^2S^2}[ABC], \qquad [CERG] = \frac{S_C^2(S^2+S_AS_B)}{4a^2b^2S^2}[ABC]$$

therefore

$$[AFPJ]+[BDQH]+[CERG] = \frac{a^2S_A^2(S^2+S_BS_C) + b^2S_B^2(S^2+S_AS_C) + c^2S_C^2(S^2+S_AS_B)}{4a^2b^2c^2S^2}[ABC]$$

but $[ABC] = 20$ then

$$[AFPJ]+[BDQH]+[CERG] = 5\frac{S^2(a^2S_A^2 + b^2S_B^2 + c^2S_C^2) + S_AS_BS_C(a^2S_A + b^2S_B + c^2S_C)}{a^2b^2c^2S^2}$$

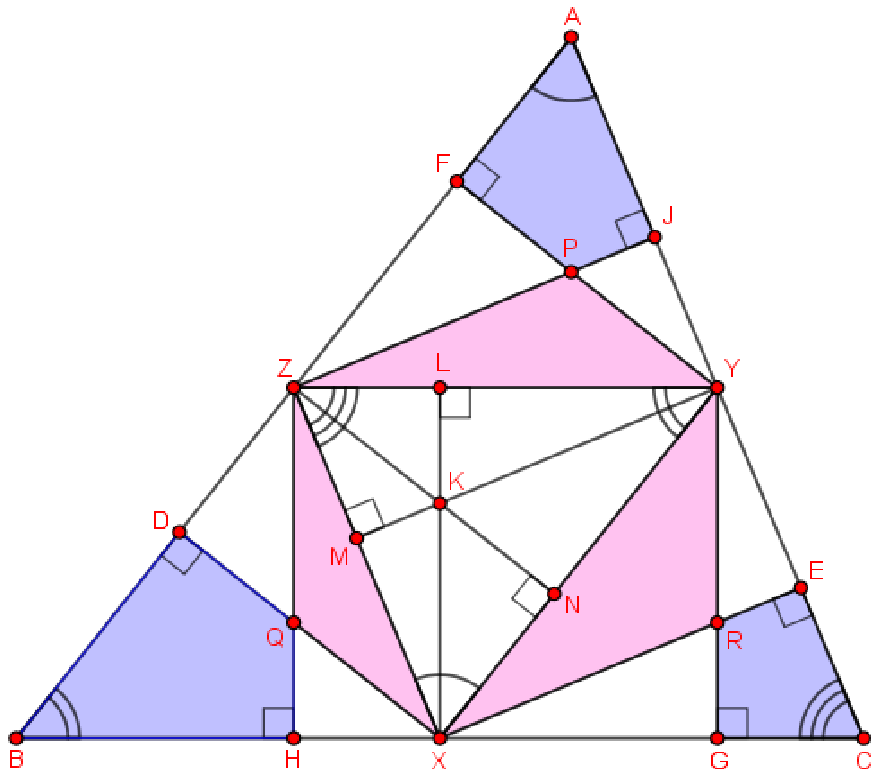
now $a^2S_A + b^2S_B + c^2S_C = 2S^2$ then

$$[AFPJ] + [BDQH] + [CERG] = 5\frac{a^2S_A^2 + b^2S_B^2 + c^2S_C^2 + 2S_AS_BS_C}{a^2b^2c^2}$$

finally $a^2S_A^2 + b^2S_B^2 + c^2S_C^2 + 2S_AS_BS_C = a^2b^2c^2$ so we have that also

$$[AFPJ] + [BDQH] + [CERG] = 5.$$

Solution 3 by Nikos Kalapodis, Patras, Greece



We denote with $[S]$ the area of shape S.
Let XL, YM and ZN be the heights of triangle XYZ and K its orthocenter.
Then the quadrilaterals $PZKY$, $QXKZ$ and $RYKX$ are parallelograms.
It follows that $[PZY] = [KZY]$, $[QZX] = [KZX]$, and $[RXY] = [KXY]$.
Therefore $[PZY] + [QZX] + [RXY] = [KZY] + [KZX] + [KXY] = [XYZ]$ (1).
Furthermore since the triangles AZY, BXZ, CYX and XYZ are congruent with orthocenters P, Q, R and K respectively, it easily follows that $[AFPJ] = [XNKM]$, $[BHQD] = [YLKN]$ and $[CERG] = [ZMKL]$.
Therefore
$[AFPJ] + [BHQD] + [CERG] = [XNKM] + [YLKN] + [ZMKL] = [XYZ]$ (2).
From (1) and (2) we get that
$[AFPZ] + [BHQD] + [CERG] = [PZY] + [QZX] + [RXY] = 5$.

Solution 4 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Triangles ΔAZY, ΔZBX and ΔYXC are equal. Also the sum of the areas of the saded triangles is equal to the area of for example ΔAZY. Also the sum of the areas of the three shaded quadrilaterals is equal to the area of one of the triangles, for example ΔAZY. Therefore, the requested sum is equal to 5.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai, Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Sachit Misra, Nelhi, India; Neculai Stanciu,"George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA and the proposer.

- **5460:** *Proposed by Ángle Plaza,Universidad de Las Palmas de Gran Canaria, Spain*

If $a, b > 0$ and $x, y > 0$ then prove that

$$\frac{a^3}{ax+by}+\frac{b^3}{bx+ay}\geq\frac{a^2+b^2}{x+y}.$$

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo,TX

Since $a, b, x, y > 0$, we have

$$\begin{aligned}
&a^3\left(bx+ay\right)\left(x+y\right)+b^3\left(ax+by\right)\left(x+y\right)-\left(a^2+b^2\right)\left(ax+by\right)\left(bx+ay\right)\\
&=a^2\left(bx+ay\right)\left[a\left(x+y\right)-\left(ax+by\right)\right]+b^2\left(ax+by\right)\left[b\left(x+y\right)-\left(bx+ay\right)\right]\\
&=a^2\left(a-b\right)y\left(bx+ay\right)+b^2\left(b-a\right)y\left(ax+by\right)\\
&=\left(a-b\right)y\left[a^2\left(bx+ay\right)-b^2\left(ax+by\right)\right]\\
&=\left(a-b\right)y\left[ab\left(a-b\right)x+\left(a^3-b^3\right)y\right]\\
&=\left(a-b\right)^2y\left[abx+\left(a^2+ab+b^2\right)y\right]\\
&\geq 0, \qquad (1)
\end{aligned}$$

with equality if and only if $a = b$.

Since $a, b, x, y > 0$, we need only to divide (1) by the positive quantity $\left(ax+by\right)\left(bx+ay\right)\left(x+y\right)$ and to re-arrange terms to obtain the desired inequality. Further, equality is attained if and only if $a = b$.

Solution 2 by Henry Ricardo, Westchester Area Math Circle, NY

Using the Engel form of the Cauchy-Schwarz inequality (or Bergström's inequality) and the AGM inequality, we see that

$$\begin{aligned}
\frac{a^3}{ax+by}+\frac{b^3}{bx+ay} &= \frac{a^4}{a^2x+aby}+\frac{b^4}{b^2x+aby}\\
&\geq \frac{(a^2+b^2)^2}{(a^2+b^2)x+2aby}\\
&\geq \frac{(a^2+b^2)^2}{(a^2+b^2)x+(a^2+b^2)y} = \frac{a^2+b^2}{x+y}.
\end{aligned}$$

Solution 3 by Anna Valkova Tomova, Varna, Bulgaria

We move the expression to the left of the right side of the inequality. Now we have to prove that the new left part is non-negative. Again we will use the capabilities of the mathematical site <http://www.worframalpha.com> to examine the transformed look of this new left-hand side of the inequality.

$$\begin{aligned}
& \frac{a^3}{ax+by}+\frac{b^3}{bx+ay}-\frac{a^2+b^2}{x+b} \\
= \quad & \frac{y(a^4y+a^3bx-a^3by-2a^2b^2x+ab^3x-ab^2y+b^4y)}{(x+y)(ay+bx)(ax+by)}. \\
= \quad & \frac{y(a-b)^2(a^2y+abx+aby+b^2y)}{(x+y)ay+bx)(ax+by)}.
\end{aligned}$$

Since the numbers involved in the expression are conditionally positive, we have proved the inequality because the equivalent expression is positive too.

Conclusion: The application of information technology enhances the quality of education in mathematics in all of its stages of study. Of course, it should be checked at every stage so as not to allow ridiculous errors. In this sense, "E-Mathematics" does not replace the classic, it continues development with new, more efficient vehicles.

Editor's Comments : **Brian Bradie of Christopher Newport University in Newport News VA** stated that this problem is a generalization of two inequalities that appeared in Problem B-1201 in the February 2017 issue of the Fibonacci Quarterly:

$$\frac{a^3}{aF_n+bF_{n+1}}+\frac{b^3}{bF_n+aF_{n+1}}\geq\frac{a^2+b^2}{F_{n+2}}$$

$$\frac{a^3}{aL_n+bL_{n+1}}+\frac{b^3}{bL_n+aL_{n+1}}\geq\frac{a^2+b^2}{L_{n+2}}$$

Three other generalizations of this problem were made by **D.M.Bătinetu-Giurgiu of the "Matei Basarab" National College in Bucharest, Romania.**

1. A generalization with "two variables:"

If $m\geq 0$ and $a,b,x,y>0$, then $\dfrac{a^{m+2}}{(ax+by)^m}+\dfrac{b^{m+2}}{(bx+ay)^m}\geq\dfrac{a^2+b^2}{(x+y)^m}$.

Proof:

$$\frac{a^{m+2}}{(ax+by)^m}+\frac{b^{m+2}}{(bx+ay)^m}=\frac{a^{2m+2}}{(a^2x+aby)^m}+\frac{b^{2m+2}}{(b^2x+aby)^m}$$

$$=\frac{(a^2)^{m+1}}{(a^2x+aby)^m}+\frac{(b^2)^{m+1}}{(b^2x+aby)^m}\overbrace{\geq}^{J.Radon}\frac{(a^2+b^2)^{m+1}}{(a^2+b^2)x+2aby)^m}\overbrace{\geq}^{AM\geq GM}\frac{(a^2+b^2)^{m+1}}{((a^2+b^2)x+(a^2+b^2)y)^m}$$

$$= \quad \frac{(a^2+b^2)^{m+1}}{(a^2+b^2)^m(x+y)^m} = \frac{a^2+b^2}{(x+y)^m}. \quad \text{Q.E.D.}$$

Corollary 1. If $m = 1$, then we obtain the problem 5460.

2. A generalization with "three variables:"

If $m \geq 0$ and $a, b, c, x, y, z > 0$, then

$$\frac{a^{m+2}}{(ax+by+cz)^m} + \frac{b^{m+2}}{(bx+cy+az)^m} + \frac{c^{m+2}}{(cx+ay+bz)^m} \geq \frac{a^2+b^2+c^2}{(x+y+z)^m}.$$

Proof:

$$\frac{a^{m+2}}{(ax+by+cz)^m} + \frac{b^{m+2}}{(bx+cy+az)^m} + \frac{c^{m+2}}{(cx+ay+bz)^m}$$

$$= \quad \frac{(a^2)^{m+1}}{(a^2x+aby+acz)^m} + \frac{(b^2)^{m+1}}{(b^2x+bcy+abz)^m} + \frac{(c^2)^{m+1}}{(c^2x+acy+bcz)^m}$$

$$\overbrace{\geq}^{J.Radon} \quad \frac{(a^2+b^2+c^2)^{m+1}}{((a^2+b^2+c^2)x+(ab+bc+ca)y+(bc+ca+ab)z)^m}$$

$$\geq \quad \frac{(a^2+b^2+c^2)^{m+1}}{(a^2+b^2+c^2)^m(x+y+z)^m} = \frac{a^2+b^2+c^2}{(x+y+z)^m}, \quad \text{Q.E.D.}$$

In the last inequality we are utilizing the fact that $a^2+b^2+c^2 \geq ab+bc+ca$ where $a, b, c > 0$.

3. A generalization with "n variables:"

If $t, x, y, a_k > 0, n \in N, n \geq 2, n \in \{1, 2, \ldots, n\}$ such that
$t\sum_{t=1}^{n} a_k^2 \geq \sum_{k=1}^{n} a_k a_{k+1},\ a_{n+1} = a_1$, then

$$\sum_{k=1}^{n} \frac{a_k^3}{xa_k + ya_{k+1}} \geq \frac{1}{x+ty}\sum_{k=1}^{n} a_k^2.$$

Proof:

$$\sum_{k=1}^{n} \frac{a_k^3}{xa_k + ya_{k+1}} = \sum_{k=1}^{n} \frac{(a_k^2)^2}{xa_k^2 + ya_k a_{k+1}} \overbrace{\geq}^{Bergstrom} \frac{\left(\sum_{k=1}^{n} a_k^2\right)^2}{\sum_{k=1}^{n}(xa_k^2 + ya_k a_{k+1})}$$

$$\frac{\left(\sum_{k=1}^{n} a_k^2\right)^2}{x\sum_{k=1}^{n} a_k^2 + y\sum_{k=1}^{n} a_k a_{k+1}} = \frac{\left(\sum_{k=1}^{n} a_k^2\right)^2}{x\sum_{k=1}^{n} a_k^2 + ty\sum_{k=1}^{n} a_k^2} = \frac{1}{x+ty}\sum_{k=1}^{n} a_k^2, \quad \text{Q.E.D.}$$

Also solved by Arkady Alt (3 solutions), San Jose, CA; Bruno Salgueiro Fanego Viveiro, Spain; D.M.Bătinetu-Giurgiu of the "Matei Basarab" National College in Bucharest, Romania; D.M.Bătinetu-Giurgiu of the "Matei Basarab" National College in Bucharest, Romania with Neculai Stanciu, "George Emil Palade" School, Buzău, Romania; Brian Bradie, Christopher Newport University, Newport News, VA; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai, Lau, Hong Kong, China; Nikos Kalapodis, Patras, Greece; Moti Levy, Rehovot, Israel; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Neculai Stanciu, "George Emil Palade" School in Buzau, Romanina with Titu Zvonaru of Comănesti, Romania; and the proposer.

- **5461:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Compute the following sum:

$$\sum_{n=1}^{\infty} \frac{\cos(2n-1)}{(2n-1)^2}.$$

Solution 1 by Brian Bradie, Christopher Newport University, Newport, VA

Consider the function $f(x) = \frac{\pi}{2} - x$ on the interval $[0, \pi]$. Because

$$\frac{2}{\pi}\int_0^{\pi} \left(\frac{\pi}{2} - x\right) dx = 0$$

and, for positive integer n,

$$\frac{2}{\pi}\int_0^{\pi} \left(\frac{\pi}{2} - x\right) \cos nx\, dx = \begin{cases} \dfrac{4}{n^2\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

it follows that the Fourier cosine series for f is

$$\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}.$$

The function f is continuous at $x = 1$, so

$$f(1) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)}{(2n-1)^2};$$

therefore,

$$\sum_{n=1}^{\infty} \frac{\cos(2n-1)}{(2n-1)^2} = \frac{\pi}{4} f(1) = \frac{\pi}{4}\left(\frac{\pi}{2} - 1\right).$$

Solution 2 by Ed Gray, Highland Beach, FL

Many of these infinite series can be solved by finding a function whose Fourier series expansion results in the given series. The series at hand reprsents an even function so suitable candidates are functions like $f(x) = x^2$, $f(x) = |x|$, etc. A perusal of some functions reveals that the function $f(x) = |x|$, $\pi < x < \pi$, seems just was we need.

The expression is:

1. $f(x) = \dfrac{\pi}{2} - \dfrac{4}{\pi} \displaystyle\sum_{k \geq 1, \text{odd}}^{\infty} \frac{\cos(kx)}{k^2}.$

Since the sum involves odd terms only, we let $k = 2n - 1$. Further, we eliminate x by letting $x = 1$.(Since an even function, $x = -1$ would do just as well.) In either case, $f(1) = f(-1) = |1| = 1$ and equation (1) becomes:

2. $1 = \dfrac{\pi}{2} - \dfrac{4}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{\cos(2n-1)}{(2n-1)^2}$, or

3. $\dfrac{4}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{\cos(2n-1)}{(2n-1)^2} = \frac{\pi}{2} - 1.$

Multiplying by $\dfrac{\pi}{4}$.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(2n-1)}{(2n-1)^2} = \frac{\pi^2}{8} - \frac{\pi}{4}.$

Solution 3 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\cos(2n-1)}{(2n-1)^2} &= \frac{1}{2}\sum_{n=1}^{\infty}(e^{i(2n-1)} + e^{-i(2n-1)}) \int_0^1 t^{2n-2} dt \int_0^1 u^{2n-2} du \\
&= \frac{e^{-i}}{2} \int_0^1 \frac{dt}{t^2} \int_0^1 \frac{du}{u^2} \sum_{n=1}^{\infty} (tue^{i})^{2k} + \frac{e^{i}}{2} \int_0^1 \frac{dt}{t^2} \int_0^1 \frac{du}{u^2} \sum_{n=1}^{\infty} (tue^{-i})^{2k} \\
&= \frac{e^{-i}}{2} \int_0^1 dt \int_0^1 du \frac{e^{2i}}{1-(tu)^2 e^{2i}} + \frac{e^{i}}{2} \int_0^1 dt \int_0^1 du \frac{e^{-2i}}{1-(tu)^2 e^{-2i}}.
\end{aligned}$$

The change $x = tu$, $y = u$ yields

$$\frac{e^{i}}{2} \int_0^1 \frac{dy}{y} \int_0^y dx \frac{1}{1 - x^2 e^{2i}} + \frac{e^{-i}}{2} \int_0^1 \frac{dy}{y} \int_0^y dx \frac{1}{1 - x^2 e^{-2i}}$$

$$
\begin{aligned}
&= \frac{e^i}{4}\int_0^1 \frac{dy}{y}\int_0^y dx\left(\frac{1}{1-xe^i}+11+xe^i\right)+\frac{e^{-i}}{4}\int_0^1 \frac{dy}{y}\int_0^y dx\left(\frac{1}{1-xe^{-i}}+\frac{1}{1+xe^{-i}}\right)\\
&= \frac{1}{4}\int_0^1 \frac{dy}{y}\Big[\mathrm{Ln}(1-xe^i\Big]\Big|_y^0+\frac{1}{4}\int_0^1 \frac{dy}{y}\Big[\mathrm{Ln}(1+xe^i\Big]\Big|_0^y\\
&= \frac{1}{4}\int_0^1 \frac{dy}{y}\Big[\mathrm{Ln}(1-xe^{-i}\Big]\Big|_y^0+\frac{1}{4}\int_0^1 \frac{dy}{y}\Big[\mathrm{Ln}(1+xe^{-i}\Big]\Big|_0^y\\
&= \frac{-1}{4}\int_0^1 \frac{\mathrm{Ln}(1-ye^i)}{y}dy+\frac{1}{4}\int_0^1 \frac{\mathrm{Ln}(1+ye^i)}{y}dy+\frac{-1}{4}\int_0^1 \frac{\mathrm{Ln}(1-ye^{-i})}{y}dy+\frac{1}{4}\int_0^1 \frac{\mathrm{Ln}(1+ye^{-i})}{y}dy\\
&= \frac{-1}{4}\int_0^{e^i} \frac{\mathrm{Ln}(1-y)}{y}dy+\frac{1}{4}\int_0^{-e^i} \frac{\mathrm{Ln}(1-y)}{y}dy+\frac{-1}{4}\int_0^{e^{-i}} \frac{\mathrm{Ln}(1-y)}{y}dy+\frac{1}{4}\int_0^{-e^{-i}} \mathrm{Ln}(1-y)ydy\\
&= \frac{1}{4}\mathrm{Li}_2(e^i)-\frac{1}{4}\mathrm{Li}_2(-e^i)+\frac{1}{4}\mathrm{Li}_2(e^{-i})-d\frac{1}{4}\mathrm{Li}_2(-e^{-i}).
\end{aligned}
$$

The relation

$$
\mathrm{Li}_2(\frac{1}{z})+\mathrm{Li}_2(z)=-\frac{\pi^2}{6}-\frac{(\mathrm{Ln}(-z))^2}{2}
$$

gives

$$
\frac{1}{4}\left(-\frac{\pi^2}{6}-\frac{(\mathrm{Ln}(-e^i))^2}{2}\right)-\frac{1}{4}\left(-\frac{\pi^2}{6}-\frac{(\mathrm{Ln}(e^i))^2}{2}\right)=-\frac{1}{8}(i(\pi-1))^2+\frac{i^2}{8}=\frac{\pi^2-2\pi}{8}.
$$

Solution 4 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

A given 2π-periodic function f can be represented as by the convergent series

$$
f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos(nx)+b_n\sin(nx)].
$$

The convergence of the series means that the sequence $(s_n(x))$ of partial sums, defined by

$$
s_n(x)=\frac{a_0}{2}+\sum_{k=1}^{n}[a_n\cos(kx)+b_n\sin(kx)],
$$

converges at a given point x to $f(x)$, $s_n(x)\to f(x)$. Consider f to be a 2π-periodic function defined by $f(x)=|x|$ for $x\in[-\pi,\pi]$. Note that f is even. Since the product of even functions with the odd function is odd, it follows that $\displaystyle\int_{-n}^{n} f(x)\sin(nx)dx=0$.

Hence $b_n=0$ for all $n\geq 1$. To compute a_n note that the product of an even function with an even function is even, so that

$$
a_n=\frac{1}{n}\int_{-n}^{n} f(x)\cos(nx)dx=\frac{2}{n}\int_0^{\pi} f(x)\cos(nx)dx=\frac{2}{n}\int_0^{\pi} x\cos(nx)dx.
$$

If $n = 0$ then $a_0 = \dfrac{2}{n}\displaystyle\int_0^\pi x dx = \pi$. Hence $a_n = 0$ if n is even and $a_n = -\dfrac{4}{n^2\pi}$ when n is odd, and hence

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}.$$

Since the series $\sum \dfrac{1}{n^2}$ converges, the M$-$Weierstarss test implies that the above series converges. Furthermore, f is continuous at every point and it is smooth except at points $m\pi$, with m odd. Hence the Fourier series of f converges to f at every point. In particular,

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2},$$

for $x \in [-\pi, \pi]$. Substituting $x = 0$, we find that

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{n^2}{8},$$

and substituting $x = 1$, we find that

$$\sum_{n=1}^{\infty}\frac{\cos(2n-1)}{(2n-1)^2} = \frac{\pi}{4}\left(\frac{\pi}{2} - 1\right) \approx 0.448302.$$

Also solved by Bruno Salgueiro Fanego Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Henry Ricardo, Westchester Area Math Circle, NY; Albert Stadler, Herrliberg, Switzerland; Anna V. Tomova, Varna, Bulgaria, and the proposer.

- **5462:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $n \geq 1$ be an integer. Calculate

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\left(1 + \sqrt{\sin(2x)}\right)^n}\mathrm{d}x.$$

Solution 1 by Moti Levy, Rehovot, Israel

First simplification by setting $t = 4x - \pi$,

$$I_n := \int_0^{\frac{\pi}{2}} \frac{\cos x}{\left(1+\sqrt{\sin 2x}\right)^n}dx = \frac{1}{4}\int_{-\pi}^{\pi}\frac{\cos\left(\frac{t}{4}+\frac{\pi}{4}\right)}{\left(1+\sqrt{\cos\frac{t}{2}}\right)^n}dt = \frac{\sqrt{2}}{4}\int_0^{\pi}\frac{\cos\frac{t}{4}}{\left(1+\sqrt{\cos\frac{t}{2}}\right)^n}dt$$

Further simplification by the change of variable $w = \frac{1-\sqrt{\cos\frac{t}{2}}}{1+\sqrt{\cos\frac{t}{2}}}$:

$$\cos\frac{t}{2} = \left(\frac{1-w}{1+w}\right)^2, \quad \cos\frac{t}{4} = \frac{\sqrt{2}}{2}\sqrt{1+\left(\frac{1-w}{1+w}\right)^2}, \quad \sin\frac{t}{2} = \sqrt{1-\left(\frac{1-w}{1+w}\right)^4}$$

$$I_n = \frac{1}{2^n}\int_0^1 (1+w)^{n-2}\left(\frac{1}{\sqrt{w}} - \sqrt{w}\right) dw.$$

By the binomial theorem,

$$(1+w)^{n-2} = \sum_{k=0}^{n-2}\binom{n-2}{k} w^k, \quad n \ge 2$$

after interchanging integration and summation,

$$\begin{aligned} I_n &= \frac{1}{2^n}\sum_{k=0}^{n-2}\binom{n-2}{k}\int_0^1 \left(w^{k-\frac{1}{2}} - w^{k+\frac{1}{2}}\right) dw \\ &= \frac{1}{2^n}\sum_{k=0}^{n-2}\binom{n-2}{k}\left(\frac{1}{k+\frac{1}{2}} - \frac{1}{k+\frac{3}{2}}\right) \\ &= \frac{1}{2^n}\sum_{k=0}^{n-2}\frac{\binom{n-2}{k}}{\left(k+\frac{1}{2}\right)\left(k+\frac{3}{2}\right)}, \qquad n \ge 2. \end{aligned}$$

For $n = 1$,

$$\begin{aligned} I_1 &= \frac{1}{2}\int_0^1 \frac{1-w}{1+w}\frac{1}{\sqrt{w}} dw = \int_0^1 \frac{1-x^2}{1+x^2} dx \\ &= \int_0^1 \left(\frac{2}{1+x^2} - 1\right) dx = \frac{\pi}{2} - 1. \end{aligned}$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

The transformation $x \to \frac{\pi}{2} - x$ yields

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\left(1+\sqrt{\sin(2x)}\right)^n} dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\left(1+\sqrt{\sin(2x)}\right)^n} dx.$$

Therefore

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\left(1+\sqrt{\sin(2x)}\right)^n} dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{\left(1+\sqrt{\sin(2x)}\right)^n} dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sqrt{(\cos x + \sin x)^2}}{\left(1+\sqrt{\sin(2x)}\right)^n} dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sqrt{(1+\sin(2x))}}{\left(1+\sqrt{\sin(2x)}\right)^n} dx = \int_0^{\frac{\pi}{4}} \frac{\sqrt{(1+\sin(2x))}}{\left(1+\sqrt{\sin(2x)}\right)^n} dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sqrt{1+\sin x}}{\left(1+\sqrt{\sin x}\right)^n} dx$$

$$\underbrace{=}_{y=\sqrt{\sin x}} \int_0^1 \frac{\sqrt{1+y^2}}{(1+y)^n} \cdot \frac{y}{\sqrt{1-y^4}} dy = \int_0^1 \frac{1}{(1+y)^n} \cdot \frac{y}{\sqrt{1-y^2}} dy$$

$$\underbrace{=}_{y=\frac{2z}{1+z^2}} \int_0^1 \frac{1}{1+\frac{2z}{(1+z^2)^n}} \cdot \frac{2z}{1+z^2} \cdot \frac{1+z^2}{1-z^2} \cdot \frac{2(1-z^2)}{(1+z^2)^2} dz = 4\int_0^1 \frac{z(1+z^2)^{n-2}}{(1+z)^{2n}} dz = \int_0^\infty \frac{z(1+z^2)^{n-2}}{(1+z)^{2n}} dz,$$

where we have used that $\int_0^1 \frac{z(1+z^2)^{n-2}}{(1+z)^n} dz = \int_0^\infty \frac{z(1+z^2)^{n-2}}{(1+z)^n} dz,$

as follows by performing the change of variables $z \to 1/z$.

Obviously, for $n=0$, $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{\left(1+\sqrt{(2x)}\right)^0} dx = 1.$

For $n=1$ we have

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\left(1+\sqrt{(2x)}\right)} dx = 2\int_0^{\infty} \frac{z}{(1+z)^2(1+z^2)} dz = \int_0^{\infty} \left(\frac{1}{1+z^2} - \frac{1}{(1+z)^2}\right) dz = \frac{\pi}{2} - 1.$$

For $n \geq 2$ we use the Binomial Theorem to expand the integrand.

$$2\int_0^{\infty} \frac{z(1+z^2)^{n-2}}{(1+z)^{2n}} dz = 2\int_0^{\infty} \frac{z(1+2z+z^2-2z)^{n-2}}{(1+z)^{2n}} dz$$

$$=2\sum_{j=0}^{n-2} \binom{n-2}{j} (-2)^j \int_0^{\infty} \frac{z^{j+1}}{(1+z)^{2j+4}} dz = 2\sum_{j=0}^{n-2} \binom{n-2}{j} (-2)^j \frac{(j+1)!^2}{(2j+3)!}$$

$$= -(n-2)! \sum_{j=0}^{n-2} (-2)^{j+1}(j+1) \frac{(j+1)!}{(n-2-j)!(2j+3)!},$$

where we have used that

$$\int_0^{\infty} \frac{z^{j+1}}{(1+z)^{2j+4}} dz = \frac{j+1}{2j+3} \int_0^{\infty} \frac{z^j}{(1+z)^{2j+3}} dz = \frac{(j+1)j}{(2j+3)(2j+2)} \int_0^{\infty} \frac{z^{(j-1)}}{(1+z)^{(2j+2)}} dz =$$

$$\ldots = \frac{(j+1)j}{(2j+3)(2j+2)\cdots(j+3)} \int_0^{\infty} \frac{1}{(1+z)^{(j+3)}} dz = \frac{(j+1)!^2}{(2j+3)!},$$ applying repeated integration by parts.

So the integral evaluates to a rational number for all natural numbers n except for $n=1$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

Denote the integral of the problem by I_n. We show that

$$I_n = \begin{cases} \dfrac{\pi-2}{2}, & \text{for } n=1 \\[2ex] \dfrac{n\displaystyle\sum_{k=0}^{n-1} \frac{\binom{n-1}{k}}{2k+1} - 2^{n-1}}{(n-1)2^{n-1}}, & \text{for } n \geq 2. \end{cases} \quad (1)$$

Let $J_n = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\left(1+\sqrt{\sin(2x)}\right)^n} dx$. By substituting $x = \dfrac{\pi}{2} - y$ into I_n, we see that $I_n = J_n$. Since $1+\sin(2x) = (\cos x + \sin x)^2$, so

$$I_n = \frac{I_n + J_n}{2} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sqrt{1+\sin(2x)}}{\left(1+\sqrt{\sin(2x)}\right)^n} dx = \frac{1}{4}\int_0^{\pi} \frac{\sqrt{1+\sin x}}{\left(1+\sqrt{\sin x}\right)^n} dx.$$

By putting $x = \pi - y$ we see that

$$\int_{\frac{\pi}{2}}^{\pi} \frac{\sqrt{1+\sin x}}{\left(1+\sqrt{\sin x}\right)^n} dx = \int_0^{\pi/2} \frac{\sqrt{1+\sin y}}{(1+\sqrt{\sin y})^n} dy.$$

Hence, $I_n = \dfrac{1}{2}\displaystyle\int_0^{\pi/2} \frac{\sqrt{1+\sin x}}{\left(1+\sqrt{\sin x}\right)^n} dx$. By putting $\sin x = \cos^2\theta$ so that

$\cos x dx = -2\sin\theta\cos\theta d\theta$ and so $I_n = \displaystyle\int_0^{\pi/2} \frac{\cos\theta d\theta}{(1+\cos\theta)^n}$. We have

$$I_1 = \int_0^{\pi/2} \frac{\cos\theta d\theta}{1+\cos\theta} = \frac{\pi}{2} - \int_0^{\pi/2} \frac{d\theta}{1+\cos\theta} = \frac{\pi}{2} - \frac{1}{2}\int_0^{/2} \sec^2\frac{\theta}{2} d = \frac{\pi-2}{2}.$$

For $n \geq 2$, integrating by parts, we have

$$I_n = \int_0^{\pi/2} \frac{\cos\theta d\theta}{(1+\cos\theta)^n} = \int_0^{\pi/2} \frac{d(\sin\theta)}{(1+\cos\theta)^n} = 1 - n\int_0^{\pi/2} \frac{\sin^2\theta d\theta}{(1+\cos\theta)^{n+1}} = 1 - n\int_0^{\pi/2} \frac{(1-\cos\theta)d\theta}{(1+\cos\theta)^n}.$$

Hence $(1-n)I_n = 1 - n\displaystyle\int_0^{\pi/2} \frac{d\theta}{(1+\cos\theta)^n} = 1 - \frac{n}{2^{n-1}}\int_0^{\pi/4} \sec^{2n}\theta d\theta$.

By putting $t = \tan\theta$, we obtain

$$\int_0^{\pi/4} \sec^{2n}\theta\theta = \int_0^1 \left(1+t^2\right)^{n-1} dt + \int_0^1 \left(\sum_{k=0}^{n-1} \binom{n-1}{k} t^{2k}\right) dt = \sum_{k=0}^{n-1} \frac{\binom{n-1}{k}}{2k+1}.$$

Thus (1) holds and this completes the solution.

Also solved by Arkady Alt, San Jose, CA; Ed Gray, Highland Beach, FL; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy, and the proposer.