

Problems

Ted Eisenberg, Section Editor

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <<http://ssmj.tamu.edu>>.

*Solutions to the problems stated in this issue should be posted before
May 15, 2011*

- **5152:** *Proposed by Kenneth Korbin, New York, NY*

Given prime numbers x and y with $x > y$. Find the dimensions of a primitive Pythagorean Triangle which has hypotenuse equal to $x^4 + y^4 - x^2y^2$.

- **5153:** *Proposed by Kenneth Korbin, New York, NY*

A trapezoid with sides $(1, 1, 1, x)$ and a trapezoid with sides $(1, x, x, x)$ are both inscribed in the same circle. Find the diameter of the circle.

- **5154:** *Proposed by Andrei Răzvan Băleanu (student, George Cosbuc National College) Motru, Romania*

Let a, b, c be the sides, m_a, m_b, m_c the lengths of the medians, r the in-radius, and R the circum-radius of the triangle ABC . Prove that:

$$\frac{m_a^2}{1 + \cos A} + \frac{m_b^2}{1 + \cos B} + \frac{m_c^2}{1 + \cos C} \geq 6Rr \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \right).$$

- **5155:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b, c, d be the roots of the equation $x^4 + 6x^3 + 7x^2 + 6x + 1 = 0$. Find the value of

$$\frac{3-2a}{1+a} + \frac{3-2b}{1+b} + \frac{3-2c}{1+c} + \frac{3-2d}{1+d}.$$

- **5156:** *Proposed by Yakub N. Aliyev, Khyrdalan, Azerbaijan*

Given two concentric circles with center O and let A be a point different from O in the interior of the circles. A ray through A intersects the circles at the points B and C . The ray OA intersects the circles at the points B_1 and C_1 , and the ray through A perpendicular to line OA intersects the circles at the points B_2 and C_2 . Prove that

$$B_1C_1 \leq BC \leq B_2C_2.$$

- **5157:** *Proposed by Juan-Bosco Romero Márquez, Madrid, Spain*

Let $p \geq 2, \lambda \geq 1$ be real numbers and let $e_k(x)$ for $1 \leq k \leq n$ be the symmetric elementary functions in the variables $x = (x_1, \dots, x_n)$ and $x^p = (x_1^p, \dots, x_n^p)$, with $n \geq 2$ and $x_i > 0$ for all $i = 1, 2, \dots, n$.

Prove that

$$e_n^{(pk/n)}(x) \leq \frac{e_k(x^p) + \lambda (e_k^p(x) - e_k(x^p))}{\binom{n}{k} + \lambda (\binom{n}{k}^p - \binom{n}{k})} \leq \left(\frac{e_1(x)}{n}\right)^{pk}, \quad 1 \leq k \leq n.$$

Solutions

- **5134:** *Proposed by Kenneth Korbin, New York, NY*

Given isosceles $\triangle ABC$ with cevian CD such that $\triangle CDA$ and $\triangle CDB$ are also isosceles, find the value of

$$\frac{AB}{CD} - \frac{CD}{AB}.$$

Solution 1 by David Stone and John Hawkins, Statesboro, GA,

Because the cevian originates at the vertex C , angle C plays a special role. We consider two cases. Then, ignoring degenerate triangles, we have the following solutions. Note that solving simple algebraic equations involving the angles (and a little trig) are all that is needed.

Case 1 – Angle C is one of the base angles of our isosceles triangle

$\angle A$	$\angle B$	$\angle C$	$\angle ACD$	$\angle BCD$	$\angle ADC$	$\angle BDC$	$\frac{AB}{CD} - \frac{CD}{AB}$
$\frac{2\pi}{5}$	$\frac{\pi}{5}$	$\frac{2\pi}{5}$	$\frac{\pi}{5}$	$\frac{\pi}{5}$	$\frac{2\pi}{5}$	$\frac{3\pi}{5}$	1
$\frac{3\pi}{7}$	$\frac{\pi}{7}$	$\frac{3\pi}{7}$	$\frac{2\pi}{7}$	$\frac{\pi}{7}$	$\frac{2\pi}{7}$	$\frac{5\pi}{7}$	$2 \cos\left(\frac{\pi}{7}\right) - \frac{1}{2 \cos\left(\frac{\pi}{7}\right)}$
							≈ 1.24698

Case 2 – Angles A and B are the base angles

$\angle A$	$\angle B$	$\angle C$	$\angle ACD$	$\angle BCD$	$\angle ADC$	$\angle BDC$	$\frac{AB}{CD} - \frac{CD}{AB}$
$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{3}{2}$
$\frac{\pi}{5}$	$\frac{\pi}{5}$	$\frac{3\pi}{5}$	$\frac{\pi}{5}$	$\frac{2\pi}{5}$	$\frac{3\pi}{5}$	$\frac{2\pi}{5}$	$\sqrt{5}$

The derivation:

Case 1: $\angle C$ is one of the base angles

Without loss of generality we may assume $\angle A = \angle C$. For convenience, let $\angle A = \alpha$ and $\angle ADC = \beta$.

$\triangle CDA$ must be isosceles, so we have three possibilities:

- (a) $\alpha = \angle ACD$
- (b) $\alpha = \beta$
- (c) $\angle ACD = \beta$

Subcase (a) is impossible because $\angle ACD < \text{the base angle } \angle C = \alpha$,

Subcase (b):

Because $\alpha = \beta$, we have $\angle ACD = \pi - 2\alpha$, so

$$\angle BCD = \angle C - \angle ACD = \alpha - (\pi - 2\alpha) = 3\alpha - \pi.$$

We also see that $\alpha = \beta$ must be acute angles forcing $\angle BDC$ to be obtuse. Thus the isosceles triangle $\triangle BDC$ must have $\angle B = \angle BCD$; i.e. $\pi - 2\alpha = 3\alpha - \pi$.

Hence, $\alpha = \frac{2\pi}{5} = \beta$. The values of all angles then follow.

Applying the Law of Sines, we learn that $\frac{AB}{CD} = \frac{1}{2 \cos \alpha}$. Because $\alpha = \frac{2\pi}{5}$ and $\cos\left(\frac{2\pi}{5}\right) = \frac{-1 + \sqrt{5}}{2}$, we calculate $\frac{AB}{CD} - \frac{CD}{AB} = \frac{1}{2 \cos \alpha} - 2 \cos \alpha = 1$.

Subcase (c):

Because $\angle ACD = \beta$, we have $\alpha = \pi - 2\beta$, and because $\angle A = \angle C = \alpha$, we have

$$\angle B = \pi - 2\alpha = \pi - 2(\pi - 2\beta) = 4\beta - \pi.$$

Since β is the size of two angles in the triangle it must be acute. Thus the supplement $\angle BDC$ is obtuse. Therefore the equal angles in the isosceles triangle $\triangle CDB$ must be $\angle B$ and $\angle DCB$ which equals $4\beta - \pi$.

Hence, in $\triangle BDC$, we have

$$\pi = \angle B + \angle DCB + \angle BDC = (4\beta - \pi) + (4\beta - \pi) + (\pi - \beta),$$

forcing $\beta = \frac{2\pi}{7}$.

Thus $\alpha = \pi - 2\beta = \pi - 2\left(\frac{2\pi}{7}\right) = \frac{3\pi}{7}$. The other angles follow and by the Law of Sines in $\triangle CDB$, we see that $\frac{CD}{AB} = \frac{1}{2 \cos\left(\frac{\pi}{7}\right)}$.

Thus

$$\frac{AB}{CD} - \frac{CD}{AB} = 2 \cos\left(\frac{\pi}{7}\right) - \frac{1}{2 \cos\left(\frac{\pi}{7}\right)} \approx 1.24698$$

Case 2: Angles A and B are the base angles.

We let $\angle A = \angle B = \alpha$ and $\angle ADC = \beta$. Again, $\triangle CDA$ must be isosceles, so we have three possibilities:

- (a) $\alpha = \angle ACD$
- (b) $\alpha = \beta$
- (c) $\beta = \angle ACD$

In an analysis similar to that above, subcases (b) and (c) lead to degenerate triangles. Subcase (a) must be split again, depending upon the isosceles nature of $\triangle CDB$, but the two possibilities lead to the two triangles presented in the table.

Editor's comment: Most of the solutions received to this problem essentially followed the above track, but the solvers often missed one of the possible answers; Michael Fried's solution and Boris Rays' solution were exceptions, they too were complete. But **Paul M. Harms** took a different approach. He placed the isosceles triangle on a coordinate system and then considered cevians from a given vertex to various points on the opposite side. The conditions of the problem led him to solving a system of equations which then enabled him to find the lengths of the sides. His method also missed one of the solutions and I spent hours in vain trying to understand why, but for the sake of seeing an alternative analysis, I present his solution.

Solution 2 by Paul M. Harms, North Newton, KS

Since some similar triangles would give the same ratios required in the problem, I will fix one side of the large triangle and check for cevians which make the small triangles isosceles.

Case 1: Let A be at $(-1, 0)$, B be at $(1, 0)$, and let C be at $(0, c)$ where $c > 0$. If D is at $(0, 0)$ and C is at $(0, 1)$, then the conditions of the problem are satisfied with

$$\frac{AB}{CD} - \frac{CD}{AB} = \frac{2}{1} - \frac{1}{2} = \frac{3}{2}.$$

Case 2: Keeping the coordinates A, B as above and letting C having the coordinates $(0, c)$, we let D be at $(d, 0)$ where $0 < d < 1$. To get the two smaller triangles to be isosceles we need $AD = AC$ and $CD = DB$.

The distance equations that need to be satisfied are

$$\begin{cases} \sqrt{c^2 + d^2} = 1 - d \\ 1 + d = \sqrt{c^2 + 1}. \end{cases}$$

Solving this system:

$$d = -2 + \sqrt{5}, \quad c = \sqrt{5 - 2\sqrt{5}}.$$

Hence, $CD = \sqrt{14 - 6\sqrt{5}}$ and $\frac{AB}{CD} - \frac{CD}{AB} = \sqrt{5}$.

Case 3: Now consider C at a vertex other than the intersection of the equal sides.

Let C be at $(2, 0)$, A be at $(0, 0)$ and B be at $(1, b)$.

a) If $AC > AB = BC$, then $CD > BC > DB$ and $\triangle CDB$ cannot be isosceles.

b) If $AB = BC > AC$, then the smaller triangles would be isosceles when $CD = DB$ and $AC = AD$. Let D be at (d, bd) . From the distance equations we have

$$\begin{cases} (d-1)^2 + (bd-b)^2 = b^2d^2 + (d-2)^2 \\ b^2d^2 + d^2 = 4. \end{cases}$$

From the second equation $(b^2 + 1) = \frac{4}{d^2}$. The first equation is

$(d-1)^2(b^2+1) = (b^2+1)d^2 - 4d + 4$. Substituting for b^2+1 we obtain

$$d^3 - d^2 - 2d + 1 = 0.$$

Using approximations, the value of d between 0 and 1 is approximately $d = 0.4451$.

Then $b = 4.3807$, $AB = \sqrt{b^2+1} = 4.4934$ and $CD = \sqrt{(2-d)^2 + b^2d^2} = 2.4939$. Then $\frac{AB}{CD} - \frac{CD}{AB} = 1.2467$.

Also solved by Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Michael N. Fried, Kibbutz Revivim, Israel; Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriptel, Germany (jointly); Antonio Ledesma López, Mathematical Club of the Instituto de Educación Secundaria-Nº 1, Requena-Valencia, Spain; Boris Rays, Brooklyn, NY; Raúl A. Simón, Santiago, Chile; Ercole Suppa, Teramo, Italy, and the proposer.

- **5135:** Proposed by Kenneth Korbin, New York, NY

Find a, b , and c such that

$$\begin{cases} ab + bc + ca = -3 \\ a^2b^2 + b^2c^2 + c^2a^2 = 9 \\ a^3b^3 + b^3c^3 + c^3a^3 = -24 \end{cases}$$

with $a < b < c$.

Solution 1 by Kee-Wai Lau, Hong Kong, China

Let $x = ab, y = bc, z = ca$, so that $x + y + z = -3, x^2 + y^2 + z^2 = 9$ and $x^3 + y^3 + z^3 = -24$. We have

$$xy + yz + zx = \frac{(x + y + z)^2 - x^2 - y^2 - z^2}{2} = 0 \quad \text{and}$$

$$xyz = \frac{x^3 + y^3 + z^3 - (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)}{3} = 1.$$

Thus $abc(a + b + c) = 0$ and $(abc)^2 = 1$. Hence, either

$$\begin{cases} a + b + c = 0 \\ ab + bc + ca = -3 \\ abc = 1 \end{cases} \quad \text{or} \quad \begin{cases} a + b + c = 0 \\ ab + bc + ca = -3 \\ abc = -1. \end{cases}$$

In the former case, a, b, c are the roots of the equation $t^3 - 3t - 1 = 0$ and in the latter case, the roots of the equation $t^3 - 3t + 1 = 0$. By standard formula, we obtain respectively

$$a = -2 \cos\left(\frac{2\pi}{9}\right), \quad b = -2 \cos\left(\frac{4\pi}{9}\right), \quad c = 2 \cos\left(\frac{\pi}{9}\right)$$

and

$$a = -2 \cos\left(\frac{\pi}{9}\right), \quad b = 2 \cos\left(\frac{4\pi}{9}\right), \quad c = 2 \cos\left(\frac{\pi}{9}\right)$$

Solution 2 by Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie, San Angelo, TX

If $a = 0$, then the given system of equations becomes

$$bc = -3, \quad b^2c^2 = 9, \quad b^3c^3 = -24,$$

which is impossible. Thus, $a \neq 0$, and similarly $b \neq 0$ and $c \neq 0$.

Let $x = ab$, $y = bc$, and $z = ca$, then

$$\begin{aligned} (ab + bc + ca)^2 &= (x + y + z)^2 \\ &= x^2 + y^2 + z^2 + 2(xy + yz + zx) \\ &= a^2b^2 + b^2c^2 + c^2a^2 + 2(ab^2c + bc^2a + ca^2b) \\ &= a^2b^2 + b^2c^2 + c^2a^2 + 2abc(a + b + c) \quad (1) \end{aligned}$$

and $(ab + bc + ca)^3 = (x + y + z)^3$

$$\begin{aligned} &= x^3 + y^3 + z^3 + 3(xy^2 + xz^2 + yx^2 + yz^2 + zx^2 + zy^2) + 6xyz \\ &= 3(x^3 + xy^2 + xz^2 + yx^2 + y^3 + yz^2 + zx^2 + zy^2 + z^3) - 2(x^3 + y^3 + z^3) + 6xyz \\ &= 3x(x^2 + y^2 + z^2) + 3y(x^2 + y^2 + z^2) + 3z(x^2 + y^2 + z^2) - 2(x^3 + y^3 + z^3) + 6xyz \\ &= 3(x + y + z)(x^2 + y^2 + z^2) - 2(x^3 + y^3 + z^3) + 6xyz \\ &= 3(ab + bc + ca)(a^2b^2 + b^2c^2 + c^2a^2) - 2(a^3b^3 + b^3c^3 + c^3a^3) + 6a^2b^2c^2. \quad (2) \end{aligned}$$

Using (1) and (2), it can easily be shown that the given system of equations is equivalent to

$$\begin{cases} a + b + c = 0 \\ ab + bc + ca = -3 \\ a^2b^2c^2 = 1 \end{cases} \quad (3)$$

If $abc = 1$, then the solutions of (3) are roots of the cubic equation

$$\begin{aligned} 0 &= (t - a)(t - b)(t - c) \\ &= t^3 - (a + b + c)t^2 + (ab + bc + ca)t - abc \\ &= t^3 - 3t - 1, \quad (4) \end{aligned}$$

and must be strictly between -2 and 2 by the Upper and Lower Bound theorem.

Let $t = 2 \cos \theta$, with $0 < \theta < \pi$, by (4),

$$\begin{aligned} 0 &= 8 \cos^3 \theta - 6 \cos \theta - 1 \\ \frac{1}{2} &= 4 \cos^3 \theta - 3 \cos \theta \\ \frac{1}{2} &= \cos(3\theta) \\ \theta &= \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}. \end{aligned}$$

Thus,

$$a = 2 \cos\left(\frac{7\pi}{9}\right) = -2 \cos\left(\frac{2\pi}{9}\right), \quad b = 2 \cos\left(\frac{5\pi}{9}\right) = -2 \cos\left(\frac{4\pi}{9}\right), \quad c = 2 \cos\left(\frac{\pi}{9}\right).$$

Similarly, when $abc = -1$,

$$a = -2 \cos\left(\frac{\pi}{9}\right), \quad b = 2 \cos\left(\frac{4\pi}{9}\right), \quad c = 2 \cos\left(\frac{2\pi}{9}\right).$$

Also solved by **Brian D. Beasley**, Clinton, SC; **Bruno Salgueiro Fanego**, Viveiro, Spain; **Paul M. Harms**, North Newton, KS; **Enkel Hysnelaj**, University of Technology, Sydney, Australia and **Elton Bojaxhiu**, Kriftel, Germany (jointly); **Antonio Ledesma López**, Mathematical Club of the Instituto de Educación Secundaria-Nº 1, Requena-Valencia, Spain; **David E. Manes**, Oneonta, NY; **Paolo Perfetti**, Department of Mathematics, University “Tor Vergata,” Rome, Italy; **David Stone** and **John Hawkins** (jointly), Statesboro, GA, and the proposer.

- **5136:** Proposed by *Daniel Lopez Aguayo* (student, Institute of Mathematics, UNAM), Morelia, Mexico

Prove that for every natural n , the real number

$$\left(\sqrt{19} - 3\sqrt{2}\right)^{1/n} + \left(\sqrt{19} + 3\sqrt{2}\right)^{1/n}$$

is irrational.

Solution 1 by Kee-Wai Lau, Hong Kong, China

Let $x = \left(\sqrt{19} - 3\sqrt{2}\right)^{1/n}$, $y = \left(\sqrt{19} + 3\sqrt{2}\right)^{1/n}$ and $z = x + y$.

It is easy to check that $y > x$, $xy = 1$ and that x and y are the two roots of the

equation, $t + \frac{1}{t} = z$. Solving for t , we obtain $x = \frac{z - \sqrt{z^2 - 4}}{2}$ and $y = \frac{z + \sqrt{z^2 - 4}}{2}$.

Applying the binomial theorem, we have

$$\begin{aligned} & 2^{n+1}\sqrt{19} \\ &= 2^n(x^n + y^n) \\ &= \left(z - \sqrt{z^2 - 4}\right)^n + \left(z + \sqrt{z^2 - 4}\right)^n \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} z^{n-k} (z^2 - 4)^{k/2} + \sum_{k=0}^n \binom{n}{k} z^{n-k} (z^2 - 4)^{k/2} \\ &= 2 \sum_{j=0}^m \binom{n}{2j} z^{n-2j} (z^2 - 4)^j, \end{aligned}$$

where m is the greatest integer not exceeding $\frac{n}{2}$. Hence, if z is rational, then $\sqrt{19}$ is also rational, which is false. Thus z is in fact irrational and this completes the solution.

Solution 2 by Valmir Bucaj (student, Texas, Lutheran University), Seguin, TX

To contradiction, suppose that

$$\left(\sqrt{19} - 3\sqrt{2}\right)^{1/n} + \left(\sqrt{19} + 3\sqrt{2}\right)^{1/n}$$

is rational.

Then it is easy to see that both $\left(\sqrt{19} - 3\sqrt{2}\right)^{1/n}$ and $\left(\sqrt{19} + 3\sqrt{2}\right)^{1/n}$ have to be rational. Let

$$\left(\sqrt{19} - 3\sqrt{2}\right)^{1/n} = \frac{a}{b} \quad \text{and} \quad \left(\sqrt{19} + 3\sqrt{2}\right)^{1/n} = \frac{c}{d} \quad (1)$$

where $a, b, c, d \in \mathbb{Z}$.

Raising both expressions in (1) to the power of n , we get

$$\left(\sqrt{19} - 3\sqrt{2}\right) = \left(\frac{a}{b}\right)^n \quad \text{and} \quad \left(\sqrt{19} + 3\sqrt{2}\right) = \left(\frac{c}{d}\right)^n \quad (2)$$

Adding the left sides and the right sides of the expressions in (2) and dividing by 2 we get

$$\sqrt{19} = \frac{1}{2} \left[\left(\frac{a}{b}\right)^n + \left(\frac{c}{d}\right)^n \right]. \quad (3)$$

However, since $\frac{a}{b} \in \mathbb{Q}$ and $\frac{c}{d} \in \mathbb{Q}$, it follows that $\frac{1}{2} \left[\left(\frac{a}{b}\right)^n + \left(\frac{c}{d}\right)^n \right] \in \mathbb{Q}$; that is, the right-hand side of the expression in (3) is a rational number, while the left-hand side, namely $\sqrt{19}$, is an irrational number.

Therefore, the contradiction that we arrived at shows that our initial assumption that $\left(\sqrt{19} - 3\sqrt{2}\right)^{1/n} + \left(\sqrt{19} + 3\sqrt{2}\right)^{1/n}$ is a rational number is not correct, hence the statement of the problem holds.

Solution 3 by Pedro H.O. Pantoja (student, UFRN), Natal, Brazil;

Using the identity

$$x^{n+1} + \frac{1}{x^{n+1}} = \left(x + \frac{1}{x}\right) \left(x^n + \frac{1}{x^n}\right) - \left(x^{n-1} + \frac{1}{x^{n-1}}\right)$$

if $x + \frac{1}{x}$ is rational, then so would be $x^n + \frac{1}{x^n}$, where $x = \left(\sqrt{19} + \sqrt{18}\right)^{\frac{1}{n}}$

$$\Rightarrow \frac{1}{x} = \left(\sqrt{19} - \sqrt{18}\right)^{\frac{1}{n}}.$$

Hence,

$$x^n + \frac{1}{x^n} = \sqrt{19} + \sqrt{18} + \sqrt{19} - \sqrt{18} = 2\sqrt{19}$$

which is irrational.

It follows that $x + \frac{1}{x}$ must be irrational too.

Editor's comment: **David E. Manes** cited the paper "On A Substitution Made In Solving Reciprocal Equations" by Arnold Singer that appeared in the **Mathematics Magazine** [38(1965), p. 212] as being helpful in solving such equations. This paper starts off as follows: "The standard procedure employed in solving reciprocal equations requires the substitution $y = x + 1/x$. One is then required to write $x^2 + 1/x^2, x^3 + 1/x^3 \dots$, as polynomials in y . Many texts give the relationships up to $x^4 + 1/x^4$ or so, and some give the recurrence relation..... This note derives the expression for $x^n + 1/x^n$ as a polynomial in y for general n ."

Also solved by Michael N. Fried, Kibbutz Revivim, Israel: David E. Manes, Oneonta, NY, and the proposer.

- **5137:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b, c be positive numbers such that $abc \geq 1$. Prove that

$$\prod_{cyclic} \frac{1}{a^5 + b^5 + c^2} \leq \frac{1}{27}.$$

Solution 1 by Ercole Suppa, Teramo, Italy

By AM-GM inequality we have:

$$a^5 + b^5 + c^2 \geq 3\sqrt[3]{a^5 b^5 c^2}$$

$$a^2 + b^5 + c^5 \geq 3\sqrt[3]{a^2 b^5 c^5}$$

$$a^5 + b^2 + c^5 \geq 3\sqrt[3]{a^5 b^2 c^5}$$

Therefore

$$\prod_{cyclic} (a^5 + b^5 + c^2) \geq 27\sqrt[3]{a^{12} b^{12} c^{12}} = 27(abc)^4 \geq 27 \quad \Rightarrow$$

$$\prod_{cyclic} \frac{1}{a^5 + b^5 + c^2} \leq \frac{1}{27}$$

and the conclusion follows.

Solution 2 by Valmir Bucaj (student, Texas Lutheran University), Seguin TX

Editor's comment: All solutions received were similar to the above. But Valmir Bucaj submitted two solutions to the problem. One solution was similar to the above, the other is listed below.

The inequality to be proved is equivalent to

$$(a^5 + b^5 + c^2)(a^5 + c^5 + b^2)(b^5 + c^5 + a^2) \geq 27.$$

Multiplying out the left-hand side and using the fact that $abc \geq 1$, after rearranging we get

$$(a^5 + b^5 + c^2)(a^5 + c^5 + b^2)(b^5 + c^5 + a^2) \geq \left(\frac{a^5}{b^5} + \frac{b^5}{c^5} + \frac{c^5}{a^5}\right) + \left(\frac{a^5}{c^5} + \frac{c^5}{b^5} + \frac{b^5}{a^5}\right)$$

$$\begin{aligned}
& + \left(\frac{a^2}{b^5} + \frac{b^5}{a^2} \right) + \left(\frac{a^5}{c^2} + \frac{c^2}{a^5} \right) + \left(\frac{a^2}{c^5} + \frac{c^5}{a^2} \right) \\
& + \left(\frac{b^2}{c^5} + \frac{c^5}{b^2} \right) + \left(\frac{c^2}{b^5} + \frac{b^5}{c^2} \right) + \left(\frac{a^5}{b^2} + \frac{b^2}{a^5} \right) \\
& + a^{12} + b^{12} + c^{12} + 3 + \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \\
& \geq 21 + a^{12} + b^{12} + c^{12} + \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}.
\end{aligned}$$

The last inequality follows from the fact that $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq 3$ and from $\frac{x}{y} + \frac{y}{x} \geq 2$, where x, y, z are positive real numbers.

It remains to show that

$$a^{12} + b^{12} + c^{12} + \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \geq 6.$$

But this follows immediately from the AM-GM inequality and from the fact that $abc \geq 1$. That is

$$a^{12} + b^{12} + c^{12} + \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \geq 6 \cdot \sqrt[6]{(a \cdot b \cdot c)^{12}} \cdot \frac{1}{(a \cdot b \cdot c)^3} = 6 \cdot \underbrace{\sqrt[6]{(abc)^9}}_{\geq 1} \geq 6.$$

This proves $(a^5 + b^5 + c^2)(a^5 + c^5 + b^2)(b^5 + c^5 + a^2) \geq 27$, and thus the statement of the problem.

Also solved by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany (jointly); Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Pedro H.O. Pantoja (student, UFRN), Natal, Brazil; Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy; Boris Rays, Brooklyn, NY, and the proposer.

- **5138:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let $n \geq 2$ be a positive integer. Prove that

$$\frac{n}{F_n F_{n+1}} \leq \frac{1}{(n-1)F_1^2 + F_2^2} + \cdots + \frac{1}{(n-1)F_n^2 + F_1^2} \leq \frac{1}{n} \sum_{k=1}^n \frac{1}{F_k^2},$$

where F_n is the n^{th} Fibonacci number defined by $F_0 = 0, F_1 = 1$ and for all $n \geq 2, F_n = F_{n-1} + F_{n-2}$.

Solution 1 by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany

First we prove the LHS inequality. The function $f(x) = \frac{1}{x}$ is a convex function since the second derivative is positive, and so according to Jensen's inequality we have

$$\sum_{k=1}^n f(x) \geq n f\left(\frac{\sum_{k=1}^n x}{n}\right) \text{ and implying this together with the known fact that}$$

$$\sum_{k=1}^n F_k^2 = F_n F_{n+1} \text{ we have}$$

$$\begin{aligned} \frac{1}{(n-1)F_1^2 + F_2^2} + \dots + \frac{1}{(n-1)F_n^2 + F_1^2} &\geq n \frac{1}{\frac{(n-1)F_1^2 + F_1^2 + \dots + (n-1)F_n^2 + F_1^2}{n}} \\ &= \frac{n}{\sum_{k=1}^n F_k^2} \\ &= \frac{n}{F_n F_{n+1}} \end{aligned}$$

Now we will prove the RHS inequality. First using the AM-GM inequality we have

$$\begin{aligned} \frac{1}{(n-1)F_1^2 + F_2^2} + \dots + \frac{1}{(n-1)F_n^2 + F_1^2} &\leq \frac{1}{n \sqrt[n]{F_1^{2(n-1)} F_2^2}} + \dots + \frac{1}{n \sqrt[n]{F_n^{2(n-1)} F_1^2}} \\ &= \frac{\sum_{cyclic} F_1^{\frac{2}{n}} F_2^{\frac{2(n-1)}{n}} \dots F_n^2}{n \prod_{k=1}^n F_k^2} \end{aligned}$$

Now since the sequence $[2, 0, 2, \dots, 2]$ majorizes the sequence $[\frac{2}{n}, \frac{2(n-1)}{n}, 2, \dots, 2]$, using Muirhead's Inequality we have

$$\begin{aligned} \frac{1}{(n-1)F_1^2 + F_2^2} + \dots + \frac{1}{(n-1)F_n^2 + F_1^2} &\leq \frac{\sum_{cyclic} F_1^{\frac{2}{n}} F_2^{\frac{2(n-1)}{n}} \dots F_n^2}{n \prod_{k=1}^n F_k^2} \\ &\leq \frac{\sum_{cyclic} F_1^2 F_2^0 \dots F_n^2}{n \prod_{k=1}^n F_k^2} \\ &= \sum_{k=1}^n \frac{1}{F_k^2} \end{aligned}$$

and this is the end of the proof.

Solution 2 by the Proposer

Applying AM-HM inequality to the positive numbers $\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}$, we have

$$\frac{n}{(n-1)x_k + x_{k+1}} = \frac{n}{\underbrace{x_k + \dots + x_k}_{n-1} + x_{k+1}} \leq \frac{1}{n} \left(\underbrace{\frac{1}{x_k} + \frac{1}{x_k} + \dots + \frac{1}{x_k}}_{n-1} + \frac{1}{x_{k+1}} \right)$$

or

$$\frac{n}{(n-1)x_k + x_{k+1}} \leq \frac{1}{n} \left(\frac{n-1}{x_k} + \frac{1}{x_{k+1}} \right)$$

Adding the preceding inequalities for $1 \leq k \leq n$, and putting $x_{n+1} = x_1$, yields

$$\sum_{k=1}^n \frac{n}{(n-1)x_k + x_{k+1}} \leq \frac{1}{n} \sum_{k=1}^n \left(\frac{n-1}{x_k} + \frac{1}{x_{k+1}} \right) = \sum_{k=1}^n \frac{1}{x_k}$$

On the other hand, applying again AM-HM inequality to the positive numbers $\frac{1}{(n-1)x_k + x_{k+1}}$, $1 \leq k \leq n$, ($x_{n+1} = x_1$) we have

$$\sum_{k=1}^n \frac{1}{(n-1)x_k + x_{k+1}} \geq \frac{n^2}{\sum_{k=1}^n ((n-1)x_k + x_{k+1})} = \frac{n}{\sum_{k=1}^n x_k}$$

Combining the preceding results, we obtain

$$\left(\frac{1}{n^2} \sum_{k=1}^n x_k \right)^{-1} \leq \sum_{k=1}^n \frac{n}{(n-1)x_k + x_{k+1}} \leq \sum_{k=1}^n \frac{1}{x_k}$$

Setting $x_k = F_k^2$, $1 \leq k \leq n$, in the preceding inequalities, we get

$$\left(\frac{1}{n^2} \sum_{k=1}^n F_k^2 \right)^{-1} \leq \sum_{k=1}^n \frac{n}{(n-1)F_k^2 + F_{k+1}^2} \leq \sum_{k=1}^n \frac{1}{F_k^2}$$

or

$$\left(\frac{F_n F_{n+1}}{n^2} \right)^{-1} \leq \sum_{k=1}^n \frac{n}{(n-1)F_k^2 + F_{k+1}^2} \leq \sum_{k=1}^n \frac{1}{F_k^2}$$

on account of the well-known formulae $\sum_{k=1}^n F_k^2 = F_n F_{n+1}$. From the above the statement follows. Equality holds when $n = 2$, and we are done.

Editor's comment: **Valmir Bucaj (student, Texas Lutheran University), Seguin TX**, solved a slight variation of the given inequality. He showed that

$$\frac{n}{F_n F_{n+1}} \leq \frac{1}{(n-1)F_1^2 + F_n^2} + \dots + \frac{1}{(n-1)F_n^2 + F_1^2} \leq \frac{1}{n} \sum_{k=1}^n \frac{1}{F_k^2}.$$

- **5139:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Calculate

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\zeta(n+m) - 1}{n+m},$$

where ζ denotes the Riemann Zeta function.

Solution 1 by Kee-Wai Lau, Hong Kong, China

We show that the value of the required sum is $\gamma = 0.577\dots$, the Euler constant.

Since $\frac{1}{(n+m)k^{n+m}} > 0$ for positive integers, m, n and k , we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\zeta(n+m) - 1}{n+m} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n+m} \sum_{k=2}^{\infty} \frac{1}{k^{n+m}} = \sum_{k=2}^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(n+m)k^{n+m}}.$$

For each integer $t > 1$, the number of solutions of the equation $n+m = t$ in positive integers n and m is $t-1$. Hence, $\sum_{k=2}^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(n+m)k^{n+m}} = \sum_{k=2}^{\infty} \sum_{t=2}^{\infty} \frac{t-1}{tk^t}$.

For real $x > 1$, we have the well known series $\sum_{t=1}^{\infty} \frac{1}{x^t} = \frac{1}{x-1}$ and

$$\sum_{t=1}^{\infty} \frac{1}{tx^t} = -\ln\left(1 - \frac{1}{x}\right) \text{ so that } \sum_{t=1}^{\infty} \frac{t-1}{tk^t} = \frac{1}{k-1} + \ln\left(\frac{k-1}{k}\right).$$

It follows that for any integer $M > 1$, we have

$$\sum_{k=2}^M \sum_{t=2}^{\infty} \frac{t-1}{tk^t} = \sum_{k=1}^{M-1} \frac{1}{k} - \ln(M),$$

which tends to γ as M tends to infinity. This proves our claim.

Solution 2 by Paolo Perfetti, Department of Mathematics, University “Tor Vergata,” Rome, Italy

Answer: γ

Proof: We will need the well known $\sum_{k=1}^n \frac{1}{k} = \ln n + \gamma + o(1)$ and γ is of course the Euler–Mascheroni constant.

Setting $n+m = k$ the series is

$$\sum_{k=2}^{\infty} \sum_{m=1}^{k-1} \frac{\zeta(k) - 1}{k} = \sum_{k=2}^{\infty} (k-1) \frac{\zeta(k) - 1}{k} = \sum_{k=2}^{\infty} (\zeta(k) - 1) - \sum_{k=2}^{\infty} \frac{\zeta(k) - 1}{k}$$

$$\sum_{k=2}^{\infty} (\zeta(k) - 1) = \sum_{k=2}^{\infty} \sum_{p=2}^{\infty} \frac{1}{p^k} = \sum_{p=2}^{\infty} \sum_{k=2}^{\infty} \frac{1}{p^k} = \sum_{p=2}^{\infty} \frac{1}{p^2} \frac{p^2}{p^2 - 1} = \sum_{p=2}^{\infty} \left(\frac{1}{p-1} - \frac{1}{p+1} \right) = 1$$

$$- \sum_{k=2}^{\infty} \frac{\zeta(k) - 1}{k} = - \sum_{k=2}^{\infty} \sum_{p=2}^{\infty} \frac{1}{k} \frac{1}{p^k} = \sum_{p=2}^{\infty} \left(\ln\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right) = \lim_{N \rightarrow \infty} \sum_{p=2}^N \left(\ln\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right)$$

$$= \lim_{N \rightarrow \infty} \sum_{p=2}^N \left(\ln(p-1) - \ln p + \frac{1}{p} \right) = \lim_{N \rightarrow \infty} ((-\ln N) + \ln N + \gamma + o(1)) = \gamma$$

Thus the result easily follows

Also solved by the proposer.