Problems

Ted Eisenberg, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before
May 15, 2010

- 5104: *Proposed by Kenneth Korbin, New York, NY*

  There are infinitely many primitive Pythagorean triangles with hypotenuse of the form $4x^4 + 1$ where x is a positive integer. Find the dimensions of all such triangles in which at least one of the sides has prime integer length.

- 5105: *Proposed by Kenneth Korbin, New York, NY*

  Solve the equation
  $$x + y - \sqrt{x^2 + xy + y^2} = 2 + \sqrt{5}$$
  if x and y are of the form $a + b\sqrt{5}$ where a and b are positive integers.

- 5106: *Proposed by Michael Brozinsky, Central Islip, NY*

  Let a, b, and c be the sides of an acute-angled triangle ABC. Let H be the orthocenter and let d_a, d_b and d_c be the distances from H to the sides BC,CA, and AB respectively.

  Show that
  $$d_a + d_b + d_c \le \frac{3}{4}D$$
  where D is the diameter of the circumcircle.

- 5107: *Proposed by Tuan Le (student, Fairmont, H.S.), Anaheim, CA*

  Let a, b, c be positive real numbers. Prove that
  $$\frac{\sqrt{a^3 + b^3}}{a^2 + b^2} + \frac{\sqrt{b^3 + c^3}}{b^2 + c^2} + \frac{\sqrt{c^3 + a^3}}{c^2 + a^2} \ge \frac{6(ab + bc + ac)}{(a + b + c)\sqrt{(a + b)(b + c)(c + a)}}$$

- 5108: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Compute
  $$\lim_{n\to\infty} \frac{1}{n} \tan\left[\sum_{k=1}^{4n+1} \arctan\left(1 + \frac{2}{k(k+1)}\right)\right].$$

- 5109 *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $k \geq 1$ be a natural number. Find the value of

$$\lim_{n\to\infty} \frac{(k\sqrt[n]{n} - k + 1)^n}{n^k}.$$

Solutions

- 5086: *Proposed by Kenneth Korbin, New York, NY*

Find the value of the sum

$$\frac{2}{3} + \frac{8}{9} + \cdots + \frac{2N^2}{3^N}.$$

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

If $x \neq 1$, the formula for a geometric sum yields

$$\sum_{k=0}^{N} x^k = \frac{x^{N+1} - 1}{x - 1}.$$

If we differentiate and simplify, we obtain

$$\sum_{k=1}^{N} kx^{k-1} = \frac{Nx^{N+1} - (N+1)\,x^N + 1}{(x-1)^2}.$$

Next, multiply by x and differentiate again to get

$$\sum_{k=1}^{N} kx^k = \frac{Nx^{N+2} - (N+1)\,x^{N+1} + x}{(x-1)^2}$$

and

$$\sum_{k=1}^{N} k^2x^{k-1} = \frac{N^2x^{N+2} - (2N^2 + 2N - 1)\,x^{N+1} + (N+1)^2\,x^N - x - 1}{(x-1)^3}.$$

Finally, multiply by x once more to yield

$$\sum_{k=1}^{N} k^2x^k = \frac{N^2x^{N+3} - (2N^2 + 2N - 1)\,x^{N+2} + (N+1)^2\,x^{N+1} - x^2 - x}{(x-1)^3}.$$

In particular, when we substitute $x = \dfrac{1}{3}$ and simplify, the result is

$$\sum_{k=1}^{N} \frac{k^2}{3^k} = \frac{3^{N+1} - (N^2 + 3N + 3)}{2 \cdot 3^N}.$$

Therefore, the desired sum is

$$\sum_{k=1}^{N} \frac{2k^2}{3^k} = \frac{3^{N+1} - (N^2 + 3N + 3)}{3^N}.$$

Solution 2 by Ercole Suppa, Teramo, Italy

The required sum can be written as $S_N = \dfrac{2}{3^N} \cdot x_n$, where x_n denotes the sequence

$$x_n = 1^2 \cdot 3^{n-1} + 2^2 \cdot 3^{n-2} + 3^2 \cdot 3^{n-3} + \cdots + n^2 \cdot 3^0.$$

Since

$$x_{n+1} = 1^2 \cdot 3^n + 2^2 \cdot 3^{n-1} + 3^2 \cdot 3^{n-2} + \cdots + n^2 \cdot 3^1 + (n+1)^2 \cdot 3^0,$$

such a sequence satisfies the linear recurrence

$$x_{n+1} - 3x_n = (n+1)^2. \qquad (*)$$

Solving the characteristic equation $\lambda - 3 = 0$, we obtain the homogeneous solutions $x_n = A \cdot 3^n$, where A is a real parameter. To determine a particular solution, we look for a solution of the form $x_n^{(p)} = Bn^2 + Cn + D$. Substituting this into the difference equation, we have

$$\begin{aligned} B(n+1)^2 + C(n+1) + D - 3\left[Bn^2 + Cn + D\right] &= (n+1)^2 \Leftrightarrow \\ -2Bn^2 + 2(B-C)n + B + C - 2D &= n^2 + 2n + 1. \end{aligned}$$

Comparing the coefficients of n and the constant terms on the two sides of this equation, we obtain

$$B = -\frac{1}{2}, \qquad C = -\frac{3}{2}, \qquad D = -\frac{3}{2}$$

and thus

$$x_n^{(p)} = -\frac{1}{2}n^2 + -\frac{3}{2}n - \frac{3}{2}$$

The general solution of $(*)$ is simply the sum of the homogeneous and particular solutions, i.e.,

$$x_n = A \cdot 3^n - \frac{1}{2}n^2 + -\frac{3}{2}n - \frac{3}{2}$$

From the boundary condition $x_1 = 1$, the constant is determined as $\dfrac{3}{2}$.
Finally, the desired sum is

$$S_N = \frac{3^{N+1} - N^2 - 3N - 3}{3^N}$$

and we are done.

Also solved by Brian D. Beasley, Clinton, SC; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Pat Costello, Richmond, KY; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, Sydney, Australia & Elton Bojaxhiu, Germany; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; John Nord, Spokane, WA; Paolo Perfetti, Department of Mathematics, Tor Vergata Universtiy, Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA; Taylor University Problem Solving Group, Upland, IN, and the proposer.

- **5087:** *Proposed by Kenneth Korbin, New York, NY*

  Given positive integers $a, b, c,$ and d such that $(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2)$ with $a < b < c < d$. Rationalize and simplify

$$\frac{\sqrt{x+y}-\sqrt{x}}{\sqrt{x+y}+\sqrt{x}} \quad \text{if} \quad \begin{cases} x = bc+bd+cd, & \text{and} \\ y = ab+ac+ad. \end{cases}$$

Solution by Paul M. Harms, North Newton, KS

From the equation given in the problem we have

$$(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd = 2\left(a^2+b^2+c^2+d^2\right).$$

From the last equation we have

$$2\,(ab+ac+ad+bc+bd+cd) = a^2+b^2+c^2+d^2.$$

We note that,

$$\begin{aligned} x+y &= ab+ac+ad+bc+bd+cd, \text{ then} \\ 2\,(x+y) &= a^2+b^2+c^2+d^2 \end{aligned}$$

From the identity in the problem,

$$\begin{aligned} 2\,(x+y) &= \frac{(a+b+c+d)^2}{2} \text{ or} \\ (x+y) &= \frac{(a+b+c+d)^2}{2^2} \end{aligned}$$

Also note that,

$$\begin{aligned} y &= a(b+c+d) \text{ or} \\ \frac{y}{a} &= b+c+d. \text{ Then} \\ x+y &= \frac{(a+(y/a))^2}{2^2} = \frac{(a^2+y)^2}{(2a)^2}. \end{aligned}$$

We have,

$$\begin{aligned} x &= (x+y)-y \\ &= \frac{(a^2+y)^2}{(2a)^2} - y \\ &= \frac{a^4+2a^2y+y^2-4a^2y}{4a^2} \\ &= \frac{(a^2-y)^2}{(2a)^2}. \end{aligned}$$

From $a < b < c < d$, we see that

$$\begin{aligned} a^2 - y &= a^2 - a\,(b + c + d) < 0. \text{ Thus} \\ \sqrt{(a^2 - y)^2} &= y - a^2. \end{aligned}$$

Working with the expression to be simplified, we have

$$\begin{aligned} \frac{\sqrt{x+y} - \sqrt{x}}{\sqrt{x+y} + \sqrt{x}} &= \frac{(\sqrt{x+y} - \sqrt{x})^2}{y} \\ &= \frac{\left[(a^2+y)/(2a) - (y - a^2)/(2a)\right]^2}{y} \\ &= \frac{(2a^2/2a)^2}{y} \\ &= \frac{a^2}{y} \\ &= \frac{a}{b+c+d}. \end{aligned}$$

Also solved by Brian D. Beasley, Clinton, SC; G. C., Greubel, Newport News, VA; Enkel Hysnelaj, Sydney, Australia & Elton Bojaxhiu, Germany; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5088: *Proposed by Isabel Iriberri Díaz and José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b be positive integers. Prove that

$$\frac{\varphi(ab)}{\sqrt{\varphi^2(a^2) + \varphi^2(b^2)}} \leq \frac{\sqrt{2}}{2},$$

where $\varphi(n)$ is Euler's totient function.

Solution by Tom Leong, Scotrun, PA

We show

$$\varphi(ab) \leq \sqrt{\varphi(a^2)\varphi(b^2)} \leq \sqrt{\frac{\varphi^2 a^2) + \varphi^2(b^2)}{2}}$$

which implies the desired result. The second inequality used here is simply the AM-GM Inequality. To prove the first inequality, let p_i denote the prime factors of both a and b, and let q_j denote the prime factors of a only and r_k the primes factors of b only. Then

$$\begin{aligned} \varphi(ab) &= ab \prod_i \left(1 - \frac{1}{p_i}\right) \prod_j \left(1 - \frac{1}{q_j}\right) \prod_k \left(1 - \frac{1}{r_k}\right) \\ \varphi(a^2)\varphi(b^2) &= \left[a^2 \prod_i \left(1 - \frac{1}{p_i}\right) \prod_j \left(1 - \frac{1}{q_j}\right)\right] \left[b^2 \prod_i \left(1 - \frac{1}{p_i}\right) \prod_k \left(1 - \frac{1}{r_k}\right)\right] \end{aligned}$$

where we understand the empty product to be 1. Then $\varphi(ab) \le \sqrt{\varphi(a^2)\varphi(b^2)}$ reduces to

$$\prod_j \left(1 - \frac{1}{q_j}\right) \prod_k \left(1 - \frac{1}{r_k}\right) \le 1$$

which is obviously true.

Editor's comment: **Kee-Wai Lau of Hong Kong, China** mentioned in his solution to this problem that in the *Handbook of Number Theory I* (Section 1.2 of Chapter I by J. Sándor, D.S. Mitrinovi, and B. Crstic, Springer, 1995), the proof of $(\varphi(mn))^2 \le \varphi(m^2)\varphi(n^2)$, for positive integers m and n is attributed to a 1940 paper by T. Popoviciu. Kee-Wai then wrote $\sqrt{\varphi^2(a^2) + \varphi^2(b^2)} \ge \sqrt{2\varphi(a^2)\varphi(b^2)} \ge \sqrt{2}\varphi(ab)$, proving the inequality.

Also solved by Brian D. Beasley, Clinton, SC; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Enkel Hysnelaj, Sydney, Australia & Elton Bojaxhiu, Germany; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA; Ercole Suppa, Teramo, Italy; and the proposers.

- 5089: *Proposed by Panagiote Ligouras, Alberobello, Italy*

  In $\triangle ABC$ let $AB = c, BC = a, CA = b, r =$ the in-radius and r_a, r_b, and r_c= the ex-radii, respectively.

  Prove or disprove that

$$\frac{(r_a - r)(r_b + r_c)}{r_a r_c + r r_b} + \frac{(r_c - r)(r_a + r_b)}{r_c r_b + r r_a} + \frac{(r_b - r)(r_c + r_a)}{r_b r_a + r r_c} \ge 2\left(\frac{ab}{b^2 + ca} + \frac{bc}{c^2 + ab} + \frac{ca}{a^2 + bc}\right).$$

Solution by Kee-Wai Lau, Hong Kong, China

We prove the inequality.

Let s and S be respectively the semi-perimeter and area of $\triangle ABC$. It is well known that

$$r = \frac{S}{s},\ r_a = \frac{S}{s-a},\ r_b = \frac{S}{s-b},\ r_c = \frac{S}{s-c}.$$

Using these relations, we readily simplify

$$\frac{(r_a - r)(r_b + r_c)}{r_a r_c + r r_b} \text{ to } \frac{a}{c},\ \frac{(r_c - r)(r_a + r_b)}{r_c r_b + r r_a} \text{ to } \frac{c}{b},\ \text{ and } \frac{(r_b - r)(r_c + r_a)}{r_b r_a + r r_c} \text{ to } \frac{b}{a}.$$

Since $b^2 + ca \ge 2b\sqrt{ca}$, $c^2 + ab \ge 2c\sqrt{ab}$, and $a^2 + bc \ge 2a\sqrt{bc}$, so

$$2\left(\frac{ab}{b^2 + ca} + \frac{bc}{c^2 + ab} + \frac{ca}{a^2 + bc}\right) \le \sqrt{\frac{a}{c}} + \sqrt{\frac{b}{a}} + \sqrt{\frac{c}{b}}.$$

By the Cauchy-Schwarz inequality, we have

$$\sqrt{\frac{a}{c}} + \sqrt{\frac{b}{a}} + \sqrt{\frac{c}{b}} \le \sqrt{3\left(\frac{a}{c} + \frac{b}{a} + \frac{c}{b}\right)},$$

and by the arithmeic mean-geometric mean inequality we have

$$3 = 3\left(\sqrt[3]{\left(\frac{a}{c}\right)\left(\frac{b}{a}\right)\left(\frac{c}{b}\right)}\right) \le \frac{a}{c} + \frac{b}{a} + \frac{c}{b}.$$

It follows that $\sqrt{\frac{a}{c}} + \sqrt{\frac{b}{a}} + \sqrt{\frac{c}{b}} \le \frac{a}{c} + \frac{b}{a} + \frac{c}{b}$ and this completes the solution.

Also solved by Tom Leong, Scotrun, PA; Ercole Suppa, Teramo, Italy, and the proposer.

- 5090: *Proposed by Mohsen Soltanifar (student), University of Saskatchewan, Canada*

Given a prime number p and a natural number n. Calculate the number of elementary matrices $E_{n\times n}$ over the field Z_p.

Solution by Paul M. Harms, North Newton, KS

The notation 0 and 1 will be used for the additive and multiplicative identities, respectively.

There are three types of matrices which make up the set of elementary matrices. One type is a matrix where two rows of the identity matrix are interchanged. Since there are n rows and we interchange two at a time, the number of elementary matrices of this type is $\frac{n(n-1)}{2}$, the combination of n things taken two at a time.

Another type of elementary matrix is a matrix where one of the elements along the main diagonal is replaced by an element which is not 0 or 1. There are $(p-2)$ elements which can replace a 1 on the main diagonal. The number of elementary matrices of this type is $(p-2)n$.

The third type of elementary matrix is the identity matrix where at most one position, not on the main diagonal, is replaced by a non-zero element. There are (n^2-n) positions off the main diagonal and $(p-1)$ non-zero elements. Then there are $(n^2-n)(p-1)$ different elementary matrices where a non-zero element replaces one zero element in the identity matrix. If the identity matrix is included here, the number of elementary matrices of this type is $(n^2-n)(p-1)+1$.

The total number of elementary matrices is

$$\frac{n(n-1)}{2} + (p-2)n + (n^2-n)(p-1) + 1 = n^2\left(p - \frac{1}{2}\right) - \frac{3n}{2} + 1.$$

Comment by David Stone and John Hawkins of Statesboro, GA. There doesn't seem to be any need to require that p be prime as we form and count these elementary matrices. However, if m were not prime then Z_m would not be a field and the algebraic properties would be affected. For instance, it's preferable that any elementary matrix be invertible and the appearance of non-invertible scalars would produce non-invertible elementary matrices such as $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ over Z_4.

Also solved by David E. Manes, Oneonta, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5091: *Proposed by Ovidiu Furdui, Cluj, Romania*

  Let $k, p \geq 0$ be nonnegative integers. Evaluate the integral

$$\int_{-\pi/2}^{\pi/2} \frac{\sin^{2p} x}{1+\sin^{2k+1} x+\sqrt{1+\sin^{4k+2} x}} dx.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

We show that the integral equals $\dfrac{(2p-1)!!}{(2p)!!}\dfrac{\pi}{2}$, independent of k.

Here $(-1)!! = 0!! = 1,\ n!! = n(n-2)\ldots(3)(1)$ if n is a positive odd integer and $n!! = n(n-2)\ldots(4)(2)$ if n is a positive even integer.

By substituting $x = -y$, we have

$$\int_{-\pi/2}^{0} \frac{\sin^{2p} x}{1+\sin^{2k+1} x+\sqrt{1+\sin^{4k+2} x}} dx = \int_{0}^{\pi/2} {}_0 \frac{\sin^{2p} y}{1-\sin^{2k+1} y+\sqrt{1+\sin^{4k+2} y}} \text{ so that}$$

$$\begin{aligned}
&\int_{-\pi/2}^{\pi/2} \frac{\sin^{2p} x}{1+\sin^{2k+1} x+\sqrt{1+\sin^{4k+2} x}} dx \\
=\ &\int_{0}^{\pi/2} \sin^{2p} x \left(\frac{1}{1+\sin^{2k+1} x+\sqrt{1+\sin^{4k+2} x}}+\frac{1}{1-\sin^{2k+1} x+\sqrt{1+\sin^{4k+2} x}}\right) dx \\
=\ &2\int_{0}^{\pi/2} \sin^{2p} x \left(\frac{1+\sqrt{1+\sin^{4k+2} x}}{\left(1+\sin^{2k+1} x+\sqrt{1+\sin^{4k+2} x}\right)\left(1-\sin^{2k+1} x+\sqrt{1+\sin^{4k+2} x}\right)}\right) dx \\
=\ &\int_{0}^{\pi/2} \sin^{2p} x dx.
\end{aligned}$$

The last integral is standard and its value is well known to be $\dfrac{(2p-1)!!}{(2p)!!}\dfrac{\pi}{2}$.

Solution 2 by Paolo Perfetti, Department of Mathematics, Tor Vergata, Rome, Italy

The answer is: $\dfrac{(2p)!}{2^{2p}(p!)^2}\dfrac{\pi}{2}$ for any k.

Proof Let's substitute $\sin x = t$

$$\int_{-1}^{1} \frac{t^{2p}}{1+t^{2k+1}+\sqrt{1+t^{4k+2}}} \frac{dt}{\sqrt{1-t^2}} = \int_{-1}^{1} \frac{t^{2p}(1+t^{2k+1}-\sqrt{1+t^{4k+2}})}{2t^{2k+1}} \frac{dt}{\sqrt{1-t^2}}$$

Now

$$\int_{-1}^{1} \frac{t^{2p}}{2t^{2k+1}} \frac{dt}{\sqrt{1-t^2}} = \int_{-1}^{1} \frac{t^{2p}\sqrt{1+t^{4k+2}}}{2t^{2k+1}} \frac{dt}{\sqrt{1-t^2}} = 0$$

since the integrands are odd functions. It remains

$$\frac{1}{2}\int_{-1}^{1}\frac{t^{2p}}{\sqrt{1-t^2}}dt = \frac{1}{2}\int_{-\pi/2}^{\pi/2}(\sin x)^{2p}dx$$

after changing variable $t = \sin x$. Integrating by parts we obtain

$$\begin{aligned}\int_{-\pi/2}^{\pi/2}(\sin x)^{2p}dx &= \int_{-\pi/2}^{\pi/2}(-\cos x)'(\sin x)^{2p-1}dx\\ &= -\cos x(\sin x)^{2p-1}\Big|_{-\pi/2}^{\pi/2}+(2p-1)\int_{-\pi/2}^{\pi/2}\cos^2 x(\sin x)^{2p-2}dx\\ &= (2p-1)\int_{-\pi/2}^{\pi/2}(\sin x)^{2p-2}dx-(2p-1)\int_{-\pi/2}^{\pi/2}(\sin x)^{2p}dx\end{aligned}$$

and if we call $I_{2p} = \int_{-\pi/2}^{\pi/2}(\sin x)^{2p}dx$, then we have $I_{2p} = \frac{2p-1}{2p}I_{2p-2}$. It results that $I_{2p} = \frac{(2p-1)!!}{(2p)!!}\pi = \frac{(2p)!}{2^{2p}(p!)^2}\pi$ and then $\frac{1}{2}\int_{-1}^{1}\frac{t^{2p}}{\sqrt{1-t^2}}dt = \frac{\pi}{2}\frac{(2p-1)!!}{(2p)!!} = \frac{(2p)!}{2^{2p}(p!)^2}\frac{\pi}{2}$

Editor's comment: The two solutions presented, $\frac{(2p-1)!!}{(2p)!!}\frac{\pi}{2}$ and $\frac{(2p)!}{2^{2p}(p!)^2}\frac{\pi}{2}$, are equivalent to one another.

Also solved by Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.