Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before May 15, 2012

- **5200:** *Proposed by Kenneth Korbin, New York, NY*

  Given positive integers (a, b, c, d) such that,

$$(a+b+c+d)^2 = 2\left(a^2+b^2+c^2+d^2\right)$$

  with $a < b < c < d$. Find positive integers x, y and z such that

$$\begin{aligned} x &= \sqrt{ab+ad+bd} - \sqrt{ab+ac+bc}, \\ y &= \sqrt{bc+bd+cd} - \sqrt{bc+ab+ac}, \\ z &= \sqrt{bc+bd+cd} - \sqrt{ac+ad+cd}. \end{aligned}$$

- **5201:** *Proposed by Kenneth Korbin, New York, NY*

  Given convex cyclic quadrilateral ABCD with integer length sides where $\left(\overline{AB}, \overline{BC}, \overline{CD}\right) = 1$ and with $\overline{AB} < \overline{BC} < \overline{CD}$.

  The inradius, the circumradius, and both diagonals have rational lengths. Find the possible dimensions of the quadrilateral.

- **5202:** *Proposed by Neculai Stanciu, Buzău, Romania*

  Solve in $\Re^2$,

$$\begin{cases} \ln\left(x+\sqrt{x^2+1}\right) = \ln \dfrac{1}{y+\sqrt{y^2+1}} \\ 2^{y-x}\left(1-3^{x-y+1}\right) = 2^{x-y+1}-1. \end{cases}$$

- **5203:** *Proposed by Pedro Pantoja, Natal-RN, Brazil*

  Evaluate,

$$\int_0^{\pi/4} \ln\left(\frac{1+\sin^2 2x}{\sin^4 x+\cos^4 x}\right) dx.$$

- **5204:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let $f : \Re \to \Re$ be a non-constant function such that,

$$f(x+y) = \frac{f(x)+f(y)}{1+f(x)f(y)}$$

  for all $x, y \in \Re$. Show that $-1 < f(x) < 1$ for all $x \in \Re$.

- **5205:** *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

  Find the sum,

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{(-1)^{n-1}}{n} - \ln 2\right) \cdot \ln \frac{n+1}{n}.$$

Solutions

- **5182:** *Proposed by Kenneth Korbin, New York, NY*

  Part I: An isosceles right triangle has perimeter P and its Morley triangle has perimeter x. Find these perimeters if $P = x + 1$.

  Part II: An isosceles right triangle has area K and its Morley triangle has area y. Find these areas if $K = y + 1$

  Solution by David E. Manes, Oneonta, NY

  For part I, $P = \dfrac{2(2+\sqrt{2})}{4-4\sqrt{2}+3\sqrt{6}}$ and $x = \dfrac{3\sqrt{2}\left(2-\sqrt{3}\right)}{4-4\sqrt{2}+3\sqrt{6}}$.

  For part II, $K = \dfrac{4\left(16+7\sqrt{3}\right)}{109}$ and $y = \dfrac{28\sqrt{3}-45}{109}$.

  Denote the isosceles right triangle by ABC with the right angle at vertex C and sides a, b, c opposite the vertices A, B, C respectively. Then $a = b$ and $c = \sqrt{2}a$, whence $P = (2+\sqrt{2})a$. The side length s of the Morley triangle of ABC is given by $s = 8R \sin\dfrac{A}{3} \sin\dfrac{B}{3} \sin\dfrac{C}{3}$ where R is the circumradius of triangle ABC. Then

$$\begin{aligned} R &= \frac{abc}{\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}} \\ &= \frac{\sqrt{2}a^3}{\sqrt{a(2+\sqrt{2})\left(\sqrt{2}a\right)^2 a\left(2-\sqrt{2}\right)}} \\ &= \frac{\sqrt{2}a}{2}. \end{aligned}$$

  Using the identity $\sin^2\dfrac{z}{2} = \dfrac{1-\cos z}{2}$, one calculates

$$\sin\frac{A}{3} \cdot \sin\frac{B}{3} = \sin^2 15^\circ = \frac{1-\cos 30^\circ}{2} = \frac{2-\sqrt{3}}{4}.$$

Therefore,

$$s = 8R\sin^2 15^\circ \sin 30^\circ = 8\left(\frac{\sqrt{2}a}{2}\right)\left(\frac{2-\sqrt{3}}{4}\right)\frac{1}{2} = \frac{\sqrt{2}}{2}\left(2-\sqrt{3}\right)a$$

so that the perimeter x of the Morley triangle is given by $x = 3s = \dfrac{3\sqrt{2}}{2}\left(2-\sqrt{3}\right)a$.

The equation $P = x+1$ implies $\left(2+\sqrt{2}\right)a = \dfrac{3\sqrt{2}}{2}\left(2-\sqrt{3}\right)a + 1$ or
$a = \dfrac{2}{4-4\sqrt{2}+3\sqrt{6}}$. Hence,

$$\begin{aligned} P &= (2+\sqrt{2})a = \frac{2(2+\sqrt{2})}{4-4\sqrt{2}+3\sqrt{6}} \quad \text{and} \\ x &= \frac{3\sqrt{2}}{2}\left(2-\sqrt{3}\right)a = \frac{3\sqrt{2}\left(2-\sqrt{3}\right)}{4-4\sqrt{2}+3\sqrt{6}}. \end{aligned}$$

In part II,

$$\begin{aligned} K &= \frac{a^2}{2} \text{ and} \\ y &= \frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}\left[\frac{\sqrt{2}}{2}\left(2-\sqrt{3}\right)a\right]^2 \\ &= \frac{\sqrt{3}}{8}\left(7-4\sqrt{3}\right)a^2. \end{aligned}$$

The equation $K = y+1$ implies $\dfrac{a^2}{2} = \dfrac{\sqrt{3}}{8}\left(7-4\sqrt{3}\right)a^2 + 1$; that is, $a^2 = \dfrac{8(16+7\sqrt{3})}{109}$.
Hence,

$$K = \frac{4\left(16+7\sqrt{3}\right)}{109} \text{ and } y = \frac{28\sqrt{3}-45}{109}.$$

Also solved by Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Charles McCracken, Dayton, OH; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5183:** *Proposed by Kenneth Korbin, New York, NY*

A convex pentagon ABCDE, with integer length sides, is inscribed in a circle with diameter $\overline{AE}$.

Find the minimum possible perimeter of this pentagon.

Solution by Kee-Wai Lau, Hong Kong, China

We show that the minimum possible perimeter of this pentagon is 164.

Suppose that O is the center of the circle. Let $AB = p, BC = q, CD = r, DE = s, AE = d, \angle AOB = 2\alpha, \angle BOC = 2\beta, \angle COD = 2\gamma, \angle DOE = 2\delta$, where p, q, r, s, d are positive integers and $\alpha, \beta, \gamma, \delta$ are positive numbers such that $\alpha + \beta + \gamma + \delta = \dfrac{\pi}{2}$.

We have

$$\begin{aligned} \sin\alpha &= \frac{p}{d},\ \cos\alpha = \frac{\sqrt{d^2 - p^2}}{d},\ \sin\beta = \frac{q}{d},\ \cos\beta = \frac{\sqrt{d^2 - q^2}}{d}, \\ \sin\gamma &= \frac{r}{d},\ \cos\gamma = \frac{\sqrt{d^2 - r^2}}{d},\ \text{and}\ \sin\delta = \frac{s}{d}. \end{aligned}$$

Since $\sin\delta = \cos(\alpha + \beta + \gamma)$, so

$$d^2 s = \sqrt{d^2 - p^2}\sqrt{d^2 - q^2}\sqrt{d^2 - r^2} - \left(\sqrt{d^2 - p^2}\right) qr - \left(\sqrt{d^2 - q^2}\right) rp - \left(\sqrt{d^2 - r^2}\right) pq.$$

It is not hard to see that if at least one of the $\sqrt{d^2 - p^2}\sqrt{d^2 - q^2}\sqrt{d^2 - r^2}$ is irrational, then $d^2 s$ is also irrational. Hence we seek the primitive Pythagorean triples with $d = m^2 + n^2,\ p \in \{2mn, m^2 - n^2\}$ such that

$$\frac{\sqrt{d^2 - p^2}\sqrt{d^2 - q^2}\sqrt{d^2 - r^2} - \left(\sqrt{d^2 - p^2}\right) qr - \left(\sqrt{d^2 - q^2}\right) rp - \left(\sqrt{d^2 - r^2}\right) pq}{d^2}$$

is a positive integer. We now find with a computer that the minimum perimeter is 164, given by $(d, p, q, s) = (65, 16, 25, 25, 33)$ and other combinations.

Comment by Editor. **David Stone and John Hawkins of Statesboro, GA** approached the problem in the above manner. They ran a MATLAB program checking all integer combinations for p, q, r, s, and $d \leq 5000$, and using the data from this program they showed that they, like Kee-Wai, had actually found the smallest solution. They then went on to list some additional pentagons satisfying the requirements of the problem, but with larger perimeters. (I have substituted Kee-Wai's notation into David and John's matrix.)

d	p	q	r	s	*perimeter*
85	13	36	40	40	214
125	35	35	44	75	314
130	32	50	50	66	328
145	17	24	87	87	360
170	26	72	80	80	428
185	57	60	60	104	466
195	48	75	75	99	492
205	45	45	84	133	512
221	21	85	85	140	552
250	70	70	88	150	628
255	39	108	120	120	642
260	64	100	100	132	656
265	23	96	140	140	664
290	34	48	174	174	720
305	55	55	136	207	758
325	36	80	91	260	792
325	36	80	165	204	810
325	36	91	253	91	796
325	36	91	165	195	812
325	80	80	125	204	814
325	36	125	165	165	816
325	80	125	195	91	816
325	80	125	125	165	820

David and John went on to state that they didn't know if an analytical proof exists (as opposed to a computer one) that shows that the minimum perimeter is given by $(d, p, q, s) = (65, 16, 25, 25, 33)$. They then made the following comments.

There is interesting territory for further exploration of these integer-valued, convex, pentagons inscribed in a semi-circle ("Korbin pentagons"?). We could define (p, q, r, s, d) to be *primitive* if there is no common factor and hope to find a parametric-type formula for generating the primitive ones. Based on our few examples, it appears that in a primitive pentagon, s must be the non-prime hypotenuse of a primitive Pythagorean triple, that d is the product of primes congruent to $1(\text{mod } 4)$ and p, q, r and s must be legs of some right triangle having d as its hypotenuse. It also seems that d must be expressible in more than one way as the sum of squares. It looks like many primitive pentagons exist. For example with $d = 325 = 5^2 \cdot 13$, there are several primitive pentagons (and one multiple of our minimal example). Two of these primitive pentagons even have the same perimeter!

Also solved by David Stone and John Hawkins of Statesboro, GA, the proposer.

- **5184:** *Proposed by Neculai Stanciu, Buzău, Romania*

If x, y and z are positive real numbers, then prove that

$$\frac{(x+y)(y+z)(z+x)}{(x+y+z)(xy+yz+zx)} \geq \frac{8}{9}.$$

Solution 1 by Pedro Pantoja, Natal-RN, Brazil

If x, y, z are positive real numbers then,

$$(x+y)(y+z)(z+x) = (xy+yz+zx)(x+y+z) - xyz.$$

So,

$$\frac{(x+y)(y+z)(z+x)}{(xy+yz+zx)(x+y+z)} = 1 - \frac{xyz}{(xy+yz+zx)(x+y+z)}.$$

By the AM-GM inequality,

$$x+y+z \geq 3\sqrt[3]{xyz} \text{ and } xy+yz+zx \geq 3\sqrt[3]{(xyz)^2},$$

which implies that $(xy+yz+zx)(x+y+z) \geq 9xyz$. And this implies that

$$\begin{aligned} \frac{xyz}{(xy+yz+zx)(x+y+z)} &\leq \frac{1}{9}. \text{ So,} \\ 1 - \frac{xyz}{(xy+yz+zx)(x+y+z)} &\geq 1 - \frac{1}{9} = \frac{8}{9}. \text{ Therefore,} \\ \frac{(x+y)(y+z)(z+x)}{(xy+yz+zx)(x+y+z)} &\geq \frac{8}{9}. \end{aligned}$$

Solution 2 by Bruno Salgueiro Fanego, Viveiro Spain

$$\begin{aligned} &\frac{9(x+y)(y+z)(z+x)}{8(x+y+z)(xy+yz+zx)} \\ &= \frac{9(xy+yz+zx+y^2)(z+x)}{8(x+y+z)(xy+yz+zx)} \\ &= 9\frac{(xy+yz+zx)(z+x)+y^2(z+x)}{8(z+y+z)(xy+yz+zx)} \\ &= 9\frac{(x+y+z)(xy+yz+zx)-(xy+yz+zx)y+y^2(z+x)}{8(x+y+z)(xy+yz+zx)} \\ &= 9\frac{(x+y+z)(xy+yz+zx)-(xy^2-y^2z-xyz+y^2z+xy^2)}{8(x+y+z)(xy+yz+zx)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{9(x+y+z)(xy+yz+zx)-9xyz}{8(x+y+z)(xy+yz+zx)} \\
&= \frac{8(x+y+z)(xy+yz+zx)+(x+y+z)(xy+yz+zx)-8xyz-xyz}{8(x+y+z)(xy+yz+zx)} \\
&= \frac{8(x+y+z)(xy+yz+zx)-8xyz+(x+y+z)(xy+yz+zx)-xyz}{8(x+y+z)(xy+yz+zx)} \\
&\geq \frac{8(x+y+z)(xy+yz+zx)-8xyz+3\sqrt[3]{xyz}3\sqrt[3]{xyyzzx}-xyz}{8(x+y+z)(xy+yz+zx)} \\
&= \frac{8(x+y+z)(xy+yz+zx)-8xyz+9xyz-xyz}{8(x+y+z)(xy+yz+zx)} \\
&= \frac{8(x+y+z)(xy+yz+zx)}{8(x+y+z)(xy+yz+zx)} \\
&= 1,
\end{aligned}
$$

which is equivalent to the proposed inequality, where the $AM-GM$ inequality has been applied. We note that equality holds if, and only if, $x=y=z$.

Solution 3 by Paul M. Harms, North Newton, KS

Let $y=tx$ and $z=rx$ where x, r, and t are positive teal numbers. Then the inequality to be proved becomes

$$\frac{(1+t)(t+r)(1+r)}{(1+r+t)(r+rt+t)} \geq \frac{8}{9}.$$

The following inequalities are equivalent to the above inequality.

$$
\begin{aligned}
9(1+t)(t+r)(1+r) &\geq 8(1+r+t)(r+rt+t) \\
9(t+t+2rt+t^2+r^2+rt^2+r^2t) &\geq 8(r+3rt+t+r^2+t^2+rt^2+r^2t \\
r+t-6rt+t^2+r^2+rt^2+r^2t &\geq 0.
\end{aligned}
$$

To prove the last inequality we write the left side of this inequality as follows:

$$(t^2-2rt+r^2)+r(t^2-2t+1)+t(r^2-2r+1)=(t-r)^2+r(t-1)^2+t(r-1)^2.$$

Since all three terms are non-negative, the above inequalities are correct. Thus the inequality in the problem has been proved.

Solution 4 by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany

Normalizing the LHS one can assume that $x+y+z=1$ and so our inequality will become equivalent to proving

$$\frac{(x+y)(y+z)(z+x)}{(xy+yz+zx)} \geq \frac{8}{9}$$

subject to $x+y+z=1$ and the fact that x, y and z are positive real numbers.

We will use the Lagrange Multiplier method to find the minimum of the function $f(x,y,z) = \dfrac{(x+y)(y+z)(z+x)}{(xy+yz+zx)}$ subject to $g(x,y,z) = x+y+z = 1$ and the fact that x, y and z are positive real numbers.

Doing easy manipulations we have that

$$\bigtriangledown f(x,y,z) \quad = \quad < f_x, f_y, f_z >$$

$$< \quad \frac{x(y+z)(xy+2yz+zx}{(xy+yz+zx)^2}, \frac{y(z+x)(xy+yz+2zx}{(xy+yz+zx)^2}, \frac{z(x+y)(2xy+yz+zx}{(xy+yz+zx)^2} >$$

and

$$\bigtriangledown g(x,y,z) =< g_x, g_y, g_z >=< 1,1,1 >$$

applying the Lagrange Multiplier method, the extremes will be the solutions of the system of the equations

$$\begin{cases} \bigtriangledown f(x,y,z) = \lambda \bigtriangledown g(x,y,z) \\ g(x,y,z) = 1 \end{cases}$$

which is equivalent to

$$\begin{cases} \dfrac{x(y+z)(xy+2yz+zx)}{(xy+yz+zx)^2} = \lambda \\ \\ \dfrac{y(z+x)(xy+yz+2zx)}{(xy+yz+zx)^2} = \lambda \\ \\ \dfrac{z(x+y)(2xy+yz+zx)}{(xy+yz+zx)^2} = \lambda \\ \\ x+y+z=1 \end{cases}$$

where λ is a real number.

Solving this easy system of equations we have that the solutions will be

$$(x,y,z,\lambda) = \{(1,1,-1,0),(1,-1,1,0),(-1,1,1,0),\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{8}{9}\right)\}$$

Using the fact that x, y and z are positive real numbers, the only point of interest will be

$$(x,y,z) = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$$

and the value of the function at that point will be

$$f\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) = \frac{\left(\frac{1}{3}+\frac{1}{3}\right)\left(\frac{1}{3}+\frac{1}{3}\right)\left(\frac{1}{3}+\frac{1}{3}\right)}{\left(\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{3}\right)} = \frac{8}{9}$$

Getting the value of the function $f(x,y,z)$ at another point, let say $(x,y,z) = \left(\frac{1}{2},\frac{1}{4},\frac{1}{4}\right)$ we have

$$f\left(\frac{1}{2},\frac{1}{4},\frac{1}{4}\right) = \frac{\left(\frac{1}{2}+\frac{1}{4}\right)\left(\frac{1}{4}+\frac{1}{4}\right)\left(\frac{1}{4}+\frac{1}{2}\right)}{\left(\frac{1}{2}\times\frac{1}{4}+\frac{1}{4}\times\frac{1}{4}+\frac{1}{4}\times\frac{1}{2}\right)} = \frac{9}{10} > \frac{8}{9}$$

we have that the extreme point $(x,y,z) = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ is a minimum and this is the end of the proof.

Solution 5 by Kee-Wai Lau, Hong Kong, China

It can be checked readily that

$$\frac{(x+y+)(y+z)(z+x)}{(x+y+z)(xy+yz+zx)} = \frac{x(y-z)^2+y(z-x)^2+z(x-y)^2}{9(x+y+z)(xy+yz+zx)} + \frac{8}{9},$$

and the inequality of the problem follows.

Solution 6 by Andrea Fanchini, Cantú Italy

Let $p = x+y+z, q = xy+yz+zx$ and $r = xyz$. Then the given inequality becomes

$$\frac{pq-r}{pq} \geq \frac{8}{9}.$$

I.e.,

$$pq \geq 9r,$$

that we can prove easily using the AM-GM inequality,

$$pq = (x+y+z)(xy+yz+zx) \geq 3\sqrt[3]{xyz} \cdot 3\sqrt[3]{x^2y^2z^2} = 9r.$$

So the proposed inequality is proved.

Comment by Albert Stadler of Herrliberg, Switzerland

I tried to generalize this problem to n variables and conjectured the following statement: Let $x_1, x_2, \ldots, x_n$ be n real positive numbers , $n \geq 2$. Then

$$\frac{\prod_{i=1}^{n}(x_i+x_{i+1})}{x_1\cdot x_2\cdots x_n\sum_{i=1}^{n}x_i\cdot\sum_{i=1}^{n}\frac{1}{x_i}} \geq \frac{2^n}{n^2}, \text{ with the assumption that } x_{n+1}=x_1.$$

For $n=2$ this says that $\dfrac{(x_1+x_2)^2}{(x_1+x_2)^2} \geq \dfrac{2^2}{2^2}$ which is true.

For $n=3$ we have the statement of problem 5184.

For $n=4$ this says that $\dfrac{(x_1+x_2)(x_2+x_3)(x_3+x_4)(x_4+x_1)}{(x_1+x_2+x_3+x_4)(x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4)} \geq \dfrac{2^4}{4^2}$, which is equivalent to $(x_1x_3-x_2x_4)^2 \geq 0$, and this is obviously true.

However it turns out that the statement is false for $n=5$ as is evidenced by the counterexample $(x_1,x_2,x_3,x_4,x_5) = (8,3,1,2,8)$.

Also solved by Daniel Lopez Aguayo, Institute of Mathematics (UNAM) Morelia, Mexico; Dionne Bailey, Elsie Campbell and Charles Diminnie, San Angelo, TX; Scott H. Brown, Montgomery, AL; Michael Brozinsky, Central Islip, NY; David E. Manes, Oneonta NY; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Paolo Perfetti (two solutions), Department of Mathematics, University "Tor Vergata," Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5185:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Calculate, without using a computer, the value of

$$\sin\left[\arctan\left(\frac{1}{3}\right)+\arctan\left(\frac{1}{5}\right)+\arctan\left(\frac{1}{7}\right)+\arctan\left(\frac{1}{11}\right)+\arctan\left(\frac{1}{13}\right)+\arctan\left(\frac{111}{121}\right)\right].$$

Solution 1 by Andrea Fanchini, Cantú, Italy

Knowing that the argument of the product of complex numbers is the sum of the arguments of the factors, we can see that

$$\theta=\arctan\left(\frac{1}{3}\right)+\arctan\left(\frac{1}{5}\right)+\arctan\left(\frac{1}{7}\right)+\arctan\left(\frac{1}{11}\right)+\arctan\left(\frac{1}{13}\right)+\arctan\left(\frac{111}{121}\right)$$

is the argument of the following multiplication

$$(3+i)\,(5+i)\,(7+i)\,(11+i)\,(13+i)\,(121+111i)$$

multiplying in the usual way, we obtain the pure imaginary number $2696200i$, so $\theta=\dfrac{\pi}{2}$ and then finally we have $\sin\left(\dfrac{\pi}{2}\right)=1$.

Solution 2 by Anastasios Kotronis, Athens, Greece

The following identities are well known:

$$\arctan a+\arctan b=\begin{cases}\arctan\dfrac{a+b}{1-ab} & ,\ ab<1\\[2ex] \arctan\dfrac{a+b}{1-ab}+\pi & ,\ ab>1\wedge a>0\\[2ex] \arctan\dfrac{a+b}{1-ab}-\pi & ,\ ab>1\wedge a<0\end{cases}$$

$$\arctan a+\arctan\frac{1}{a}=\begin{cases}\dfrac{\pi}{2} & ,\ a>0\\[2ex] -\dfrac{\pi}{2} & ,\ a<0\end{cases}$$

Applying these formulas to the pair $\arctan\left(\dfrac{1}{3}\right), \arctan\left(\dfrac{1}{5}\right)$, and then repeating to $\arctan\left(\dfrac{1}{7}\right), \arctan\left(\dfrac{1}{11}\right)$ and to $\arctan\left(\dfrac{1}{13}\right)$, we obtain that

$$\begin{aligned}
& \sin\left(\arctan\left(\frac{1}{3}\right)+\arctan\left(\frac{1}{5}\right)+\arctan\left(\frac{1}{7}\right)+\arctan\left(\frac{1}{11}\right)+\arctan\left(\frac{1}{13}\right)+\arctan\left(\frac{111}{121}\right)\right) \\
= & \sin\left(\arctan\left(\frac{121}{111}\right)+\arctan\left(\frac{111}{121}\right)\right) \\
= & \sin\frac{\pi}{2} \\
= & 1.
\end{aligned}$$

Also solved by Brian D. Beasley, Clinton, SC; Dionne Bailey, Elsie Campbell and Charles Diminnie, San Angelo, TX; Michael C. Faleski (two solutions), University Center, MI; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Kenneth Korbin, New York, NY; David E. Manes, Oneonta NY; Kee-Wai Lau, Hong Kong, China; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy; Boris Rays (two solutions), Brooklyn, NY; Neculai Stanciu, Buzău, Romania with Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins (jointly), Statesboro, GA; Albert Stadler, Herrilberg, Switzerland, and the proposer.

- **5186:** *Proposed by John Nord, Spokane, WA*

  Find k so that $\displaystyle\int_0^k \left(-\frac{b}{a}x+b\right)^n dx = \frac{1}{2}\int_0^a \left(-\frac{b}{a}x+b\right)^n dx.$

Solution by Ángel Plaza, Department of Mathematics, University of Las Palmas de Gran Canaria, Spain

It is clear that if $b=0$, then the equation holds for every k. Assuming that the parameter $b \neq 0$, it may be removed, and the problem then becomes to find k so that

$$\int_0^k \left(\frac{a-x}{a}\right)^n dx = \frac{1}{2}\int_0^a \left(\frac{a-x}{a}\right)^n dx.$$

We may assume that $a>0$. Integrating we obtain:

$$\left(\frac{a-x}{a}\right)^n\Bigg]_0^k = \frac{1}{2}\left(\frac{a-x}{a}\right)^n\Bigg]_0^a \Rightarrow \left(\frac{a-k}{a}\right)^n - 1 = -\frac{1}{2}.$$

And, therefore $k = a\left(1-\dfrac{1}{\sqrt[n+1]{2}}\right)$ if n is even, while $k = a\left(1 \pm \dfrac{1}{\sqrt[n+1]{2}}\right)$ if n is odd.

Comment by David Stone and John Hawkins of Statesboro, GA. When a and b are positive, we have the usual area interpretation of our result. The problem asks us to determine how far along we should move to capture half of the area from 0 to a.

In this case, the graph of $y = \left(\dfrac{-b}{a}\right)(x-a)^n$ has $y-$intercept $(0,b)$, and drops off to its $x-$intercept $(a,0)$, so the integral $\displaystyle\int_0^a \left(-\frac{b}{a}x+b\right)^n dx$ actually represents the area under the curve.

For n even, the graph bottoms out at $(a, 0)$ and stays above the x-axis and we see that $k = \left(1 - \frac{1}{\sqrt[n+1]{2}}\right) a$ is the "magical" spot where we halve the area.

For n odd, the graph slices through the $x-$axis at $(a, 0)$ and is symmetric about this $x-$intercept, so we have two values of k. The first $k_1 = a - \frac{a}{\sqrt[n+1]{2}}$, actually marks the spot where half of the area from 0 to a is achieved, while the second $k_2 = a + \frac{a}{\sqrt[n+1]{a}}$ marks the spot where the net area once again equals half of the area from 0 to a.

Note that the geometrical interpretation is more complicated when a and/or b is negative, but the same values of k provide the correct area interpretation.

Also solved by Daniel Lopez Aguayo, Institute of Mathematics (UNAM) Morelia, Mexico; Brian D. Beasley, Clinton, SC; Michael C. Faleski, University Center, MI; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta NY; Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy; Boris Rays, Brooklyn, NY; Neculai Stanciu, Buzău, Romania; David Stone and John Hawkins (joinlty), Statesboro, GA and the proposer.

- **5187:** *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

Let $f : [0, 1] \to (0, \infty)$ be a continuous function. Find the value of

$$\lim_{n\to\infty} \left(\frac{\sqrt[n]{f(\frac{1}{n})} + \sqrt[n]{f(\frac{2}{n})} + \cdots + \sqrt[n]{f(\frac{n}{n})}}{n} \right)^n.$$

Solution by Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy

The answer is $e^{\int_0^1 \ln(f(x))dx}$

Proof.

$$\sqrt[n]{f(\frac{k}{n})} = \exp\{\frac{1}{n}\ln[f(\frac{k}{n})]\} = 1 + \frac{1}{n}\ln\left(f(\frac{k}{n})\right) + O(\frac{1}{n^2}).$$

We observe that being the function continuous, it is bounded above and below and the lower bound is positive by the positivity of the function namely $0 < m \le f(x) \le M$ for any $x \in [a, b]$. This allowed us to write $O(1/n^2)$ in the last term regardless the presence of the function $f(x)$. Thus,

$$\frac{\sqrt[n]{f(\frac{1}{n})} + \sqrt[n]{f(\frac{2}{n})} \ldots \sqrt[n]{f(\frac{k}{n})}}{n} = 1 + \frac{1}{n^2}\sum_{k=1}^{n} \ln(f(\frac{k}{n})) + O(\frac{1}{n^2}) \doteq 1 + \frac{p_n}{n} + O(\frac{1}{n^2})$$

and (use $\ln(1 + x) = x + O(x^2)$).

$$(1 + \frac{p_n}{n} + O(\frac{1}{n^2}))^n = \exp\{n\ln(1 + \frac{p_n}{n} + O(\frac{1}{n^2})\} = \exp\{p_n + O(\frac{p_n}{n}) + O(\frac{1}{n^2}) + O(\frac{p_n^2}{n^2})\}.$$

The quantity p_n is clearly the Riemann sum of $\int_0^1 \ln(f(x))dx$ and then the last exponential is bounded below and above since $\ln(m) \leq \int_0^1 \ln(f(x))dx \leq \ln(M)$. We obtain

$$\exp\{p_n + O(\frac{p_n}{n}) + O(\frac{1}{n^2}) + O(\frac{p_n^2}{n^2})\} \to \ e^{\int_0^1 \ln(f(x))dx}.$$

Also solved by Kee-Wai Lau, Hong Kong, China; Neculai Stanciu, Buzău, Romania; Albert Stadler, Herrilberg, Switzerland, and the proposer.

Addendum

The name of **Brian D. Beasley of Clinton, SC** was inadvertently left off the list of having solved problem 5176. Sorry Brian, mea culpa.

Also, **Albert Stadler of Herrilberg, Switzerland** noticed two typos in the February, 2012 issue of the column. In his solution to 5176, the fourth line in the first equation array lists the term y^3, but it should be y^2. And in his solution to 5178, the last line should have been $x^2 + y^2 + z^2$ and not $x^3 + y^3 + z^3$. Again, sorry, mea culpa.