# Problems Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

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*Solutions to the problems stated in this issue should be posted before*
*May 15, 2015*

- **5343:** *Proposed by Kenneth Korbin, New York, NY*

  Four different Pythagorean Triangles each have hypotenuse equal to $4p^4 + 1$ where $p$ is prime.

  Express the sides of these triangles in terms of $p$.

- **5344:** *Proposed by Y. N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan*

  Let $\triangle ABC$ be isosceles with $AB = AC$. Let $D$ be a point on side $BC$. A line through point $D$ intersects rays $AB$ and $AC$ at points $E$ and $F$ respectively. Prove that $ED \cdot DF \geq BD \cdot DC$.

- **5345:** *Proposed by Arkady Alt, San Jose, CA*

  Let $a, b > 0$. Prove that for any $x, y$ the following inequality holds

  $$|a\cos x + b\cos y| \leq \sqrt{a^2 + b^2 + 2ab\cos(x+y)},$$

  and find when equality occurs.

- **5346:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

  Show that in any triangle $ABC$, with the usual notations, the following hold,

  $$\frac{h_b + h_c}{h_a} r_a^2 + \frac{h_c + h_a}{h_b} r_b^2 + \frac{h_a + h_b}{h_c} r_c^2 \ \geq \ 2s^2,$$

  where $r_a$ is the excircle tangent to side $a$ of the triangle and $s$ is the triangle's semiperimeter.

- **5347:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $0 < a < b$ be real numbers and let $f, g : [a, b] \to R_+^*$ be continuous functions. Prove that there exists $c \in (a,b)$ such that

$$\left(\frac{1}{f(c)} + \frac{1}{\int_c^b g(t)\,dt}\right)\left(g(c) + \int_a^c f(t)\,dt\right) \geq 4$$

($R_+^*$ represents the set of non-negative real numbers.)

- **5348:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $k \geq 1$ be an integer. Prove that

  $$\int_0^1 \ln^k(1-x)\ln x\,\mathrm{d}x = (-1)^{k+1}k!(k+1-\zeta(2)-\zeta(3)-\cdots-\zeta(k+1)),$$

  where $\zeta$ denotes the Riemann zeta function.

*Solutions*

- **5325:** *Proposed by Kenneth Korbin, New York, NY*

  Given the sequence $x = (1, 7, 41, 239, 1393, 8119, \ldots)$, with $x_n = 6x_{n-1} - x_{n-2}$. Let $y = \dfrac{x_{2n} + x_{2n-1}}{x_n}$. Find an explicit formula for $y$ expressed in terms of $n$.

  **Solution by 1 D.M. Bătinetu-Giurgiu, National College "Matei Basarab," Bucharest, Romania**

  The recurrence sequence $x_n$ has the equation $r^2 - 6r + 1 = 0$ with solutions

  $$\begin{aligned} r_1 &= \left(\sqrt{2}+1\right)^2,\ r_2 = \left(\sqrt{2}-1\right)^2,\ \text{so} \\ x_n &= ur_1^n + vr_2^n = \left(\sqrt{2}+1\right)^{2n}u + \left(\sqrt{2}-1\right)^{2n}v, \end{aligned}$$

  and because $x_1 = 1,\ x_2 = 7$ yields that

  $$(u,v) = \left(\frac{\sqrt{2}-1}{2}, -\frac{\sqrt{2}+1}{2}\right).$$

  Therefore,

  $$\begin{aligned} x_n &= \frac{\left(\sqrt{2}+1\right)^{2n-1} - \left(\sqrt{2}-1\right)^{2n-1}}{2},\ \text{and} \\ y_n &= \frac{x_{2n}+x_{2n-1}}{x_n} = 2\sqrt{2}\left(\left(\sqrt{2}+1\right)^{2n-1} + \left(\sqrt{2}-1\right)^{2n-1}\right), \end{aligned}$$

and we are done.

**Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX**

In our solution to Problem 5308 (see Dec. 2014 issue of this column ), we used the techniques for solving homogeneous linear difference equations to show that the closed form expression for $x_n$ is

$$x_n = \frac{\left(\sqrt{2}+1\right)^{2n-1} - \left(\sqrt{2}-1\right)^{2n-1}}{2}$$

for all $n \geq 1$. It follows that for all $n \geq 1$,

$$\begin{aligned}
x_{2n} + x_{2n-1} &= \frac{\left(\sqrt{2}+1\right)^{4n-1} - \left(\sqrt{2}-1\right)^{4n-1}}{2} + \frac{\left(\sqrt{2}+1\right)^{4n-3} - \left(\sqrt{2}-1\right)^{4n-3}}{2} \\
&= \frac{\left(\sqrt{2}+1\right)^{4n-3}\left[\left(\sqrt{2}+1\right)^2+1\right] - \left(\sqrt{2}-1\right)^{4n-3}\left[\left(\sqrt{2}-1\right)^2+1\right]}{2} \\
&= \frac{\left(\sqrt{2}+1\right)^{4n-3}\left[2\left(2+\sqrt{2}\right)\right] - \left(\sqrt{2}-1\right)^{4n-3}\left[2\left(2-\sqrt{2}\right)\right]}{2} \\
&= \left(\sqrt{2}+1\right)^{4n-3}\left(2+\sqrt{2}\right) - \left(\sqrt{2}-1\right)^{4n-3}\left(2-\sqrt{2}\right) \\
&= \sqrt{2}\left[\left(\sqrt{2}+1\right)^{4n-2} - \left(\sqrt{2}-1\right)^{4n-2}\right] \\
&= \sqrt{2}\left[\left(\sqrt{2}+1\right)^{2n-1} + \left(\sqrt{2}-1\right)^{2n-1}\right]\left[\left(\sqrt{2}+1\right)^{2n-1} - \left(\sqrt{2}-1\right)^{2n-1}\right] \\
&= 2\sqrt{2}x_n\left[\left(\sqrt{2}+1\right)^{2n-1} + \left(\sqrt{2}-1\right)^{2n-1}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
y &= \frac{x_{2n} + x_{2n-1}}{x_n} \\
&= 2\sqrt{2}\left[\left(\sqrt{2}+1\right)^{2n-1} + \left(\sqrt{2}-1\right)^{2n-1}\right]
\end{aligned}$$

for all $n \geq 1$.

**Solution 3 by G. C. Greubel, Newport News, VA**

First consider the difference equation

$$x_{n+2} = 6x_{n-1} - x_n \tag{1}$$

which has the general solution $x_n = Aa^{2n} + Bb^{2n}$ where $a = 1+\sqrt{2}$ and $b = 1-\sqrt{2}$. For the initial conditions $x_0 = 1$ and $x_1 = 7$ the sequence $x_n$ has the solution $x_n = Q_{2n+1}/2$, where $Q_n$ are the Pell-Lucas numbers with the recurrence relation $Q_{n+2} = 2Q_{n+1} + Q_n$. The element $x_{2n+1} + x_{2n-1}$ can be determined to be $4P_{4n}$, where $P_n$ are the Pell numbers. This leads to the desired quantity being sought as

$$y_n = \frac{8P_{4n}}{Q_{2n+1}}. \tag{2}$$

**Comment by Henry Ricardo, New York Math Circle, NY.** The numbers $x_n$ in the proposed problem are the **NSQ numbers** (named for Newman, Shanks, and Williams, authors of an influential 1980 group theory paper.) The *On-Line Encyclopedia of Integer Sequences* (OEIS) lists the sequence as entry $A002315$ and gives the formula (without proof)

$$x_n = \frac{(1+\sqrt{2})(3+2\sqrt{2})^n + (1-\sqrt{2})(3-2\sqrt{2})^n}{2} = \frac{(1+\sqrt{2})^{2n+1} + (1-\sqrt{2})^{2n+1}}{2},$$

for non-negative integers $n$. In addition to many comments on the sequence itself, the connection between this sequence and other OEIS entries are also pointed out.

**Also solved by Arkady Alt, San Jose, CA; Brian D. Beasley, Presbyterian College, Clinton, SC; Bruno Salgueiro Fanego, Viveiro, Spain; Ethan Gegner (student at Taylor University), Upland IN; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Tsvetelina Karamfilova, Petko Rachov Slaveikov Secondary School, Kardzhali, Bulgaria; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Carl Libis of the University of Tennessee at Martin, TN; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Corneliu Mănescu-Avram, Transportation High School Ploiesti, Romania; Angel Plaza, Universidad de Las Palmas, de Gran, Canaria, Spain; Henry Ricardo, New York Math Circle, NY; Neculai Stanciu, "George Emil Palade School," Buzău, Romania (jointly with) Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Albert Stadler, Herrliberg, Switzerland,**

- **5326:** *Proposed by Armend Sh. Shabani, University of Prishtina, Republic of Kosova*

Find all positive integer solutions to $m! + 2^{4k-1} = l^2$.

**Solution 1 by Ed Gray, Highland Beach, FL**

We note that $2^{(4k-1)}$ always seems to end with the integer 8 for all values of $k$. We prove this by induction. The statement is obviously true for $k = 1$. Assume that the statement is true for all positive integers up to and including $k$. I.e. $2^{(4k-1)}$ ends in with the integer 8. Does this $2^{(4k-1)}$ imply that $2^{(4(k+1)-1)}$ also ends with the integer 8?

$$2^{(4(k+1)-1)} = 2^{(4k+3)} = 2^4\left(2^{(4k-1)}\right) = 16\left(2^{(4k-1)}\right).$$

But by the induction hypothesis, $2^{(4k-1)}$ ends with integer 8 and so $16\left(2^{(4n-1)}\right)$ also ends with the integer 8.

Now we note that for all integers $m \geq 5$, the integer $m!$ ends with the integer 0. So, $m! + 2^{(4k-1)}$ ends in 8 for all integers $m \geq 5$. But there is no square number whose units digit is 8. So if there are any integer solutions to $m! + 2^{4k-1} = l^2$, the value of the positive integer $m$ must be 1, 2, 3, or 4.

If $m = 4$, then $m! = 24$ ends in a 4 and so $m! + 2^{4k-1}$ ends with the unit's digit in 4+8, so $l^2$ must end in 2, but there is no integer whose square ends with a 2 So $m \neq 4$.
If $m = 3$, then $m! = 6$ and so $m! + 2^{4k-1}$ ends with a unit's digit of 6+8. That is, the units digit of $l^2$ must be 4, which implies that $l$ must be even. Suppose that $l = 2r$. Then

$$6 + 2^{4k-1} \quad = \quad l^2$$

$$
\begin{aligned}
6+2^{4k-1} &= (2r)^2 \\
6+2^{4k-1} &= 4r^2
\end{aligned}
$$

But 4 divides the right hand side and 4 divides $2^{4k-1}$, but 4 does not divide 6 so, $m \neq 3$.

If $m = 2$, then $m! = 4$ and so $m! + 2^{4k-1}$ ends with a unit's digit of 2, but there is no integer square has a units digit of 2. So,$m \neq 2$.

Finally, if $m = 1$ then $m! + 2^{4k-1}$ becomes

$$
\begin{aligned}
1+2^{4k-1} &= l^2 \\
2^{4k-1} &= l^2 - 1 \\
2^{4k-1} &= (l-1)(l+1).
\end{aligned}
$$

So, both factors $(l-1)$ and $(l+1)$ must be a power of 2.

Let $l - 1 = 2^a$ and $l + 1 = 2^b$. Subtracting gives $2 = 2^b - 2^a$ whose only solution is $b = 2$ and $a = 1$. So $l - 1 = 2^1 = 2$ and $l + 1 = 2^2$

Since

$$
\begin{aligned}
2^{4k-1} &= (l-1)(l+1) \\
2^{4k-1} &= (2)(4) \\
2^{4k-1} &= (2^3), \text{ so,} \\
k &= 1.
\end{aligned}
$$

The only solution to $m! + 2^{4k-1} = l^2$ is when $m = 1, k = 1$ and $l = 3$.

**Solution 2 by Jerry Chu, (student at Saint George's School), Spokane, WA**

We note that $2^{4k-1} \mod 3$ is 2. And $l^2 \mod 3$ is either 0 or 1. So, $m!$ must not be a multiple of 3. Therefore, $m = 1$ or 2.

When $m = 1$, $2^{4k-1} = l^2 - 1 = (l+1)(l-1)$.

Because $(l-1)$ and $(l+1)$ can only be powers of 2, $l$ must equal 3. So
$m = 1, k = 1, l = 3$.

When $m = 2$, we take all terms mod 4 and see that $2 + 0 = 0 + 1$, which is impossible.

Therefore the only solution is $m = 1, k = 1, l = 3$.

**Solution 3 by Adnan Ali (student in A.E.C.S-4), Mumbai, India**

Assume that for $m \geq 3$, there exist solutions. Then putting the equation modulo 3, we see that

$$l^2 = m! + 2^{4k-1} \equiv (-1)^{4k-1} \equiv -1 \pmod 3$$

but $-1$ is not a quadratic residue modulo 3. So we conclude that $m \leq 2$. But now we may assume that there is a solution for $m = 2$, then we simply realize the fact that

$$l^2 = 2! + 2^{4k-1} \equiv 2 + 0 \pmod 4,$$

and since 2 is not a quadratic residue modulo 4, we are left with the only option $m = 1$. So, we have

$$2^{4k-1} + 1 = l^2 \Leftrightarrow (l+1)(l-1) = 2^{4k-1}$$

and so we must have both $l+1, l-1$ as powers of 2, so we let $l+1 = 2^a > l-1 = 2^b$ for integers $a, b$ so that $a + b = 4k + 1$ and see that $2^a - 2^b = 2 = 2^b(2^{a-b} - 1)$ forcing $2^b = 2$ and $2^{a-b} - 1 = 1$ which has the only solution $(a, b) = (2, 1)$ and $2 + 1 = 3 = 4k - 1$ implies that $k = 1$.
So we conclude that the only possible solution is $(l, m, k) = (3, 1, 1)$.

**Solution 4 by Henry Ricardo, New York Math Circle, NY**

The triple $(m, k, l) = (1, 1, 3)$ is the only solution in positive integers.

To prove this assertion, we use the following easily established facts: (1) $2^{4k-1} \equiv 8 \pmod{10}$ for positive integers $k$; (2) If $l^2 \equiv r \pmod{10}$, then $r \in S = \{0, 1, 4, 5, 6, 9\}$.

First, if $m \geq 5$, then $m! \equiv 0 \pmod{10}$ so that $m! + 2^{4k-1} \equiv 8 \pmod{10}$. But $8 \notin S$. Thus $1 \leq m < 5$.

If $m = 2$, then $N = m! + 2^{4k-1} = 2(1 + 2^{4k-2})$, which can't be a perfect square since $1 + 2^{4k-2}$ is odd, implying that the prime divisor 2 does not appear with an even exponent in the prime power factorization of $N$. Similarly, if $m = 3$, then $m! + 2^{4k-1} = 2(3 + 2^{4k-2})$, which can't be a perfect square.

Finally, we eliminate $m = 4$ since $m! + 2^{4k-1} = 24 + 2^{4k-1} \equiv 4 + 8 \equiv 2 \pmod{10}$ and $2 \notin S$.

**Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Brian D. Beasley, Presbyterian College, Clinton, SC; Ethan Gegner (student, Taylor University), Upland, IN; Paul M. Harms, North Newton, KS; Bruno Salgueiro Fanego, Viveiro, Spain; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Corneliu Mănescu-Avram, Transportation High School Ploiesti, Romania; Haroun Meghaichi (student, University of Science and Technology Houari Boumediene), Algeria; Neculai Stanciu, "George Emil Palade School," Buzău, Romania (jointly with) Titu Zvonaru, Comănesti, Romania; Albert Stadler, Herrliberg, Switzerland, and the proposer.**

- **5327:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

Show that in any triangle $ABC$, with the usual notations, that

$$\left(\frac{ab}{a+b}\right)^2 + \left(\frac{bc}{b+c}\right)^2 + \left(\frac{ca}{c+a}\right)^2 \geq 9r^2.$$

**Solution 1 by Kee-Wai Lau, Hong Kong, China**

By the Cauchy-Schwarz inequality, we have

$$\left(\frac{ab}{a+b}\right)^2+\left(\frac{bc}{b+c}\right)^2+\left(\frac{ca}{c+a}\right)^2 \geq \frac{1}{3}\left(\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}\right)^2.$$

Hence it suffices to show that

$$\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a} \geq 3\sqrt{3}r. \qquad (1)$$

Let $s$ be the semiperimeter and $F$ the area of triangle $ABC$. It is well known that

$$F = rs = \frac{ab\sin C}{2} = \frac{bc\sin A}{2} = \frac{ca\sin B}{2}.$$ So (1) is equivalent to

$$(a+b+c)\left(\frac{1}{(a+b)\sin C}+\frac{1}{(b+c)\sin A}+\frac{1}{(c+a)\sin B}\right) \geq 3\sqrt{3}. \qquad (2)$$

Let $S_1 = \dfrac{1}{\sin A}+\dfrac{1}{\sin B}+\dfrac{1}{\sin C}$ and $S_2 = \dfrac{a}{\sin A(b+c)}+\dfrac{b}{\sin B(c+a)}+\dfrac{c}{\sin C(a+b)}$ so that the left side of (2) can be written as $S_1+S_2$. By the convexity of the function $\dfrac{1}{\sin x}$, for $0<x<\pi$, we have $S_1 \geq 3\left(\dfrac{1}{\sin\left(\frac{A+B+C}{3}\right)}\right) = 2\sqrt{3}$. By the sine formula, we have

$$\begin{aligned}
S_2 &= \frac{1}{\sin B+\sin C}+\frac{1}{\sin C+\sin A}+\frac{1}{\sin A+\sin B} \\
&= \frac{1}{2}\left(\frac{1}{\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}+\frac{1}{\sin\left(\frac{C+A}{2}\right)\cos\left(\frac{C-A}{2}\right)}+\frac{1}{\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}\right) \\
&\geq \frac{1}{2}\left(\frac{1}{\sin\left(\frac{B+C}{2}\right)}+\frac{1}{\sin\left(\frac{C+A}{2}\right)}+\frac{1}{\sin\left(\frac{A+B}{2}\right)}\right) \\
&= \frac{1}{2}\left(\sec\left(\frac{A}{2}\right)+\sec\left(\frac{B}{2}\right)+\left(\frac{C}{2}\right)\right).
\end{aligned}$$

Hence by the convexity of the function $\sec x$ for $0<x<\dfrac{\pi}{2}$, we have

$$S_2 \geq \frac{3}{2}\sec\left(\frac{A+B+C}{6}\right) = \sqrt{3}.$$

Thus (2) holds and this completes the solution.

**Solution 2 by Perfetti Paolo, Department of Mathematics, University Tor Vergata, Rome, Italy**

The Cauchy–Schwarz inequality yields

$$\left(\frac{ab}{a+b}\right)^2+\left(\frac{bc}{b+c}\right)^2+\left(\frac{ca}{c+a}\right)^2 \geq \frac{(ab+bc+ca)^2}{(a+b)^2+(b+c)^2+(c+a)^2} \geq 9r^2.$$

where $r=\sqrt{(s-a)(s-b)(s-c)/s}$, and $s=(a+b+c)/2$.

Letting $x=(b+c-a)/2$, using the symmetry in the statement of the problem and upon clearing the denominators we obtain

$$\frac{1}{A}\sum_{\text{sym}}\left(17x^3y^2+\frac{1}{2}x^5+7x^4y-9x^3yz-\frac{31}{2}x^2y^2z\right)\geq 0$$

and $A=(3(x^2+y^2+z^2)+5(xy+yz+zx))(x+y+z)>0$. Muirhead's theorem concludes the proof. Indeed

$$[3,2,0]\succ[2,2,1], \quad [5,0,0]\succ[3,1,1], \quad [4,1,0]\succ[3,1,1]$$

The underlying AGM's are

$$x^3y^2+x^3z^3\geq 2x^3yz, \quad 3x^5+y^5+z^5\geq 5x^3yz, \quad 9x^4y+y^4z+3z^4x\geq 13x^3yz$$

and symmetry.

**Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Moti Levy, Rehovot, Israel; Nikos Kalapodis (four solutions), Patras, Greece; Albert Stadler, Herrliberg, Switzerland; Nicusor Zlota (two solutions) "Traian Vuia" Technical College, Focsani, Romania, and the proposer.**

- **5328:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Without the aid of a computer, find the positive solutions of the equation

$$2^{x+1}\left(1-\sqrt{1+x^2+2^x}\right)=\left(x^2+2^x\right)\left(1-\sqrt{1+2^{x+1}}\right).$$

**Solution 1 by Junho Chang, Colegio Hispano-Inglés, and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain**

Multiplying both terms of the given equation by $\left(1+\sqrt{1+x^2+2^x}\right)\left(1+\sqrt{1+2^{x+1}}\right)$ and simplifying we obtain

$$\begin{aligned}\sqrt{1+2^{x+1}} &= \sqrt{1+x^2+2^x}\\ 2^{x+1} &= x^2+2^x\\ 2^x &= x^2.\end{aligned}$$

Taking logarithms, the last equation may be written as $\dfrac{\ln x}{x}=\dfrac{\ln 2}{2}$. Let us consider the function $f(x)=\dfrac{\ln x}{x}$ defined for positive real numbers $x$. Since $f'(x)=\dfrac{1-\ln x}{x^2}$, $f(x)$ is increasing for $x \in (0, e)$ and it is decreasing for $x \in (e, +\infty)$. Since $\lim_{x \to 0^+} f(x) = -\infty$, and $f(e) = 1/e > \ln 2/2$ there is a unique root to the equation $\dfrac{\ln x}{x} = \dfrac{\ln 2}{2}$ in $(0, e)$, which is $x = 2$. Also, since $\lim_{x \to +\infty} f(x) = 0$, there is a unique root to the equation $\dfrac{\ln x}{x} = \dfrac{\ln 2}{2}$ in $(e, +\infty)$, which is $x = 4$. So, $x = 2$ and 4 are the only positive solutions to the problem.

**Solution 2 by Haroun Meghaichi (student, University of Science and Technology Houari Boumediene), Algeria**

For convenience we set $a = 2^{x+1}, b = x^2 + 2^x$ then (1) is equivalent to

$$a\left(1 - \sqrt{1+b}\right) = b\left(1 - \sqrt{1+a}\right) \Leftrightarrow \frac{ab}{1+\sqrt{1+b}} = \frac{ab}{1+\sqrt{1+a}}.$$

Since $ab \neq 0$, we get

$$\begin{aligned}\frac{1}{1+\sqrt{1+b}} = \frac{1}{1+\sqrt{1+a}} &\Longrightarrow 1+\sqrt{1+b} = 1+\sqrt{1+a}\\ &\Longrightarrow a = b\end{aligned}$$

Which means that $2^{x+1} = 2^x + x^2$ then $2^x$ taking ln of both sides we get

$$\frac{\ln x}{x} = \frac{\ln 2}{2}, \quad x > 1$$

Let $f : (1, \infty) \mapsto R$ be defined by $f(x) = \frac{\ln x}{x} - \frac{\ln 2}{2}$, then $f'(x) = x^{-2}(1 - \ln x)$ then $f$ cannot have more than two roots (since $f$ increases on $(1, e)$ and decreases on $(e, +\infty)$) and since $2, 4$ are obvious roots we conclude that the only positive solutions to the equation (1) are $2, 4$.

**Also solved by Adnan Ali (student in A.E.C.S-4), Mumbai, India; Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Jerry Chu, (student at Saint George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italy; Henry Ricardo, New York Math Circle, NY; Neculai Stanciu, "George Emil Palade School," Buzău, Romania (jointly with) Titu Zvonaru, Comănesti, Romania; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.**

- **5329:** *Proposed by Arkady Alt, San Jose, CA*

Find the smallest value of $\dfrac{x^3}{x^2+y^2} + \dfrac{y^3}{y^2+z^2} + \dfrac{z^3}{z^2+x^2}$ where real $x, y, z > 0$ and $xy + yz + zx = 1$.

**Solution 1 by Kee-Wai Lau, Hong Kong, China**

Since

$$\begin{aligned}
& \frac{x^3}{x^2+y^2}+\frac{y^3}{y^2+z^2}+\frac{z^3}{z^2+x^2} \\
= \; & \frac{1}{2}\left(\left(2x-y+\frac{y(x-y)^2}{x^2+y^2}\right)+\left(2y-z+\frac{z(y-z)^2}{y^2+z^2}\right)+\left(2z-x+\frac{x(z-x)^2}{z^2+x^2}\right)\right) \\
\geq \; & \frac{1}{2}\left((2x-y)+(2y-z)+(2z-x)\right) \\
= \; & \frac{x+y+z}{2} \\
= \; & \frac{1}{2\sqrt{2}}\sqrt{6(xy+yz+zx)+(x-y)^2+(y-z)^2+(z-x)^2} \\
= \; & \frac{1}{2\sqrt{2}}\sqrt{6} \\
= \; & \frac{\sqrt{3}}{2},
\end{aligned}$$

and equality holds when $x=y=z=\dfrac{1}{\sqrt{3}}$, so the smallest value required is $\dfrac{\sqrt{3}}{2}$.

**Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain**

Since real $x,y,z>0$ and $xy+yz+zx=1$, there is an acute triangle $ABC$ such that $\cot A=x, \cot B=y$ and $\cot C=z$ so

$$\begin{aligned}
& \frac{x^3}{x^2+y^2}+\frac{y^3}{y^2+z^2}+\frac{z^3}{z^2+x^2} \\
= \; & \cot A-\frac{\cot A\cot^2 B}{\cot^2 A+\cot^2 B}+\cot B-\frac{\cot B\cot^2 C}{\cot^2 B+\cot^2 C}+\cot C-\frac{\cot C\cot^2 A}{\cot^2 C+\cot^2 A} \\
\geq \; & \cot A+\cot B+\cot C-\frac{\cot A\cot^2 B}{2\cot A\cot B}-\frac{\cot B\cot^2 C}{2\cot B\cot C}-\frac{\cot C\cot^2 A}{2\cot C\cot A} \\
= \; & \frac{1}{2}\left(\cot A+\cot B+\cot C\right) \\
\geq \; & \frac{\sqrt{3}}{2}
\end{aligned}$$

with equality iff $\cot A=\cot B=\cot C$ and $A=B=C=\pi/3$, that is iff $x=y=z=\dfrac{1}{\sqrt{3}}$, where we have used that $\cot A, \cot B > 0, (\cot A - \cot B)^2 \geq 0$ with equality iff $\cot A = \cot B$ and cyclically, and inequality 2.38 page 28, *Geometric Inequalities,* Bottema O., Djordjević, R.Ž., Janić, R.R., Mitrinović, D.S. Vasić, P.M., Wolters-Noordhoff, , Groningen, 1969.

**Solution 3 by Henry Ricardo, New York Math Circle, NY**

The AGM inequality gives us

$$\frac{x^3}{x^2+y^2} \;=\; x - \frac{xy^2}{x^2+y^2} \;\geq\; x - \frac{xy^2}{2xy} \;=\; x - \frac{y}{2}.$$

Similarly, we get

$$\frac{y^3}{y^2+z^2} \;\geq\; y - \frac{z}{2} \quad \text{and} \quad \frac{z^3}{z^2+x^2} \;\geq\; z - \frac{x}{2}.$$

Adding these three inequalities, we see that

$$f(x,y,z) \;=\; \frac{x^3}{x^2+y^2} + \frac{y^3}{y^2+z^2} + \frac{z^3}{z^2+x^2} \;\geq\; \frac{x+y+z}{2}. \qquad (A)$$

Now we have

$$(x+y+z)^2 \;=\; x^2+y^2+z^2+2(xy+yz+zx) \;=\; x^2+y^2+z^2+2,$$

so $x+y+z = \sqrt{x^2+y^2+z^2+2} \;\geq\; \sqrt{3}$, where we have used the well-known inequality $x^2+y^2+z^2 \;\geq\; xy+yz+zx$.

Thus $f(x,y,z) \;\geq\; \dfrac{\sqrt{3}}{2}$, with equality if and only if $x=y=z=1/\sqrt{3}$.

*Editor's comment* : The author also provided a second solution to the above problem. It starts off exactly as the one above up until statement $A$. Then:

$$\begin{aligned} f(x,y,z) \;&=\; \frac{x^3}{x^2+y^2} + \frac{y^3}{y^2+z^2} + \frac{z^3}{z^2+x^2} \;\geq\; \frac{x+y+z}{2} \;=\; \frac{3}{2}\left(\frac{x+y+z}{3}\right) \\ &\geq \frac{3}{2}\left(\frac{xy+yz+zx}{3}\right)^{1/2} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}. \end{aligned}$$

Thus $f(x,y,z) \;\geq\; \dfrac{\sqrt{3}}{2}$, with equality if and only if $x=y=z=1/\sqrt{3}$.

**Solution 4 by Albert Stadler, Herrliberg, Switzerland**

Suppose that $xy+y+zx=1$. We claim that

$$\frac{x^3}{x^2+y^2} + \frac{y^3}{y^2+z^2} + \frac{z^3}{z^2+x^2} \quad \geq \frac{\sqrt{3}}{2}, \qquad (1)$$

with equality if and only if $x=y=z=\dfrac{1}{\sqrt{3}}$. By homogeneity, (1) is equivalent to the unconditional inequality

$$\frac{1}{\sqrt{xy+yz+zx}}\left(\frac{x^3}{x^2+y^2} + \frac{y^3}{y^2+z^2} + \frac{z^3}{z^2+x^2} \geq \frac{\sqrt{3}}{2}\right). \qquad (2)$$

We first note that

$$(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2zx \geq 3\left(xy+yz+zx\right),$$

since by the Cauchy-Schwarz Inequality, $x^2+y^2+z^2 \geq xy+yz+zx$, with equality if and only if $x=y=z$.

So

$$\frac{1}{\sqrt{xy+yz+zx}}\left(\frac{x^3}{x^2+y^2}+\frac{y^3}{y^2+z^2}+\frac{z^3}{z^2+x^2}\right) \geq \frac{\sqrt{3}}{x+y+z}\left(\frac{x^3}{x^2+y^2}+\frac{y^3}{y^2+z^2}+\frac{z^3}{z^2+x^2}\right).$$

To prove (2) it is therefore enough to prove that

$$\frac{1}{x+y+z}\left(\frac{x^3}{x^2+y^2}+\frac{y^3}{y^2+z^2}+\frac{z^3}{z^2+x^2}\right) \geq \frac{1}{2} \tag{3}$$

with equality if and only if $x=y=z$.

Clearing denominators we see that (3) is equivalent to

$$\sum_{cycl} x^5y^2 + \sum_{cycl} x^2y^5 + \sum_{cycl} x^4y^3 \geq \sum_{cycl} x^3y^4 + \sum_{cycl} x^4y^2z + \sum_{cycl} x^4yz^2. \tag{4}$$

By the weighted AM-GM inequality,

$$\frac{1}{2}x^2y^2 + \frac{1}{2}x^4y^3 \geq x^3y^4,$$

$$\frac{3}{19}x^2y^5 + \frac{2}{19}y^3z^5 + \frac{14}{19}z^2x^5 \geq x^4yz^2,$$

$$\frac{1}{2}x^5y^2 + \frac{1}{2}x^3z^4 \geq x^4y^{z2},$$

$$\frac{10}{19}x^5y^5 + \frac{3}{76}y^5z^2 + \frac{7}{38}z^5x^2 + \frac{1}{4}x^4y^3 \geq x^4y^2z.$$

We conclude that

$$\frac{1}{2}\sum_{cycl} x^2y^5 + \frac{1}{2}\sum_{cycl} x^4y^3 \geq \sum_{cycl} x^3y^4, \tag{5}$$

$$\frac{1}{2}\sum_{cycl} x^2y^5 = \frac{1}{2}\left(\frac{3}{19}\sum_{cycl} x^2y^5 + \frac{2}{19}\sum_{cycl} y^2z^5 + \frac{14}{19}\sum_{cycl} z^2x^5\right) \geq \frac{1}{2}\sum_{cycl} x^4yz^2, \tag{6}$$

$$\frac{1}{4}\sum_{cycl} x^5y^2 + \frac{1}{4}\sum_{cycl} x^4y^3 = \frac{1}{4}\sum_{cycl} x^5y^2 + 14\sum_{cycl} x^3z^4 \geq \frac{1}{2}\sum_{cycl} x^4yz^3, \tag{7}$$

$$\frac{3}{4}\sum_{cycl} x^5y^2 + \frac{1}{4}\sum_{cycl} x^4y^3 = \frac{10}{19}\sum_{cycl} x^5y^2 + \frac{3}{76}\sum_{cycl} x^5z^2 + \frac{7}{38}\sum_{cycl} z^5x^2 + \frac{1}{4}\sum_{cycl} x^4y^3 \geq 4\sum_{cycl} x^4y^2z. \qquad (8)$$

Condition (4) follows by adding (5),(6),(7), and (8). Equality holds if and only if $x = y = z$. (This is the equality condition for weighted AM-GM inequalities.)

**Also solved by Adnan Ali (student, in A.E.C.S-4), Mumbai, India; Michael Brozinsky, Central Islip, NY; Ed Gray, Highland Beach, FL; Moti Levy, Rehovot, Israel; Paolo Perfetti, Department of Mathematics, Tor Vergata Roma University, Rome, Italy; Neculai Stanciu, "George Emil Palade School," Buzău, Romania (jointly with) Titu Zvonaru, Comănesti, Romania; Nicusor Zlota (plus a generalization) "Traian Vuia" Technical College, Focsani, Romania, and the proposer.**

- **5330:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $B(x) = \begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix}$ and let $n \geq 2$ be an integer.

Calculate the matrix product

$$B(2)B(3)\cdots B(n).$$

**Solution 1 by Neculai Stanciu, "George Emil Palade School," Buzău, Romania (jointly with) Titu Zvonaru, Comănesti, Romania**

We denote A(n)=B(1)B(2)...B(n). We have

$$A(1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ A(2) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}.$$

We assume that

$$A(n) = \begin{pmatrix} \dfrac{(n+1)!}{2} & \dfrac{(n+1)!}{2} \\[2ex] \dfrac{(n+1)!}{2} & \dfrac{(n+1)!}{2} \end{pmatrix}. \qquad (1)$$

Since

$$A(n) = \begin{pmatrix} \dfrac{(n+1)!}{2} & \dfrac{(n+1)!}{2} \\[2ex] \dfrac{(n+1)!}{2} & \dfrac{(n+1)!}{2} \end{pmatrix}\begin{pmatrix} n+1 & 1 \\[2ex] 1 & n+1 \end{pmatrix} = \begin{pmatrix} \dfrac{(n+2)!}{2} & \dfrac{(n+2)!}{2} \\[2ex] \dfrac{(n+2)!}{2} & \dfrac{(n+2)!}{2} \end{pmatrix},$$

we have shown, by mathematical induction that (1) holds for all integers $n \geq 2$.

**Solution 2 by Moti Levy, Rehovot, Israel**

Let $B(x) = xI + A$, where $A$ is an involute matrix (i.e., $A^2 = I$).
Let $P_n = B\,(2)\,B\,(3)\cdots B\,(n)$.

Since $A$ is an involute matrix then

$$P_n = \alpha_n I + \beta_n A,$$
$$P_2 = 2I + A.$$

$$P_{n+1} = P_n B(n+1) = (\alpha_n I + \beta_n A)\left((n+1)\,I + A\right).$$

A recurrence formula for $\alpha_n, \beta_n$ is

$$\begin{aligned} \alpha_{n+1} &= (n+1)\,\alpha_n + \beta_n \\ \beta_{n+1} &= (n+1)\,\beta_n + \alpha_n \\ \alpha_2 &= 2, \quad \beta_2 = 1. \end{aligned}$$

Let $x_n = \alpha_n - \beta_n$ and $y_n = \alpha_n + \beta_n$, then

$$\begin{aligned} x_{n+1} &= n x_n, \\ y_{n+1} &= (n+2)\,y_n, \\ x_2 &= 1, \quad y_2 = 3. \end{aligned}$$

The solution for $x_n$, $y_n$ is

$$x_n = (n-1)!, \quad y_n = \frac{1}{2}\,(n+1)!.$$

Solving for $\alpha_n, \beta_n$,

$$\begin{aligned} \alpha_n &= \frac{1}{4}\,(n+1)! + \frac{1}{2}\,(n-1)!, \\ \beta_n &= \frac{1}{4}\,(n+1)! - \frac{1}{2}\,(n-1)!. \end{aligned}$$

For any involutory matrix $A$,

$$P_n = \left(\frac{1}{4}\,(n+1)! + \frac{1}{2}\,(n-1)!\right) I + \left(\frac{1}{4}\,(n+1)! - \frac{1}{2}\,(n-1)!\right) A.$$

For the special case $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the solution is

$$B\,(2)\,B\,(3)\cdots B\,(n) = \begin{pmatrix} \frac{1}{4}\,(n+1)! + \frac{1}{2}\,(n-1)! & \frac{1}{4}\,(n+1)! - \frac{1}{2}\,(n-1)! \\ \frac{1}{4}\,(n+1)! - \frac{1}{2}\,(n-1)! & \frac{1}{4}\,(n+1)! + \frac{1}{2}\,(n-1)! \end{pmatrix}.$$

**Also solved by Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Brian D. Beasley, Presbyterian College, Clinton, SC; Jerry Chu (student at Saint George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Ethan Gegner (student at Taylor University), Upland IN; Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News, VA; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kee-Wai Lau, Hong Kong, China; Carl Libis of the University of Tennessee at Martin, TN; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Haroun**

**Meghaichi (student, University of Science and Technology Houari Boumediene), Algeria; Corneliu Mănescu-Avram, Transportation High School Ploiesti, Romania; Henry Ricardo, New York Math Circle, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Aoxi Yao (student at Saint George's School), Spokane, WA; Ricky Wang, (student at Saint George's School), Spokane, WA, and the proposer.**

*Late Solutions*

A late solution was received to problem #5319 and to 5321 by **Adnan Ali (student in A.E.C.S-4), Mumbai, India.**

Solutions to problems 5322, 5323 and 5324 were received from **Arkady Alt of San Jose, CA.** They were received on time but misfiled by me, and his name was inadvertently not listed as having solved these problems in the February 2015 issue of the column. Arkady, I am sorry; mea culpa (once again.)

Solutions to problems 5313, 5314, 5315, and 5318 were also received from **Carl Libis of the University of Tennessee at Martin, TN.** They too were received on time but misfiled by me–again, mea culpa.

Solutions 5320 and to 5322 were also submitted on time by **Albert Stadler of Herrliberg, Switzerland,** and inadvertently and misfiled by me.

And for the files submitted by **Moubinool Omarjee of Lyce Henri IV, Paris, France** my computer of its own accord, placed them into a "junk file." But he deserves credit for having solved problems 5257, 5269, 5275, 5276, and 5281.

To Albert and to Moubinool, and to all others for whom this might have also happened, mea culpa.