Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
May 15, 2018

5487: *Proposed by Kenneth Korbin, New York, NY*

Given that $\dfrac{(x+1)^4}{x(x-1)^2} = a$ with $x = \dfrac{b+\sqrt{b-\sqrt{b}}}{b-\sqrt{b-\sqrt{b}}}$. Find positive integers a and b.

5488: *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta, Turnu-Severin, Mehedinti, Romania*

Let a, and b be complex numbers. Solve the following equation:

$$x^3 - 3ax^2 + 3(a^2 - b^2)x - a^3 + 3ab^2 - 2b^3 = 0.$$

5489: *Proposed by D.M. Bătinetu-Giurgiu, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School Buzău, Romania*

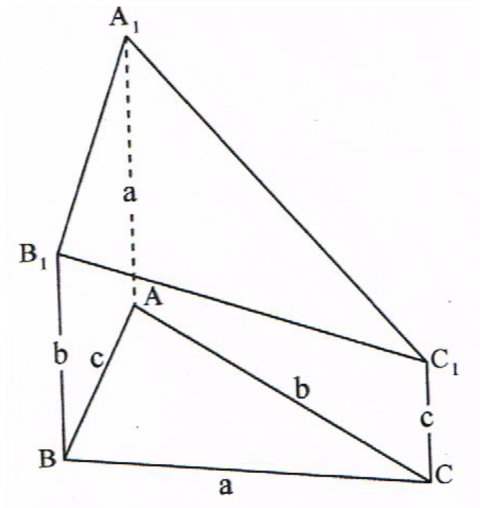
If $a > 0$, compute $\displaystyle\int_0^a \left(x^2 - ax + a^2\right) \arctan(e^x - 1)dx.$

5490: *Proposed by Moshe Stupel, "Shaanan" Academic College of Education and Gordon Academic College of Education, and Avi Sigler, "Shaanan" Academic College of Education, Haifa, Israel*

Triangle ABC whose side lengths are a, b, and c lies in plane P. The segment$A_1A,\ BB_1,\ CC_1$ satisfy:

$$A_1A \perp P,\ B_1B \perp P,\ C_1C \perp P,$$

where $A_1A = a,\ B_1B = b$ and $C_1C = c$, as shown in the figure. Prove that $\triangle A_1B_1C_1$ is acute -angled.



5491: *Proposed by Roger Izard,Dallas, TX*

Let O be the orthocenter of isosceles triangle ABC, $AB = AC$. Let OC meet the line segment AB at point F. If $m = FO$, prove that $c^4 \geq m^4 + 11m^2c^2$.

5492: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let a, b, c, d be four positive numbers such that $ab + ac + ad + bc + bd + cd = 6$. Prove that

$$\sqrt{\frac{abc}{a+b+c+3d}} + \sqrt{\frac{bcd}{b+c+d+3a}} + \sqrt{\frac{cda}{c+d+a+3b}} + \sqrt{\frac{dab}{d+a+b+3c}} \leq 2\sqrt{\frac{2}{3}}.$$

Solutions

5469: *Proposed by Kenneth Korbin, New York, NY*

Let x and y be positive integers that satisfy the equation $3x^2 = 7y^2 + 17$. Find a pair of larger integers that satisfy this equation expressed in terms of x and y.

Solution 1 by Bruno Salugueiro Fanego, Viveiro, Spain

It suffices to find a pair of the type $(ax+by, cx+dy)$ where $a, b, c,$ and d are positive integers and $3(ax+by)^2 - 7(cx+dy)^2 = 3x^2 - 7y^2$.

Since $(ax+by)^2 - 7(cx+dy)^2 = (3a^2 - 7c^2)x^2 + (3b^2 - 7d^2)y^2 + 2(3ab - 7cd)xy$, it is sufficient that $a, b, c,$ and d satisfy the relations:

$$\begin{cases} 3a^2 - 7b^2 = 3 \\ 3b^2 - 7d^2 = -7 \\ 3ab - 7cd = 0. \end{cases}$$

The pair $(a, c) = (55, 36)$ of positive integers verifies $3a^2 - 7b^2 = 3$, and if it assumed that $d = a = 55$ then it only remains to find a positive integer b such that
$3 \cdot b - 7 \cdot 36 = 0$ and $3b^2 - 7 \cdot 55^2 = -7$.

Since $b = 84$ satisfies $3b^2 - 7 \cdot 55^2 = -7$, the pair of larger integers
$(55x + 84y, 36x + 55y)$ solves the problem.

Solution 2 by Ed Gray, Highland Beach, FL

Clearly, by inspection, the equation is satisfied by $x = 8, y = 5$. Let the larger integers which satisfy the equation be $x + k = 8 + k$ and $y + m = 5 + m$ then we have:

1. $3(8+k)^2 = 7(5+m)^2 + 17,$ expanding
2. $3(64 + 16k + k^2) = 7(25 + 10m + m^2) + 17$
3. $192 + 48k + 3k^2 = 175 + 70m + 7m^2 + 17$
4. $48k + 3k^2 = 70m + 7m^2$. Let $(k, m) = r$. So,
5. $k = ra, m = rb,\ (a, b) = 1,$ and substituting into 4)
6. $48ra + 3r^2a^2 = 70rb + 7r^2b^2$
7. $48a + 3ra^2 = 70b + 70rb^2$
8. $3a(16 + ra) = 7b(10 + rb)$. Suppose
9. $3a = 10 + rb$ and that
10. $7b = 16 + ra$. Multiply step 9 by a and step10 by b
11. $3a^2 = 10a + arb$
12. $7b^2 = 16b + arb$
13. $3a^2 - 7b^2 = 10a - 16b$
14. $3a^2 - 10a = 7b^2 - 16b,$ and by inspection $b\ = 4, a\ = 6$
15. $3a^2 - 10a = 3(6)^2 - (10)(6) = 108a^2 - 60 = 48$
16. $7b^2 - 16b = 7(4)^2 - (16)(4) = 112 - 64 = 48$. From step 10
17. $7b = 16 + ra$ or $(7)(4) = 28 = 16 + 6r,$ and $r = 2$. From step 5
18. $k = ra = (2)(16) = 12,\ m = rb = (2)(4) = 8$.

Hence the larger integers are $x + k = x + 12 = 20$ and $y + m = y + 8 = 13$.

As a check, substituting $(20, 13)$ into $3x^2 = 7y^2 + 17$, gives equality.

Solution 3 by David E. Manes, Oneonta, NY

We will show that if (x, y) is a solution of $3x^2 = 7y^2 + 17$, then $(55x + 84y, 36x + 55y)$ is another solution of this equation. Furthermore, all positive integer solutions of this equation are given by

$$x_n = \left(4 + \frac{5\epsilon\sqrt{21}}{6}\right)(55+12\sqrt{21})^n + \left(4 - \frac{5\epsilon\sqrt{21}}{6}\right)(55-12\sqrt{21})^n,$$
$$y_n = \left(\frac{4\sqrt{21}}{7} + \frac{5\epsilon}{2}\right)(55+12\sqrt{21})^n + \left(\frac{-4\sqrt{21}}{7} + \frac{5\epsilon}{2}\right)(55-12\sqrt{21})^n,$$

where $n \geq 1$ and $\epsilon = \pm 1$.
Re-write the equation as (1) $3x^2 - 7y^2 - 17 = 0$ and note that the two least positive solutions (x, y) are $(8, 5)$ and $(20, 13)$. Construct the recurrent sequences

$$\begin{aligned} x_{n+1} &= \alpha x_n + \beta y_n \\ y_{n+1} &= \gamma x_n + \delta y_n, \end{aligned}$$

where α, β, γ and δ are unknowns and assume that (x_{n+1}, y_{n+1}) is a solution of (1). Then $3(\alpha x_n + \beta y_n)^2 - 7(\gamma x_n + \delta y_n)^2 - 17 = 0$ which expands to

$$(3\alpha^2 - 7\gamma^2)x_n^2 + (3\beta^2 - 7\delta^2)y_n^2 + (6\alpha\beta - 14\gamma\delta)x_n y_n - 17 = 0.$$

Comparing this equation with (1), we get

$$(2)\ 3\alpha^2 - 7\gamma^2 = 3, \quad (3)\ 3\beta^2 - 7\delta^2 = -7, \quad (4)\ 3\alpha\beta = 7\gamma\delta.$$

Squaring equation (4), we get $(3\alpha^2)(3\beta^2) = 49\gamma^2\delta^2$. Using (2) and (3), one obtains

$$(3 + 7\gamma^2)(-7 + 7\delta^2) = 49\gamma^2\delta^2 \quad \text{or} \quad 3\delta^2 - 7\gamma^2 = 3.$$

Subtracting this equation from (2) results in $\alpha = \pm\delta$. Substituting this value in (4), we get $3(\pm\delta)\beta = 7\gamma\delta$ or $\beta = \pm\frac{7}{3}\gamma$. Finally, substituting this value in (3) along with $\delta = \pm\alpha$ yields the equation $3\alpha^2 - 7\gamma^2 = 3$ which is equation (2). The smallest positive integer solution of this equation is $\alpha = 55, \gamma = 36$ so that $\beta = \frac{7}{3}\gamma = 84$. Since we want positive solutions only, we let

$$\begin{aligned} x_{n+1} &= \alpha x_n + \frac{7}{3}\gamma y_n = 55x_n + 84y_n \\ y_{n+1} &= \gamma x_n + \alpha y_n = 36x_n + 55y_n. \end{aligned}$$

Define the 2×2 matrix A over the reals as follows: $A = \begin{pmatrix} 55 & 84 \\ 36 & 55 \end{pmatrix}$. By construction, if $\begin{pmatrix} x \\ y \end{pmatrix}$ is a positive integer solution vector of equation (1), then $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 55x + 84y \\ 36x + 55y \end{pmatrix}$ is also a solution vector with larger integers. Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$, and $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 20 \\ 13 \end{pmatrix}$. Then all positive integer solutions of equation (1) are given by

$$\begin{pmatrix} x_n' \\ y_n' \end{pmatrix} = A^n \begin{pmatrix} 8 \\ 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_n'' \\ y_n'' \end{pmatrix} = A^n \begin{pmatrix} 20 \\ 13 \end{pmatrix}$$

for $n \geq 1$. Noting that $A\begin{pmatrix} 8 \\ -5 \end{pmatrix} = \begin{pmatrix} 20 \\ 13 \end{pmatrix}$, we can shorten the above description to

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \begin{pmatrix} 8 \\ \epsilon 5 \end{pmatrix},$$

where $n \geq 1$ and $\epsilon = \pm 1$. To get a closed form expression for the positive integer solutions of equation $3x^2 - 7y^2 - 17 = 0$, we note that A has two distinct eigenvalues $\lambda_1 = 55 + 12\sqrt{21}$ and $\lambda_2 = 55 - 12\sqrt{21}$ with corresponding eigenvectors $\overrightarrow{v_1} = (\sqrt{21}, 3)$ and $\overrightarrow{v_2} = (-\sqrt{21}, 3)$. Therefore, A is diagonalizable; that is, there exists a diagonal matrix $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, where λ_1, λ_2 are the eigenvalues of A and an invertible matrix $P = \begin{pmatrix} \sqrt{21} & -\sqrt{21} \\ 3 & 3 \end{pmatrix}$ consisting of the two eigenvectors in columns such that $P^{-1}AP = D$. Therefore, $A = PDP^{-1}$ so that for each positive integer n, $A^n = PD^nP^{-1}$. Hence,

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \begin{pmatrix} 8 \\ \epsilon 5 \end{pmatrix} = \left(PD^nP^{-1}\right) \begin{pmatrix} 8 \\ \epsilon 5 \end{pmatrix},$$

which computationally yields

$$x_n = \left(4 + \frac{5\epsilon\sqrt{21}}{6}\right)(55 + 12\sqrt{21})^n + \left(4 - \frac{5\epsilon\sqrt{21}}{6}\right)(55 - 12\sqrt{21})^n,$$

$$y_n = \left(\frac{4\sqrt{21}}{7} + \frac{5\epsilon}{2}\right)(55 + 12\sqrt{21})^n + \left(\frac{-4\sqrt{21}}{7} + \frac{5\epsilon}{2}\right)(55 - 12\sqrt{21})^n,$$

where $n \geq 1$ and $\epsilon = \pm 1$.

Some examples of solutions (x_n, y_n) to the equation $3x^2 = 7y^2 - 17$ with $\epsilon = 1$ are $(x_1, y_1) = (860, 563)$, $(x_2, y_2) = (94592, 61925)$, $(x_3, y_3) = 10\,404\,260, 6\,811\,187)$ and $(x_4, y_4) = (1\,144\;374\,008, 749\,168\,645)$. Examples of solutions with $\epsilon = -1$ are $(x_1, y_1) = (20, 13)$, $(x_2, y_2) = (2192, 1435)$, $(x_3, y_3) = (241100, 157837)$ and $(x_4, y_4) = (26\,518\,808, 17\,360\,635)$.

Solution 4 by Jeremiah Bartz, University of North Dakota, Grand Forks, ND

Put $z = 3x$ and rewrite $3x^2 = 7y^2 + 17$ as $z^2 - 21y^2 = 51$. This is a Pell-like equation with two classes of solutions with initial solutions (z, y) of $(24, 5)$ and $(60, 13)$. The fundamental solution of the corresponding Pell equation $z^2 - 21y^2 = 1$ is $55 + 12\sqrt{21}$. From Pell theory, solutions can be generated via $z_{i+1} + y_{i+1}\sqrt{21} = (z_i + y_i\sqrt{21})(55 + 12\sqrt{21})$. This is equivalent to

$$z_{i+1} = 55z_i + 252y_i$$
$$y_{i+1} = 12z_i + 55y_i$$

or back substituting

$$x_{i+1} = 55x_i + 84y_i$$
$$y_{i+1} = 36x_i + 55y_i.$$

Thus (x_{i+1}, y_{i+1}) is a larger pair of positive integer solutions to $3x^2 = 7y^2 + 17$ than the positive integer solutions (x_i, y_i). The initial solutions are listed below and separated by class.

i	x_i	y_i	x_i	y_i
1	8	5	20	13
2	860	563	2192	1435
3	94592	61925	241100	157837
4	10404260	6811187	26518808	17360635
5	1144374008	749168645	2916827780	1909512013

Editor's notes: **Kenneth Korbin, proposer of problem 5469** stated the following:

Given the sequence $t = (\cdots, 33, 7, 2, 3, 13, \cdots)$ with $5t_N = t_{N-1} + t_{N+1}$. Let a and b be a pair of consecutive terms in this sequence with both odd. If $x = \dfrac{a+b}{2}, y = \dfrac{a-b}{2}$ then $3x^2 = 7y^2 + 17$. Example : $x = \dfrac{33+7}{2}, y = \dfrac{33-7}{2}$.

Brian D. Beasley of Presbyterian College in Clinton, SC noted that using Brahmagupta's identity (see [1]), which notes that if $x_1^2 - Ny_1^2 = k_1$ and $x_2^2 - Ny_2^2 = k_2$, then

$$(x_1x_2 + Ny_1y_2)^2 - N(x_1y_2 + x_2y_1)^2 = k_1k_2.$$

Since $55^2 - \left(\frac{7}{3}\right) 36^2 = 1$, if x and y are positive integers with $x^2 - \left(\frac{7}{3}\right) y^2 = \frac{17}{3}$, then

$$(55x + \tfrac{7}{3} \cdot 36y)^2 - \tfrac{7}{3}\,(55y + 36x)^2 = \tfrac{17}{3}.$$

Hence the solution (x, y) produces the larger solution $(55x + 84y, 55y + 36x)$.

Reference:

[1] https://en.wikipedia.org/wiki/Brahmagupta%27s_identity

David Stone and John Hawkins, both at Georgia Southern University in Statesboro, GA approached the problem by looking at the graph of the given statement as a hyperbola, with the question asking us to prove that this graph contains infinitely many lattice points with both coordinates being integers. They did this and then listed a few of the lattice points, two of them being $(94592, 61925)$ and $(241100, 157837)$. They went on to state the following:

The asymptotes of the given hyperbola are $y = \pm\sqrt{\dfrac{3}{7}}x \approx 0.6546536707x$. Our lattice points demonstrate the closeness of the curve to the asymptote. We compute $\dfrac{y}{x}$:

$$\frac{6129}{94592} \approx 0.6546536705$$

$$\frac{157837}{241100} \approx 0.6546536707.$$

Very close! We'd need a piece of graph paper the size of a football field to see that these points are on the hyperbola but not on the asymptote.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Anthony Bevelacqua, University of North Dakota, Grand Forks, ND; Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Athens, Greece; Trey Smith, Angelo State University, San Angelo, TX; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Anna V. Tomova, Varna, Bulgaria, and the proposer.

5470: *Proposed by Moshe Stupel, "Shaanan" Academic College of Education and Gordon Academic College of Education, and Avi Sigler, "Shaanan" Academic College of Education, Haifa, Israel*

Prove that there are an infinite number of Heronian triangles (triangles whose sides and area are natural numbers), whose side lengths are three consecutive natural numbers.

Solution 1 by Kenneth Korbin, New York, NY

Given a Heronian Triangle with consecutive integer length sides $(b-1, b, b+1)$. Then, a larger such triangle has sides (b^2-3, b^2-2, b^2-1), and another such triangle has sides $\left(-1+2b+\sqrt{3b^2-12}, 2b+\sqrt{3b^2-12}, 1+2b+\sqrt{3b^2-12}\right)$.

Therefore there are infinitely many such triangle. Examples, $(3,4,5),(13,14,15)$, etc.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

Let $(n-1, n, n+1)$ be the sides of the triangle. Then the semiperimeter is $s=\dfrac{3n}{2}$. The area is given by Heron's formula as:

$$\sqrt{s(s-n+1)(s-n)(s-n-1)}=\frac{n}{4}\sqrt{3(n+2)(n-2)}. \qquad (1)$$

Clearly n must be even if (1) represents an integer. We must show that there are infinitely many pairs of integers (m,n) such that $3(2n+2)(2n-2)=4m^2$ or equivalently $m^2-3n^2=-3$. We see that m must be divisible by 3 and we need therefore to find pairs of integers (m,n) such that $n^2-3m^2=1$. This is Pell's equation whose general solution is given by $n-\sqrt{3}m=\left(2-\sqrt{3}\right)^k$, where k is an integer. We conclude that (1) is an integer if and one if $n=\left(2-\sqrt{3}\right)^k+\left(2+\sqrt{3}\right)^k,\ k=0,1,2,$ etc.

Solution 3 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Athens, Greece

A triangle whose sides and area are rational numbers is called a *rational triangle.* If the rational triangle is right-angled, it is called a *right-angled rational triangle* or a *rational Pythagorean triangle* or *a numerical right triangle.* If the sides of a rational triangle is of integer length, it is called an *integer triangle.* If further these sides have no common factor greater than unity, the triangle called *primitive integer triangle.* If the integer triangle is right-angled, it is called a Pythagorean triangle. A *Heronian triangle* (named after Heron of Alexandria) is an integer triangle with the additional property that its area is also an integer. A Heronian triangle is called a *primitive Heronian triangle* if sides have no common factor greater than unity. In the 7^{th} century, the Indian mathematician Brahmagupta studied the special case of triangles with consecutive integer sides.

Assume that the consecutive sides of a Brahmagupta triangle are $d-1, d, d+1$, where $d>4$ is a positive integer. The semiperimeter is $s=\dfrac{3d}{2}$, and by Heron's formula the area A is

$$A=\frac{d}{2}\sqrt{3\left[\left(\frac{d}{2}\right)^2-1\right]}. \qquad (1)$$

But A must also be an integer, then the base d of the triangle must be even and the altitude h of the triangle must be an integer multiple of 3 since $16A^2=3d^2(d^2-4)$. Since $A=\dfrac{dh}{2}$, this equation reduces to

$$4h^2 = 3(d^2 - 4). \tag{2}$$

If d were odd, then the factors on the right side of (2) would all be odd, a contradiction. Thus d is even and we may write $d = 2y$ (y is a positive integer). The area of the triangle is then $A = hy$, and it follows that h is a rational number. But

$$h^2 = 3(y^2 - 4). \tag{3}$$

is an integer and h itself has to be an integer and hence a multiple of 3. If $h = 3x$, we reduce (3) to the Pell equation $y^2 - 3x^2 = 1$. The Pell equation has an infinity of integer solutions. If $(x, y) = (U, T)$, where $T > 0, U > 0$ the solution with least positive x, all the solutions are given by

$$xy\sqrt{3} = \pm\left(T + U\sqrt{3}\right)^n,$$

where n is an arbitrary integer (Mordell, 1969, p. 53).

[1] Mordell, L.J. (1969). *Diophantine equations.* London Academic Press, Inc.

Solution 4 by Paul M. Harms, North Newton, KS

Let n be a natural number and let $n-1$ and $n+1$ be the sides of the triangle. To be a triangle, n must be at least 3. If s is the semi-perimeter, the area of this triangle is

$$\sqrt{s(s-(n-1))(s-n)(s-(n+1))} = \sqrt{\frac{3n}{2}\left(\frac{n}{2}-1\right)\left(\frac{n}{2}+1\right)} = \frac{n}{2^2}\sqrt{3\left(n^2-4\right)}.$$

For area to be a natural number one requirement is that $3\left(n^2-4\right)$ be the square of a natural number. Since 3 is a factor inside the square root we need $3\left(n^2-4\right) = (3t)^2$ for some natural number t. The last equation is equivalent to $n^2 - 3t^2 = 4$. Letting $n = 2x$ and $t = 2y$ where x and y are natural numbers, the equation becomes $x^2 - 3y^3 = 1$ which is a Pell equation. One solution is $x = x_1 = 2$ and $y = y_1 = 1$. There exist an infinite number of solutions of natural numbers found by equating coefficients of the equation $x_k + y_k\sqrt{3} = \left(2+\sqrt{3}\right)^k$ for $k = 1, 2, 3, 4$, etc.

For example, when $k = 3, x_3 = 26$ and $y_3 = 15$. In this case, the n for the triangle given above is $n = 2x_3 = 52$. We $n = 2x_k$ we need to make sure that the area is a natural number. Replacing n by $2x_k$ in the area formula, we obtain

$$\frac{x_k}{2}\sqrt{4(3)\left(x_k^2-1\right)} = x_k\sqrt{3\left(x_k^2-1\right)} = x_k\left(3y_k\right).$$

Thus the area is the product of natural numbers so it is a natural number.

Editor′s comments: **Bruno Salgueiro Fanego of Viveiro, Spain** mentioned in his solution that:
"Discussions of the more general problem of finding all the Heronian triangles whose side lengths are in arithmetic progression can be found, for example, in the articles Heron Triangles with Sides in Arithmetic Progression and Heronian Triangles with Sides in Arithmetic Progression: An Inradius Perspective by J. A. MacDougall and by Herb

Bailey and William Gosnell, respectively."
(http://www.jstor.org/stable/10.4169/math.mag.85.4.290);
Mathematics Magazine Vol. 85, No. 4 (October 2012), pp. 290-294.

This generalization was also mentioned in the solution submitted by **David Stone and John Hawkins both of Georgia Southern University in Statesboro, GA.** They showed in their solution that all primitive Heronian triangles with sides in arithmetic progression had to have one as the difference in the side lengths. Having a common difference greater than 1 produced a Heronian Triangle, but not a primitive one.

Brian D. Beasley of Presbyterian College in Clinton, SC also stated that this problem is well-known, as noted in [1] (below), and that the sequence $\{n_i\}$ (giving the infinitely many Heronian triangles with side lengths $(n_i - 1, n_i, n_i + 1)$, where $\{n_i\}$ is defined by

$$n_1 = 4,\ n_2 = 14, \text{ and } n_{i+2} = 4n_{i+1} - n_i \text{ for } i \geq 1$$

may also be given in the closed form

$$n_i = (2 + \sqrt{3})^i + (2 - \sqrt{3})^i.$$

Reference:

[1] H. W. Gould, A Triangle with Integral Sides and Area, *The Fibonacci Quarterly*, Vol. 11(1) 1973, 27-39.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Bruno Salgueiro Fanego of Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Trey Smith, Angelo State University, San Angelo, TX; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposers.

5471: *Proposed by Arkady Alt, San Jose, CA*

For natural numbers p and n where $n \geq 3$ prove that

$$n^{\frac{1}{n^p}} > (n+p)^{\frac{1}{(n+1)(n+2)(n+3)\cdots(n+p)}}.$$

Solution 1 by Moti Levy, Rehovot, Israel

The function $f(x) = x^{\frac{1}{x}}$ is strictly monotone decreasing for $x \geq 3 > e$, since
$f'(x) = x^{\frac{1}{x}} \frac{1}{x^2}(1 - \ln x) < 0, \quad for \quad x > e$. Hence $n + p > n$ implies

$$n^{\frac{1}{n}} > (n+p)^{\frac{1}{(n+p)}}.$$

It follows that

$$\left(n^{\frac{1}{n}}\right)^{\frac{1}{n^{p-1}}} > \left((n+p)^{\frac{1}{(n+p)}}\right)^{\frac{1}{n^{p-1}}},$$

or

$$\left(n^{\frac{1}{n}}\right)^{\frac{1}{n^{p-1}}} = n^{\frac{1}{n^p}} > (n+p)^{\frac{1}{(n+p)n^{p-1}}}.$$

To complete the solution, we note that

$$(n+p)^{\frac{1}{n^{p-1}(n+p)}} > (n+p)^{\frac{1}{(n+1)(n+2)\cdots(n+p)}}.$$

Solution 2 by Kee-Wai Lau, Hong Kong, China

We prove the equivalent inequality

$$\frac{\ln n}{n^p} > \frac{\ln(n+p)}{(n+1)(m+2)\cdots(n+p)}, \tag{1}$$

by induction on p.

For $x \geq 3$ let $f(x) = \dfrac{\ln x}{x}$. Since $f'(x) = \dfrac{1-\ln x}{x^2} < 0$, so $f(x)$ is strictly decreasing. Hence $f(n) > f(n+1)$ and so (1) is true for $p = 1$. Suppose that (1) is true for $p = k \geq 1$. By the induction assumption, we have

$$\begin{aligned}
\frac{\ln n}{n^{k+1}} &= \frac{1}{n}\left(\frac{\ln n}{n^k}\right) > \frac{\ln(n+k)}{n(n+1)(n+2)\cdots(n+k)} = \\
&= \frac{\ln(n+k+1)}{(n+1)(n+2)\cdots(n+k)(n+k+1)} + \frac{k\ln(n+k)}{n(n+1)(n+2)\cdots(n+k)^2} + \\
&\quad + \frac{1}{(n+1)(n+2)\cdots(n+k)}\left(f(n+k) - f(n+k+1)\right) \\
&> \frac{\ln(n+k+1)}{(n+1)(n+2)\cdots)n+k)(n+k+1)}.
\end{aligned}$$

Thus (1) is true for $p = k+1$ as well and hence true for all positive integers p.

Also solved by Ed Gray, Highland Beach, FL; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5472: *Proposed by Francisco Perdomo and Ángel Plaza, both at Universidad Las Palmas de Gran Canaria, Spain*

Let α, β, and γ be the three angles in a non-right triangle. Prove that

$$\frac{1+\sin^2\alpha}{\cos^2\alpha} + \frac{1+\sin^2\beta}{\cos^2\beta} + \frac{1+\sin^2\gamma}{\cos^2\gamma} \geq \frac{1+\sin\alpha\sin\beta}{1-\sin\alpha\sin\beta} + \frac{1+\sin\beta\sin\gamma}{1-\sin\beta\sin\gamma} + \frac{1+\sin\gamma\sin\alpha}{1-\sin\gamma\sin\alpha.}$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We prove more generally that

$$\frac{1+a^2}{1-a^2} + \frac{1+b^2}{1-b^2} + \frac{1+c^2}{1-c^2} \geq \frac{1+ab}{1-ab} + \frac{1+bc}{1-bc} + \frac{1+ca}{1-ca}, \text{ if } 0 \leq a, b, c < 1.$$

The special case follows by putting $a = \sin\alpha, b = \sin\beta, c = \sin\gamma$, with $\alpha + \beta + \gamma = \pi$.

Indeed,

$$\frac{1}{2}\cdot\frac{1+x^2}{1-x^2}+\frac{1}{2}\cdot\frac{1+y^2}{1-y^2}-\frac{1+xy}{1-xy} \quad = \quad \frac{(x-y)^2(1+xy)}{(1-x^2)(1-y^2)(1-xy)}\geq 0.$$

So

$$\frac{1+a^2}{1-a^2}+\frac{1+b^2}{1-b^2}+\frac{1+c^2}{1-c^2}-\frac{1+ab}{1-ab}-\frac{1+bc}{1-bc}-\frac{1+ca}{1-ca}=\left(\frac{1}{2}\cdot\frac{1+a^2}{1-a^2}+\frac{1}{2}\cdot\frac{1+b^2}{1-b^2}-\frac{1}{2}\cdot\frac{1+ab}{1-ab}\right)+$$

$$\left(\frac{1}{2}\cdot\frac{1+b^2}{1-b^2}+\frac{1}{2}\cdot\frac{1+c^2}{1-c^2}-\frac{1}{2}\cdot\frac{1+bc}{1-bc}\right)+\left(\frac{1}{2}\cdot\frac{1+c^2}{1-c^2}+\frac{1}{2}\cdot\frac{1+a^2}{1-a^2}-\frac{1}{2}\cdot\frac{1+ca}{1-ca}\right)\geq 0.$$

Solution 2 by Moti Levy, Rehovot, Israel

Let $a=\sin\alpha,\ b=\sin\beta$ and $c=\sin\gamma$.

Then the inequality becomes

$$\frac{1+a^2}{1-a^2}+\frac{1+b^2}{1-b^2}+\frac{1+c^2}{1-c^2}\geq\frac{1+ab}{1-ab}+\frac{1+bc}{1-bc}+\frac{1+ca}{1-ca},$$

and since $1+\dfrac{2x}{1-x^2}=\dfrac{1+x^2}{1-x^2}$, then it is equivalent to the following inequality,

$$\frac{a^2}{1-a^2}+\frac{b^2}{1-b^2}+\frac{c^2}{1-c^2}\geq\frac{ab}{1-ab}+\frac{bc}{1-bc}+\frac{ca}{1-ca},\quad 0\leq a,b,c,<1.$$

The function

$$f(x):=\frac{x^2}{1-x^2}\tag{1}$$

is monotone increasing and convex in $0\leq x<1$, since $f'(x)=\dfrac{2x}{(1-x^2)^2}\geq 0$, and

$f''(x)=2\dfrac{3x^2+1}{(1-x^2)^3}>0$ for $0\leq x<1$.

By definition (1) the right hand side is

$$\frac{ab}{1-ab}+\frac{ca}{1-ca}+\frac{bc}{1-bc}=f\left(\sqrt{ab}\right)+f\left(\sqrt{ca}\right)+f\left(\sqrt{bc}\right),$$

and the left hand side is

$$\frac{a^2}{1-a^2}+\frac{b^2}{1-b^2}+\frac{c^2}{1-c^2}=f(a)+f(b)+f(c).$$

Without loss of generality, we may assume that $a\geq b\geq c$. Then the vector (a,b,c) *majorizes* the vector $\left(\dfrac{a+b}{2},\dfrac{c+a}{2},\dfrac{b+c}{2}\right)$.

By the *Majorizing Inequality*,

$$f(a)+f(b)+f(c)\geq f\left(\frac{a+b}{2}\right)+f\left(\frac{c+a}{2}\right)+f\left(\frac{b+c}{2}\right).\tag{2}$$

By the AM-GM inequality $\dfrac{a+b}{2}\geq\sqrt{ab}$, $\dfrac{c+a}{2}\geq\sqrt{ca}$ and $\dfrac{b+c}{2}\geq\sqrt{bc}$. The function $f(x)$ is monotone increasing, hence

$$f\left(\frac{a+b}{2}\right)+f\left(\frac{c+a}{2}\right)+f\left(\frac{b+c}{2}\right)\geq f\left(\sqrt{ab}\right)+f\left(\sqrt{ca}\right)+f\left(\sqrt{bc}\right).\tag{3}$$

Inequalities (2) and (3) imply the required result.
In order to make this solution self-contained, the definition of majorizing and the Majorizing Inequality are explained here.
The explanations are excerpted from a nice short article by Murray S. Klamkin (1921-2004) who was one of the greatest problems composer.
M. S. Klamkin, *On a "Problem of the Month"*, Crux Mathematicorum, Volume 28, Number 2, page 86, 2002.
"If A and B are vectors $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n)$ where $a_1 \geq a_2 \geq \ldots \geq a_n$, $b_1 \geq b_2 \geq \ldots \geq b_n$, and $a_1 \geq b_1$, $a_1 + a_2 \geq b_1 + b_2$,
$a_1 + a_2 + \cdots + a_{n-1} \geq b_1 + b_2 + \cdots + b_{n-1}$, $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$, we say that A majorizes B and write it as $A \succ B$. Then, if F is a convex function,

$$F(a_1) + F(a_2) + \cdots + F(a_n) \geq F(b_1) + F(b_2) + \cdots + F(b_n)."$$

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Soumava Chakraborty, Kolkata, India; Pedro Acosta De Leon, Massachusetts Institute of Technology Cambridge, MA; Bruno Salgueiro Fanego, Viveiro, Spain. Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Athens, Greece, and the proposers.

5473: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $x_1, \cdots, x_n$ be positive real numbers. Prove that for $n \geq 2$, the following inequality holds:

$$\left(\sum_{k=1}^{n} \frac{\sin x_k}{((n-1)x_k + x_{k+1})^{1/2}}\right)\left(\sum_{k=1}^{n} \frac{\cos x_k}{((n-1)x_k + x_{k+1})^{1/2}}\right) \leq \frac{1}{2}\sum_{k=1}^{n}\frac{1}{x_k}.$$

(Here the subscripts are taken modulo n.)

Solution 1 by Moti Levy, Rehovot, Israel

The following three facts will be used in this solution:
1)

$$\left(\sum_{k=1}^{n} a_k \sin x_k\right)\left(\sum_{k=1}^{n} a_k \cos x_k\right) \leq \frac{1}{2}\left(\sum_{k=1}^{n} a_k\right)^2. \tag{4}$$

This can be shown by expanding the left hand side and using the facts that $\sin x_k \cos x_k \leq \frac{1}{2}$ and $\sin x_j \cos x_k + \cos x_j \sin x_k \leq 1$.
2)

$$\left(\sum_{k=1}^{n} \frac{\sqrt{a_k}}{n}\right)^2 \leq \sum_{k=1}^{n} \frac{a_k}{n}. \tag{5}$$

This is implied from $M_{\frac{1}{2}} \leq M_1$ where M_k are power means.
3)

$$\frac{1}{px + qy} \leq \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right), \quad p, q \geq 0 \ \ \textit{and} \ \ p + q = 1. \tag{6}$$

This can be shown by Jensen's inequality.

Now let

$$a_k := \frac{1}{((n-1)\,x_k + x_{k+1})^{\frac{1}{2}}}.$$

Then

$$\begin{aligned}
LHS &:= \left(\sum_{k=1}^{n} \frac{\sin x_k}{((n-1)\,x_k + x_{k+1})^{\frac{1}{2}}}\right)\left(\sum_{k=1}^{n} \frac{\cos x_k}{((n-1)\,x_k + x_{k+1})^{\frac{1}{2}}}\right) \\
&= \left(\sum_{k=1}^{n} a_k \sin x_k\right)\left(\sum_{k=1}^{n} a_k \cos x_k\right) \le \frac{1}{2}\left(\sum_{k=1}^{n} a_k\right)^2.
\end{aligned}$$

By (5),

$$\begin{aligned}
LHS &\le \frac{1}{2}\left(\sum_{k=1}^{n} a_k\right)^2 \le \frac{n}{2}\sum_{k=1}^{n} a_k = \frac{n}{2}\sum_{k=1}^{n} \frac{1}{(n-1)\,x_k + x_{k+1}} \\
&= \frac{1}{2}\sum_{k=1}^{n} \frac{1}{\frac{n-1}{n}x_k + \frac{1}{n}x_{k+1}}.
\end{aligned}$$

Set $p = \frac{n-1}{n}$ and $q = \frac{1}{n}$, then by (6)

$$\frac{1}{2}\sum_{k=1}^{n} \frac{1}{\frac{n-1}{n}x_k + \frac{1}{n}x_{k+1}} \le \frac{1}{4}\sum_{k=1}^{n}\left(\frac{1}{x_k} + \frac{1}{x_{k+1}}\right) = \frac{1}{2}\sum_{k=1}^{n}\frac{1}{x_k}.$$

Solution to 2 by Kee-Wai Lau, Hong Kong, China

Since $2ab \le a^2 + b^2$ for any real numbers a and b, so by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&2\left(\sum_{k=1}^{n} \frac{\sin x_k}{((n-1)x_k + x_{k+1})^{1/2}}\right)\left(\sum_{k=1}^{n} \frac{\cos x_k}{((n-1)x_k + x_{k+1})^{1/2}}\right) \\
\le\;& \left(\sum_{k=1}^{n} \frac{\sin x_k}{((n-1)x_k + x_{k+1})^{1/2}}\right)^2 + \left(\sum_{k=1}^{n} \frac{\cos x_k}{((n-1)x_k + x_{k+1})^{1/2}}\right)^2 \\
\le\;& \left(n\sum_{k=1}^{n} \frac{\sin^2 x_k}{(n-1)x_k + x_{k+1}} + \sum_{k=1}^{n} \frac{\cos^2 x_k}{(n-1)x_k + x_{k+1}}\right) \\
=\;& n\sum_{k=1}^{n} \frac{1}{(n-1)x_k + x_{k+1}}.
\end{aligned}$$

Applying Jensen's inequality to the convex function $\dfrac{1}{x}$ for $x > 0$, we have

$$\frac{n-1}{x_k} + \frac{1}{x_{k+1}} \ge n\left(\frac{1}{\frac{(n-1)x_k + x_{k+1}}{n}}\right) = \frac{n^2}{(n-1)x_k + x_{k+1}}.$$

It follows that $n\sum_{k=1}^{n}\frac{1}{(n-1)x_k+x_{k+1}}\le\frac{1}{n}\left(\sum_{k=1}^{n}\frac{n-1}{x_k}+\sum_{k=1}^{n}\frac{1}{x_{k+1}}\right)=\sum_{k=1}^{n}\frac{1}{x}.$

Thus the inequality of the problem holds.

Solution 3 by Arkady Alt , San Jose, CA

By Cauchy Inequality $\sum\limits_{k=1}^{n}\frac{\sin x_k}{((n-1)\,x_k+x_{k+1})^{1/2}}\le\sqrt{\sum\limits_{k=1}^{n}\frac{1}{(n-1)\,x_k+x_{k+1}}}\cdot\sqrt{\sum\limits_{k=1}^{n}\sin^2 x_k}$

and $\sum\limits_{k=1}^{n}\frac{\cos x_k}{((n-1)\,x_k+x_{k+1})^{1/2}}\le\sqrt{\sum\limits_{k=1}^{n}\frac{1}{(n-1)\,x_k+x_{k+1}}}\cdot\sqrt{\sum\limits_{k=1}^{n}\cos^2 x_k}.$

Also by AM-GM inequality

$$\sqrt{\sum_{k=1}^{n}\sin^2 x_k}\cdot\sqrt{\sum_{k=1}^{n}\cos^2 x_k}\le\frac{1}{2}\left(\sum_{k=1}^{n}\sin^2 x_k+\sum_{k=1}^{n}\cos^2 x_k\right)=\frac{1}{2}\sum_{k=1}^{n}\left(\sin^2 x_k+\cos^2 x_k\right)=\frac{n}{2}.$$

Thus, $\left(\sum\limits_{k=1}^{n}\frac{\sin x_k}{((n-1)\,x_k+x_{k+1})^{1/2}}\right)\left(\sum\limits_{k=1}^{n}\frac{\cos x_k}{((n-1)\,x_k+x_{k+1})^{1/2}}\right)\le\frac{n}{2}\sum\limits_{k=1}^{n}\frac{1}{(n-1)\,x_k+x_{k+1}}$

and it remains to prove the inequality

$$\frac{n}{2}\sum_{k=1}^{n}\frac{1}{(n-1)\,x_k+x_{k+1}}\le\frac{1}{2}\sum_{k=1}^{n}\frac{1}{x_k}\iff\sum_{k=1}^{n}\frac{1}{(n-1)\,x_k+x_{k+1}}\le\frac{1}{n}\sum_{k=1}^{n}\frac{1}{x_k}.$$

By the Cauchy Inequality

$$((n-1)\,x_k+x_{k+1})\left(\frac{n-1}{x_k}+\frac{1}{x_{k+1}}\right)\ge n^2\iff\frac{1}{(n-1)\,x_k+x_{k+1}}\le\frac{1}{n^2}\left(\frac{n-1}{x_k}+\frac{1}{x_{k+1}}\right)$$

then $\sum\limits_{k=1}^{n}\frac{1}{(n-1)\,x_k+x_{k+1}}\le\frac{1}{n^2}\sum\limits_{k=1}^{n}\left(\frac{n-1}{x_k}+\frac{1}{x_{k+1}}\right)=\frac{1}{n}\sum\limits_{k=1}^{n}\frac{1}{x_k}.$

Also solved by Ed Gray, Highland Beach, FL; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5474: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a,b\in\Re, b\neq 0$. Calculate

$$\lim_{n\to\infty}\begin{pmatrix}1-\dfrac{a}{n^2} & \dfrac{b}{n}\\ \dfrac{b}{n} & 1+\dfrac{a}{n^2}.\end{pmatrix}^n.$$

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

Let $M = \begin{pmatrix} 1 - \dfrac{a}{n^2} & \dfrac{b}{n} \\ \\ \dfrac{b}{n} & 1 + \dfrac{a}{n^2} \end{pmatrix}$. The eigenvalues of M are $1 \mp \dfrac{\sqrt{b^2n^2 + a^2}}{n^2}$.

Since they are distinct, M is diagonalizable. It can be obtained by following the diagonalization of M : $\quad M = PDP^{-1}$, where $D = \begin{pmatrix} 1 - \dfrac{\sqrt{b^2n^2 + a^2}}{n^2} & 0 \\ \\ 0 & 1 + \dfrac{\sqrt{b^2n^2 + a^2}}{n^2} \end{pmatrix}$ is the diagonal matrix whose principal diagonal are its eigenvalues and

$P = \begin{pmatrix} \dfrac{-a - \sqrt{b^2n^2 + a^2}}{bn} & \dfrac{-a + \sqrt{b^2n^2 + a^2}}{bn} \\ \\ 1 & 1 \end{pmatrix}$ is the matrix whose columns are the respective eigenvectors which form a basis of R^2. Hence,

$M^n = P \cdot D^n \cdot P^{-1}$

$$= \begin{pmatrix} \frac{-a-\sqrt{b^2n^2+a^2}}{bn} & \frac{-a+\sqrt{b^2n^2+a^2}}{bn} \\ \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \left(1 - \frac{\sqrt{b^2n^2+a^2}}{n^2}\right)^n & 0 \\ \\ 0 & \left(1 + \frac{\sqrt{b^2n^2+a^2}}{n^2}\right)^n \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} \frac{-bn}{2\sqrt{b^2n^2+a^2}} & \frac{-a+\sqrt{b^2n^2+a^2}}{2\sqrt{b^2n^2+a^2}} \\ \\ \frac{bn}{2\sqrt{b^2n^2+a^2}} & \frac{a+\sqrt{b^2n^2+a^2}}{2\sqrt{b^2n^2+a^2}} \end{pmatrix}$$, that is $M^n = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ where

$$m_{11} = m_{22} = \frac{an^2}{2\sqrt{b^2n^6 + a^2n^4}} \left(\left(1 - \sqrt{\left(\tfrac{b}{n}\right)^2 + \left(\tfrac{a}{n^2}\right)^2}\right)^n - \left(1 + \sqrt{\left(\tfrac{b}{n}\right)^2 + \left(\tfrac{a}{n^2}\right)^2}\right)^n \right) + \tfrac{1}{2} \left(\left(1 - \sqrt{\left(\tfrac{b}{n}\right)^2 + \left(\tfrac{a}{n^2}\right)^2}\right)^n +$$

and

$$m_{12} = m_{21} = \frac{bn^3}{2\sqrt{b^2n^6 + a^2n^4}} \left(\left(1 + \sqrt{\left(\tfrac{b}{n}\right)^2 + \left(\tfrac{a}{n^2}\right)^2}\right)^n - \left(1 - \sqrt{\left(\tfrac{b}{n}\right)^2 + \left(\tfrac{a}{n^2}\right)^2}\right)^n \right).$$

Thus, as $n \to \infty$,

$$m_{11} = m_{22} \sim \frac{an^2}{2|b|n^3} \left(e^{-n\sqrt{\left(\frac{b}{n}\right)^2 + \left(\frac{a}{n^2}\right)^2}} - e^{n\sqrt{\left(\frac{b}{n}\right)^2 + \left(\frac{a}{n^2}\right)^2}} \right) + \frac{1}{2} \left(e^{-n\sqrt{\left(\frac{b}{n}\right)^2 + \left(\frac{a}{n^2}\right)^2}} + e^{n\sqrt{\left(\frac{b}{n}\right)^2 + \left(\frac{a}{n^2}\right)^2}} \right)$$

$$\underset{n \to \infty}{\sim} \frac{a}{2|b|n} \left(e^{-|b|} - e^{|b|}\right) + \frac{1}{2} \left(e^{-|b|} + e^{|b|}\right) \underset{n \to \infty}{\sim} 0 + \frac{1}{2} \left(e^{-|b|} + e^{|b|}\right) = \cosh|b|$$

and as $n \to \infty$

$$
\begin{aligned}
m_{12} = m_{21} \qquad & \sim \frac{bn^3}{2|b|n^3} \left(e^{n\sqrt{\left(\frac{b}{n}\right)^2 + \left(\frac{a}{n^2}\right)^2}} - e^{-n\sqrt{\left(\frac{b}{n}\right)^2) + \left(\frac{a}{n^2}\right)^2}} \right) \\
& \sim \frac{b}{2|b|n} \left(e^{|b|} - e^{-|b|} \right) = \frac{b}{|b|} \sinh |b|, \text{ so,}
\end{aligned}
$$

$$
\lim_{n\to\infty} M^n = \begin{pmatrix} \cosh |b| & \frac{b}{|b|} \sinh |b| \\ \frac{b}{|b|} \sinh |b| & \cosh |b| \end{pmatrix}.
$$

Solution 2 by Brian Bradie, Christopher Newport University, Newport News, VA

Let

$$
A = \begin{pmatrix} 1 - \dfrac{a}{n^2} & \dfrac{b}{n} \\ \dfrac{b}{n} & 1 + \dfrac{a}{n^2} \end{pmatrix}.
$$

The eigenvalues of A are

$$
\lambda_+ = 1 + \frac{1}{n^2}\sqrt{a^2 + n^2 b^2} \quad \text{and} \quad \lambda_- = 1 - \frac{1}{n^2}\sqrt{a^2 + n^2 b^2}.
$$

Because $b \neq 0$, these two eigenvalues are distinct, which implies that A is diagonalizable. An eigenvector associated with λ_+ is

$$
\begin{pmatrix} b \\ \dfrac{1}{n}(a + \sqrt{a^2 + n^2 b^2}) \end{pmatrix},
$$

and an eigenvector associated with λ_- is

$$
\begin{pmatrix} b \\ \dfrac{1}{n}(a - \sqrt{a^2 + n^2 b^2}) \end{pmatrix}.
$$

Thus, if we set

$$
P = \begin{pmatrix} b & b \\ \dfrac{1}{n}(a + \sqrt{a^2 + n^2 b^2}) & \dfrac{1}{n}(a - \sqrt{a^2 + n^2 b^2}) \end{pmatrix}
$$

and

$$
D = \begin{pmatrix} 1 + \dfrac{1}{n^2}\sqrt{a^2 + n^2 b^2} & 0 \\ 0 & 1 - \dfrac{1}{n^2}\sqrt{a^2 + n^2 b^2} \end{pmatrix},
$$

then $A = PDP^{-1}$ and $A^n = PD^nP^{-1}$. Now,

$$
\lim_{n\to\infty} P = \begin{pmatrix} b & b \\ |b| & -|b| \end{pmatrix} \quad \text{and} \quad \lim_{n\to\infty} P^{-1} = -\frac{1}{2b|b|} \begin{pmatrix} -|b| & -b \\ -|b| & b \end{pmatrix}.
$$

Moreover,

$$D^n = \begin{pmatrix} \left(1+\frac{1}{n^2}\sqrt{a^2+n^2b^2}\right)^n & 0 \\ 0 & \left(1-\frac{1}{n^2}\sqrt{a^2+n^2b^2}\right)^n \end{pmatrix},$$

so

$$\lim_{n\to\infty} D^n = \begin{pmatrix} e^{|b|} & 0 \\ 0 & e^{-|b|} \end{pmatrix}.$$

Thus,

$$\begin{aligned} \lim_{n\to\infty} A^n &= -\frac{1}{2b|b|}\begin{pmatrix} b & b \\ |b| & -|b| \end{pmatrix}\begin{pmatrix} e^{|b|} & 0 \\ 0 & e^{-|b|} \end{pmatrix}\begin{pmatrix} -|b| & -b \\ -|b| & b \end{pmatrix} \\ &= \begin{pmatrix} \cosh|b| & \frac{b}{|b|}\sinh|b| \\ \frac{b}{|b|}\sinh|b| & \cosh|b| \end{pmatrix} \\ &= \begin{pmatrix} \cosh b & \sinh b \\ \sinh b & \cosh b \end{pmatrix}. \end{aligned}$$

Remark: This problem is very similar to Problem 1113 from the November 2017 issue of *The College Mathematics Journal.*

Editor′s comment : **David Stone and John Hawkins both at Georgia Southern University in Statesboro, GA** accompanied their solution with the following comment: "At first, we accidentally used $1-\dfrac{a}{n^2}$ in the (2, 2) spot of the matrix and the limit was the same. Perhaps there is a lot of flexibility about this term (since the limit does not depend upon a)."

Also solved by Ulich Abel, Technische Hochschule Mittelhessen, Germany; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Francisco Perdomo and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Anna V. Tomova, Varna, Bulgaria, and the proposer.

Late Solutions

A late solution was received from **Paul M. Harms of Newton, KS to problem 5467.**