Problems

Ted Eisenberg, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before
October 15, 2011

- **5164:** *Proposed by Kenneth Korbin, New York, NY*

  A triangle has integer length sides (a, b, c) such that $a - b = b - c$. Find the dimensions of the triangle if the inradius $r = \sqrt{13}$.

- **5165:** *Proposed by Thomas Moore, Bridgewater, MA*

  "Dedicated to Dr. Thomas Koshy, friend, colleague and fellow Fibonacci enthusiast."

  Let $\sigma(n)$ denote the sum of all the different divisors of the positive integer n. Then n is perfect, deficient, or abundant according as $\sigma(n) = 2n, \sigma(n) < 2n$, or $\sigma(n) > 2n$. For example, 1 and all primes are deficient; 6 is perfect, and 12 is abundant. Find infinitely many integers that are not the product of two deficient numbers.

- **5166:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b, c be lengths of the sides of a triangle ABC. Prove that

$$\left(3^{a+b} + \frac{c}{b}3^{-b}\right)\left(3^{b+c} + \frac{a}{c}3^{-c}\right)\left(3^{c+a} + \frac{b}{a}3^{-a}\right) \geq 8.$$

- **5167:** *Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy*

  Find the maximum of the real valued function

$$f(x, y) = x^4 - 2x^3 - 6x^2y^2 + 6xy^2 + y^4$$

  defined on the set $D = \{(x, y) \in \Re^2 : \ x^2 + 3y^2 \leq 1\}$.

- **5168:** *Proposed by G. C. Greubel, Newport News, VA*

  Find the value of a_n in the series

$$\frac{7t + 2t^2}{1 - 36t + 4t^2} = a_0 + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots + \frac{a_n}{t^n} + \cdots.$$

- **5169:** *Proposed by Ovidiu Furdui, Cluj, Romania*

  Let $n \geq 1$ be an integer and let i be such that $1 \leq i \leq n$. Calculate:

$$\int_0^1 \cdots \int_0^1 \frac{x_i}{x_1 + x_2 + \cdots + x_n} dx_1 \cdots dx_n.$$

Solutions

- **5146:** *Proposed by Kenneth Korbin, New York, NY*

Find the maximum possible value of the perimeter of an integer sided triangle with in-radius $r = \sqrt{13}$.

Solution 1 by Kee-Wai Lau, Hong Kong, China

Let the lengths of the sides of the triangle be a, b, and c with $c \le b \le a$.

Let $x = b + c - a,\ y = c + a - b,\ z = a + b - c$ so that x, y, z are integers and $0 < x \le y \le z$.

It is well known that $\dfrac{1}{2}\sqrt{\dfrac{xyz}{x+y+z}}$ or $\dfrac{xyz}{x+y+z} = 52$.

From $xyz < xy(x+y+z)$, we see that $xy > 52$ and from $xy < \dfrac{3xyz}{x+y+z}$, we have $xy \le 156$. Since $a = \dfrac{y+z}{2},\ b = \dfrac{z+x}{2},\ c = \dfrac{x+y}{2}$, so we have to find positive integers x, y satisfying

$$\begin{cases} x & \le y \\ 1 & \le x \le 12 \\ 52 & < xy \le 156 \end{cases}$$

such that $z = \dfrac{52(x+y)}{xy - 52}$ is a positive integer greater than or equal to y and that x, y, z are of the same parity. With the help of a computer we find that

$(x, y, z) = (2, 28, 390),\ (2, 30, 208),\ (2, 40, 78),\ (2, 52, 54),\ (4, 14, 234),\ (4, 26, 30),\ (6, 10, 104),\ (6, 16, 26)$

are the only solutions. Since $a + b + c = x + y + z$, so the maximum possible value of the perimeter of an integer sided triangle with in-radius $r = \sqrt{13}$ is 420.

Solution 2 by Brian D. Beasley, Clinton, SC

We designate the integer side lengths of the triangle by a, b, and c. We also let $x = a + b - c$, $y = c + a - b$, and $z = b + c - a$ and note that $x + y + z = a + b + c$. Then the formula for the in-radius r of a triangle becomes

$$r = \frac{1}{2}\sqrt{\frac{(a+b-c)(c+a-b)(b+c-a)}{a+b+c}} = \frac{1}{2}\sqrt{\frac{xyz}{x+y+z}}.$$

For the given triangle, we thus have $52(x + y + z) = xyz$. Then xyz is even; combined with the fact that x, y, and z have the same parity, this implies that all three are even. Writing $x = 2u$, $y = 2v$, and $z = 2w$, we obtain $13(u + v + w) = uvw$. Then 13 divides uvw, so without loss of generality, we assume $w = 13k$ for some natural number k. This produces $v = (u + 13k)/(uk - 1)$. Using this equation, a computer search reveals eight solutions for (u, v, w) (with $u \le v$) and hence for (a, b, c):

$$\begin{array}{ccccc}
(u,v,w) = (2,15,13) & \Longrightarrow & (a,b,c) = (17,15,28) & \Longrightarrow & \text{perimeter } = 60 \\
(u,v,w) = (3,8,13) & \Longrightarrow & (a,b,c) = (11,16,21) & \Longrightarrow & \text{perimeter } = 48 \\
(u,v,w) = (1,27,26) & \Longrightarrow & (a,b,c) = (28,27,53) & \Longrightarrow & \text{perimeter } = 108 \\
(u,v,w) = (1,20,39) & \Longrightarrow & (a,b,c) = (21,40,59) & \Longrightarrow & \text{perimeter } = 120 \\
(u,v,w) = (3,5,52) & \Longrightarrow & (a,b,c) = (8,55,57) & \Longrightarrow & \text{perimeter } = 120 \\
(u,v,w) = (1,15,104) & \Longrightarrow & (a,b,c) = (16,105,119) & \Longrightarrow & \text{perimeter } = 240 \\
(u,v,w) = (2,7,117) & \Longrightarrow & (a,b,c) = (9,119,124) & \Longrightarrow & \text{perimeter } = 252 \\
(u,v,w) = (1,14,195) & \Longrightarrow & (a,b,c) = (15,196,209) & \Longrightarrow & \text{perimeter } = 420
\end{array}$$

Thus the maximum value of the perimeter is 420.

Also solved by Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5147:** *Proposed by Kenneth Korbin, New York, NY*

Let

$$\begin{cases} x = 5N^2 + 14N + 23 \text{ and} \\ y = 5(N+1)^2 + 14(N+1) + 23 \end{cases}$$

where N is a positive integer. Find integers a_i such that

$$a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6 = 0.$$

Solution 1 by G. C. Greubel, Newport News, VA

The equations for x and y are given by $x = 5n^2 + 14n + 23$ and $y = 5n^2 + 24n + 42$. We are asked to find the values of a_i such that the equation

$$a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6 = 0$$

is valid. In order to do so we need to calculate the values of x^2, y^2, and xy. For this we have

$$\begin{aligned} x^2 &= 25n^4 + 140n^3 + 426n^2 + 644n + 529 \\ y^2 &= 25n^4 + 240n^3 + 996n^2 + 2016n + 1764 \\ xy &= 25n^4 + 190n^3 + 661n^2 + 1140n + 966. \end{aligned}$$

Using the above results we then have the equation

$$\begin{aligned} 0 = \; & 25(a_1 + a_2 + a_3)n^4 + 10(14a_1 + 24a_2 + 19a_3)n^3 \\ & +(426a_1 + 996a_2 + 661a_3 + 5a_4 + 5a_5)n^2 \\ & +2(322a_1 + 1008a_2 + 570a_3 + 7a_4 + 12a_5)n \\ & +(529a_1 + 1764a_2 + 966a_3 + 23a_4 + 42a_5 + a_6). \end{aligned}$$

From this we have five equations for the coefficients a_i given by

$$\begin{aligned} 0 &= a_1 + a_2 + a_3 \\ 0 &= 14a_1 + 24a_2 + 19a_3 \\ 0 &= 426a_1 + 996a_2 + 661a_3 + 5a_4 + 5a_5 \\ 0 &= 322a_1 + 1008a_2 + 570a_3 + 7a_4 + 12a_5 \\ 0 &= 529a_1 + 1764a_2 + 966a_3 + 23a_4 + 42a_5 + a_6. \end{aligned}$$

From $0 = 14a_1 + 24a_2 + 19a_3$ we have $0 = 14(a_1 + a_2 + a_3) + 10a_2 + 5a_3 = 5(2a_2 + a_3)$, where we used the fact that $0 = a_1 + a_2 + a_3$. This yields $a_3 = -2a_2$. Using this result in $0 = a_1 + a_2 + a_3$ yields $a_2 = a_1$. The three remaining equations can be reduced to

$$\begin{aligned} 0 &= 20a_1 + a_4 + a_5 \\ 0 &= 190a_1 + 7a_4 + 12a_5 \\ 0 &= 361a_1 + 23a_4 + 42a_5 + a_6. \end{aligned}$$

Solving this system we see that

$$a_1 = a_1,\ a_2 = a_1,\ a_3 = -2a_1,\ a_4 = -10a_1,\ a_5 = -10a_1,\ a_6 = 289a_1.$$

We now verify that the above coefficients work.

$$\begin{aligned}
a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6 &= 0, \text{ becomes}\\
a_1\left(x^2 + y^2 - 2xy - 10x - 10y + 289\right) &= 0, \text{ and since } a_1 \neq 0\\
x^2 + y^2 - 2xy - 10x - 10y + 289 &= 0, \text{ and}\\
(x-y)^2 - 10(x+y) + 289 &= 0.
\end{aligned}$$

From the values of x and y presented to us in terms of n at the start of the problem, we see that $x - y = -(10n + 19)$ and $x + y = 10n^2 + 38n + 65$.
Substituting these values into the above equations we obtain:

$$\begin{aligned}
0 &= (x-y)^2 - 10(x+y) + 289\\
&= (10n+19)^2 - 10(10n^2 + 38n + 65) + 289\\
&= (100n^2 + 380n + 361) - (100n^2 + 380n + 650) + 289\\
&= 361 - 650 + 289\\
&= 0.
\end{aligned}$$

We have thus verified that for the coefficients we have obtained, and for the vaules of x and y that are given, $a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6 = 0$.

Solution 2 by Kee-Wai Lau, Hong Kong, China

By putting $N = 1, 2, 3, 4, 5$, we obtain the system of equations

$$\begin{cases}
1764a_1 + 5041a_2 + 2982a_3 + 42a_4 + 71a_5 + a_6 = 0\\
5041a_1 + 12100a_2 + 7810a_3 + 71a_4 + 110a_5 + a_6 = 0\\
12100a_1 + 25281a_2 + 17490a_3 + 110a_4 + 159a_5 + a_6 = 0\\
25281a_1 + 47524a_2 + 34662a_3 + 159a_4 + 218a_5 + a_6 = 0\\
47524a_1 + 82369a_2 + 62566a_3 + 218a_4 + 287a_5 + a_6 = 0.
\end{cases} \tag{1}$$

If $a_1 = 0$, then (1) reduces to

$$\begin{cases}
5041a_2 + 2982a_3 + 42a_4 + 71a_5 + a_6 = 0\\
12100a_2 + 7810a_3 + 71a_4 + 110a_5 + a_6 = 0\\
25281a_2 + 17490a_3 + 110a_4 + 159a_5 + a_6 = 0\\
47524a_2 + 34662a_3 + 159a_4 + 218a_5 + a_6 = 0\\
82369a_2 + 62566a_3 + 218a_4 + 287a_5 + a_6 = 0.
\end{cases} \tag{2}$$

Since the determinant $\begin{vmatrix} 5041 & 2982 & 42 & 71 & 1\\ 12100 & 7810 & 71 & 110 & 1\\ 25281 & 17490 & 110 & 159 & 1\\ 47524 & 34662 & 159 & 218 & 1\\ 82369 & 62566 & 218 & 287 & 1 \end{vmatrix} = -18000000$, so (2) has the unique solution $a_2 = a_3 = a_4 = a_5 = a_6 = 0$.

If $a_1 \neq 0$, we write $a_2 = a_1 b_2,\ a_3 = a_1 b_3, a_4 = a_1 b_4,\ a_5 = a_1 b_5,\ a_6 = a_1 b_6$, so that (1) becomes

$$\begin{cases} 1764 + 5041b_2 + 2982b_3 + 42b_4 + 71b_5 + b_6 = 0 \\ 5041 + 12100b_2 + 7810b_3 + 71b_4 + 110b_5 + b_6 = 0 \\ 12100 + 25281b_2 + 17490b_3 + 110b_4 + 159b_5 + b_6 = 0 \\ 25281 + 47524b_2 + 34662b_3 + 159b_4 + 218b_5 + b_6 = 0 \\ 47524 + 82369b_2 + 62566b_3 + 218b_4 + 287b_5 + b_6 = 0. \end{cases} \qquad (3)$$

By Cramer's rule, we find the unique solution of (3) to be

$$b_2 = 1,\ b_3 = -2,\ b_4 = -10,\ b_5 = -10,\ b_6 = 289.$$

It follows that the general solution to (1) is

$$a_1 = k,\ a_2 = k,\ a_3 = -2k,\ a_4 = -10k,\ a_5 = -10k,\ a_6 = 289k, \qquad (4)$$

where k is any integer. It can be checked readily by direct expansion that $kx^2 + ky^2 - 2kxy - 10kx - 10ky + 289k = 0$ for any positive integer N, and so the general solution to the equation of the problem is given by (4).

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

As in the published solutions to SSMJ problem 5144, we first compute

$$y - x = 5\left[(N+1)^2 - N^2\right] + 14(N+1-N) + 23 - 23 = 5(2N+1) + 14 = 10N + 19 \qquad (1)$$

From $x = 5N^2 + 14N + 23$ that is $5\mathrm{N}^2 + 14N + 23 - x = 0$, one obtains

$$N_{1,2,} = \frac{-14 \pm \sqrt{14^2 - 20(23-x)}}{10} = \frac{-7 \pm \sqrt{5x - 66}}{5}$$

and since N is a positive integer, we choose $N = \dfrac{-7 + \sqrt{5x-66}}{5}$ (2).

Substituting (2) into (1) gives

$$y - x = 2\left(-7 + \sqrt{5x-66}\right) + 19 = 5 + 2\sqrt{5x-66}. \qquad (3)$$

From (3) one obtains

$$(y - x - 5)^2 = \left(2\sqrt{5x-66}\right)^2, \text{that is}$$

$$x^2 + y^2 - 2xy - 10x - 10y + 289 = 0 \qquad (4)$$

Relation (4) shows that it suffices to take the following integers for a_i

$$a_1 = a_2 = 1; \quad a_3 = -2; \quad a_4 = a_5 = -10; \quad a_6 = 289$$

Comment: Relation (4) shows that for any positive integer N, all of the points with coordinates (x, y)=(u_N, u_{N+1}) for $u_N = 5N^2 + 14N + 23$, are points situated on the parabola (*) with equation

$$(1\ X\ Y)\begin{pmatrix} 289 & -5 & -5 \\ -5 & 1 & -1 \\ -5 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ X \\ Y \end{pmatrix} = 0.$$

(*) Because $\det \begin{pmatrix} 289 & -5 & -5 \\ -5 & 1 & -1 \\ -5 & -1 & 1 \end{pmatrix} = 100 \neq 0$ and $\det \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 0$.

Solution 4 by David Stone and John Hawkins, Statesboro, GA

We will show that the proscribed points (x, y) lie on the conic

$$x^2 + y^2 - 2xy - 10xy - 10y + 289 = 0.$$

This is a parabola. In fact, it is the parabola $x = \dfrac{1}{2\sqrt{2}} y^2$ roatated counterclockwise $\dfrac{\pi}{4}$ and translated "up the diagonal $y = x$" by a distance $\dfrac{289}{20}\sqrt{2}$, having its vertex at $\left(\dfrac{289}{20}, \dfrac{289}{20}\right)$.

We will actually consider the more general problem

$$\begin{cases} x = aN^2 + bN + c \\ y = a\,(N+1)^2 + b(N+1) + c \end{cases}$$

with the restrictions on N removed.
Treating these as parametric equations, we can eliminate the parameter N (without getting bogged down in the quadratic formula).

Expanding the expression for y gives

$$\begin{aligned} y &= aN^2 + 2aN + a + bN + b + c \\ &= \left(aN^2 + bN + c\right) + 2aN + a + b \\ &= x + 2aN + a + b. \end{aligned}$$

Solving for N gives $N = \dfrac{y - x - (a+b)}{2a}$.
Substituting back into the expression for x:

$$x = a\left(\frac{y - x - a - b}{2a}\right)^2 + b\left(\frac{y - x - a - b}{2a}\right) + c,$$

which simplifies to

$$(1) \quad x^2 + y^2 - 2xy - 2ax - 2ay + \left(a^2 - b^2 + 4ac\right) = 0.$$

This is our solution for the general problem. So we do indeed have a quadratic equation for our figure; the discriminate equals zero.

From calculus,we know that a 45° rotation will remove the xy term. The transformation equations are

$$x = \frac{1}{\sqrt{2}}\left(x' - y'\right) \quad \text{and} \quad y = \frac{1}{\sqrt{2}}\left(x' + y'\right)$$

Substituting these into Equation (1), we get

$$\frac{(x'-y')^2}{2} + \frac{(x'+y')^2}{2} - 2\frac{(x'-y')(x'+y')}{2} - \frac{2a}{\sqrt{2}}\left(x' - y'\right) - \frac{2a}{\sqrt{2}}\left(x' + y'\right) + \left(a^2 - b^2 + 4ac\right) = 0.$$

This simplifies to

$$2\left(y'\right)^2 - \frac{4a}{\sqrt{2}}x' + \left(a^2 - b^2 + 4ac\right) = 0.$$

This becomes more familiar as

$$x' - \frac{a^2 - b^2 + 4ac}{2a\sqrt{2}} = \frac{1}{a\sqrt{2}}\left(y'\right)^2.$$

We recognize a nice parabola in the x', y' plane. In fact, if we translate to the new origin, $\left(\dfrac{a^2 - b^2 + 4ac}{2a\sqrt{2}}, 0\right)$ (in the x', y' plane) and let

$$x'' = x' - \frac{a^2 - b^2 + 4ac}{2a\sqrt{2}} \quad \text{and} \quad y'' = y'$$

our equation becomes

$$x'' = \frac{1}{a\sqrt{2}}\left(y''\right)^2.$$

Substituting the values $a = 5, b = 14, c = 23$ produces the solution to the given problem.

Comment 1: We see that x and y are interchangeable in Equation (1), reflecting the fact that the line $y = x$ is the axis of symmetry of our parabola. Therefore, more lattice points than originally mandated fall on the parabola.

For convenience, let $u_n = aN^2 + bN + c$. By the given condition, for any integer N, the point (u_N, u_{N+1}) lies on the parabola. By symmetry, (u_{N+1}, u_N) also lies on the parabola.

Comment 2: We see that this sequence satisfies the first order non-linear recurrence: $u_{N+1} = u_N + (2N+1)\,a + b$. We have shown that the points $(u_N, u_{N+1}),\ N \in Z$, lie on the parabola given by Equation (1) (as do the points (u_{N+1}, u_N)). This is reminiscent of the result that pairs of Fibonacci numbers (F_N, F_{N+1}) lie on the hyperbolas $y^2 - xy - x^2 = \pm 1$. In fact, such pairs are the *only* lattice points on these hyperbolas.

So we wonder if the points (u_N, u_{N+1}) and (u_{N+1}, u_N) are the only lattice points on the parabola given by Equation (1).

Also solved by Brian D. Beasley, Clinton, SC; Edwin Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; Raúl A. Simón, Santiago, Chile, and the proposer.

- **5148:** *Proposed by Pedro Pantoja (student, UFRN), Natal, Brazil*

  Let a, b, c be positive real numbers such that $ab + bc + ac = 1$. Prove that

$$\frac{a^2}{\sqrt[3]{b(b+2c)}} + \frac{b^2}{\sqrt[3]{c(c+2a)}} + \frac{c^2}{\sqrt[3]{a(a+2b)}} \geq 1.$$

Solution 1 by David E. Manes, Oneonta, NY

Let $L = \dfrac{a^2}{\sqrt[3]{b(b+2c)}} + \dfrac{b^2}{\sqrt[3]{c(c+2a)}} + \dfrac{c^2}{\sqrt[3]{a(a+2b)}}$. To prove that $L \geq 1$, we will use

Jensen's inequality that states if $\lambda_1, \lambda_2 \ldots, \lambda_n$ are positive numbers satisfying $\lambda_1+\lambda_2+\ldots+\lambda_n=1$, and $x_1, x_2, \ldots, x_n$ are any n points in an interval where f is continuous and convex, then

$$\lambda_1 f(x_1)+\lambda_2 f(x_2)+\cdots+\lambda_n f(x_n) \geq f\Big(\lambda_1 x_1+\lambda_2 x_2+\cdots+\lambda_n x_n\Big).$$

The function $f(x)=\dfrac{1}{\sqrt[3]{x}}$ is continuous and convex on the interval $(0,\infty)$. Let

$$\alpha=a^2+b^2+c^2 \qquad \lambda_1=\frac{a^2}{\alpha} \qquad \lambda_2=\frac{b^2}{\alpha} \qquad \lambda_3=\frac{c^2}{\alpha}$$

$$x_1=b^2+2bc \qquad x_2=c^2+2ac \qquad x_3=a^2+2ab$$

Then $\lambda_1+\lambda_2+\lambda_3=1$ and Jensen's inequality implies

$$\begin{aligned}
\frac{1}{\alpha}L &= \frac{a^2}{\alpha}f\left(b^2+2bc\right)+\frac{b^2}{\alpha}f\left(c^2+2ac\right)+\frac{c^2}{\alpha}f\left(a^2+2ab\right)\\
&\geq f\left(\frac{a^2\left(b^2+2bc\right)+b^2\left(c^2+2ac\right)+c^2\left(a^2+2ab\right)}{\alpha}\right)\\
&= \sqrt[3]{\frac{\alpha}{a^2b^2+b^2c^2+c^2a^2+2a^2bc+2ab^2c+2abc^2}}\\
&= \sqrt[3]{\frac{\alpha}{(ab+bc+ac)^2}}=\sqrt[3]{\alpha}.
\end{aligned}$$

Hence, $L \geq \alpha^{4/3}=\left(a^2+b^2+c^2\right)^{4/3} \geq 1$ since the inequality $(a-b)^2+(b-c)^2+(c-a)^2 \geq 0$ with the constraint $ab+bc+ac=1$ implies a$^2+b^2+c^2 \geq 1$. Note that equality occurs if and only if $a=b=c=\dfrac{1}{\sqrt{3}}$.

Solution 2 by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany

Using Cauchy-Schwarz inequality we have,

$$\left(\sqrt[3]{b(b+2c)}+\sqrt[3]{c(c+2a)}+\sqrt[3]{a(a+2b)}\right)\left(\frac{a^2}{\sqrt[3]{b(b+2c)}}+\frac{b^2}{\sqrt[3]{c(c+2a)}}+\frac{c^2}{\sqrt[3]{a(a+2b)}}\right) \geq (a+b+c)^2,$$

which implies that,

$$\frac{a^2}{\sqrt[3]{b(b+2c)}}+\frac{b^2}{\sqrt[3]{c(c+2a)}}+\frac{c^2}{\sqrt[3]{a(a+2b)}} \geq \frac{(a+b+c)^2}{\sqrt[3]{b(b+2c)}+\sqrt[3]{c(c+2a)}+\sqrt[3]{a(a+2b)}}.$$

Using the fact that the function $f(x)=\sqrt[3]{x}$ is a concave function, since the second derivative is negative, we have that any three numbers x, y, z, according to Jensen's inequality, satisfy the inequality $f(x)+f(y)+f(z) \leq 3f\left(\frac{x+y+z}{3}\right)$ and applying this we have

$$\begin{aligned}
\frac{a^2}{\sqrt[3]{b(b+2c)}}+\frac{b^2}{\sqrt[3]{c(c+2a)}}+\frac{c^2}{\sqrt[3]{a(a+2b)}} &\geq \frac{(a+b+c)^2}{\sqrt[3]{b(b+2c)}+\sqrt[3]{c(c+2a)}+\sqrt[3]{a(a+2b)}} \\
&\geq \frac{(a+b+c)^2}{3\sqrt[3]{\left(\frac{b(b+2c)+c(c+2a)+a(a+2b)}{3}\right)}} \\
&= \frac{(a+b+c)^2}{\frac{3}{\sqrt[3]{3}}\sqrt[3]{(a+b+c)^2}}
\end{aligned}$$

So it is enough to prove that

$$\begin{aligned}
\frac{(a+b+c)^2}{\frac{3}{\sqrt[3]{3}}\sqrt[3]{(a+b+c)^2}} &\geq 1, \quad \text{which implies} \\
\frac{(a+b+c)^2}{\sqrt[3]{(a+b+c)^2}} &\geq \frac{3}{\sqrt[3]{3}} \\
(a+b+c)^2 &\geq 3
\end{aligned}$$

Using the given condition and the AM-GM inequality we have

$$\begin{aligned}
(a+b+c)^2 &= a^2+b^2+c^2+2ab+2bc+2ac \\
&\geq 3ab+3bc+3ac \\
&= 3(ab+bc+ac) \\
&= 3
\end{aligned}$$

and this is the end of the proof.

Solution 3 by Andrea Fanchini, Cantú, Italy

Recall Holder's inequality that states that if $a_{ij}, 1 \leq i \leq m, 1 \leq j \leq n$ are positive real numbers, then:

$$\prod_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}\right) \geq \left(\sum_{j=1}^{n}\sqrt[m]{\prod_{i=1}^{m} a_{ij}}\right)^m.$$

Setting $n=3$ and $m=4$ and using this inequality we have

$$\left(\sum_{cyc}\frac{a^2}{\sqrt[3]{b^2+2bc}}\right)\left(\sum_{cyc}\frac{a^2}{\sqrt[3]{b^2+2bc}}\right)\left(\sum_{cyc}\frac{a^2}{\sqrt[3]{b^2+2bc}}\right)\left(\sum_{cyc}a^2\left(b^2+2bc\right)\right) \geq \left(a^2+b^2+c^2\right)^4,$$

and being that $a^2+b^2+c^2 \geq ab+bc+ca$,

$$\left(\sum_{cyc}\frac{a^2}{\sqrt[3]{b^2+2bc}}\right)\left(\sum_{cyc}\frac{a^2}{\sqrt[3]{b^2+2bc}}\right)\left(\sum_{cyc}\frac{a^2}{\sqrt[3]{b^2+2bc}}\right)\left(\sum_{cyc}a^2\left(b^2+2bc\right)\right) \geq (ab+bc+ca)^4 = 1$$

because

$$\left(\sum_{cyc}a^2\left(b^2+2bc\right)\right) = (ab+bc+ca)^2 = 1,$$

and so the proposed inequality holds.

Also solved by Arkady Alt, San Jose, CA; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy, and the proposer.

- **5149:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

A regular n-gon $A_1, A_2 \cdots, A_n$ $(n \geq 3)$ has center F, the focus of the parabola $y^2 = 2px$, and no one of its vertices lies on the x axis. The rays $FA_1, FA_2, \cdots, FA_n$ cut the parabola at points $B_1, B_2, \cdots, B_n$.

Prove that

$$\frac{1}{n}\sum_{k=1}^{n} FB_k^2 > p^2.$$

Solution by Ángel Plaza (University of Las Palmas de Gran Canaria) and Javier Sánchez-Reyes (University of Castilla-La Mancha), Spain

In polar coordinates (r,θ) centered at the focus the parabola is given by $r = p/(1+\cos\theta)$. Defining the arguments $\theta_k = \theta_n + 2k\pi/n$ for $k = 1, 2, \ldots, n$ corresponding to the vertices A_k of the polygon, we have to prove that

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\frac{p^2}{(1+\cos\theta_k)^2} &> p^2,\\ \frac{1}{n}\sum_{k=1}^{n}\frac{1}{(1+\cos\theta_k)^2} &> 1,\end{aligned}$$

where $\theta_k \neq 0$ and $\theta_k \neq \pi$. Since the function $f(x) = 1/x^2$ is strictly convex and $\sum_{k=1}^{n}\cos\theta_k = 0$, for example because the sum of all the nth complex roots of unity is zero, it follows that

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{(1+\cos\theta_k)^2} > \left(1 + \frac{\displaystyle\sum_{k=1}^{n}\cos\theta_k}{n}\right)^{-2} = 1.$$

Also solved by Raúl A. Simón, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA and the proposer.

- **5150:** *Proposed by Mohsen Soltanifar (student, University of Saskatchewan), Saskatoon, Canada*

  Let $\{A_n\}_{n=1}^{\infty}, (A_n \in M_{n\times n}(C))$be a sequence of matrices such that $\det(A_n) \neq 0, 1$ for all $n \in N$. Calculate:
$$\lim_{n\to\infty} \frac{n^n \ln(|\det(A_n)|)}{\ln\left(|\det\left(adj^{\circ n}\left(A_n\right)\right)|\right)},$$
  where $adj^{\circ n}$ refers to $adj \circ adj \circ \cdots \circ adj, n$ times, the n^{th} iterate of the classical adjoint.

  Solved 1 by the proposer

  A simple calculation of $adj^{\circ n}\left(A\right),\ m = 1, 2, \cdots, 5$ using equalities:

$$(i) \qquad adj(A)\cdot A = A\cdot adj(A) = \det(A)\cdot I_{n\times n}.$$

$$(ii) \qquad \det(A^{-1}) = (det(A))^{-1}$$

$$(iii) \qquad \det(kA) = k^n \det(A)$$

  suggests the following conjecture:

$$adj^{om}\left(A\right) = \det\left(A\right)^{P_m(n)} A^{(-1)^m}; \quad P_m(n) = \frac{(n-1)^m + (-1)^{m-1}}{n},\ m, n \in N \qquad (**)$$

  We prove the conjecture by induction on the positive integer m. The assertion trivially holds for the case $m = 1$. Let it hold for some positive integer $m > 1$. Then

$$\begin{aligned}
adj^{om+1}\left(A\right) &= adj\left(adj^{om}\left(A\right)\right) \\
&= \det\left(adj^{om}\left(A\right)\right)\left(adj^{om}\left(A\right)\right)^{-1} \\
&= \det\left(\det(A)^{P_m(n)} A^{(-1)^m}\right)\left(\det\left(A\right)^{P_m(n)} A^{(-1)^m}\right)^{-1} \\
&= \det\left(A\right)^{(n-1)P_m(n)+(-1)^m}\left(A\right)^{(-1)^{m+1}}.
\end{aligned}$$

  Besides,
$$P_{m+1}(n) = (n-1)P_m(n) + (-1)^m = \frac{(n-1)^{m+1} + (-1)^m}{n},$$
  proving the assertion for positive integer $m + 1$. Accordingly, using $(**)$ we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^n \ln\left(|\det(A_n)|\right)}{\ln\left(|\det(adj^{on}(A_n))|\right)} &= \lim_{n\to\infty} \frac{n^n \ln\left(|\det(A_n)|\right)}{\ln\left(|\det(\det(A_n)^{P_n(n)} A_n^{(-1)^n})|\right)} \\
&= \lim_{n\to\infty} \frac{n^n \ln\left(|\det(A_n)|\right)}{\ln\left(|(\det(A_n)^{nP_n(n)} \det\left(A_n{}^{(-1)^n}\right)|\right)} \\
&= \lim_{n\to\infty} \frac{n^n \ln\left(|\det(A_n)|\right)}{\ln\left(|(\det(A_n)^{nP_n(n)+(-1)^n}|\right)}
\end{aligned}$$

$$\begin{aligned} &= \lim_{n\to\infty} \frac{n^n}{nP_n(n) + (-1)^n} \\ &= \lim_{n\to\infty} \left(\frac{n}{n-1}\right)^n \\ &= e. \end{aligned}$$

Solution 2 by David Stone and John Hawkins, Statesboro, GA

We shall find a formula for $adj^{\circ n}(nA)$ and then show the limit is e.

First recall some properties of the inverse and the classical adjoint, where A is $n \times n$ and invertible and c a non-zero scalar.

$$(1) \qquad adj(A) = \det(A)A^{-1}$$

$$(2) \qquad adj\,(A)^{-1} = \frac{1}{\det(A)}A = adj\left(A^{-1}\right)$$

$$(3) \qquad \det\left[adj(A)\right] = \left[\det A\right]^{n-1}$$

$$(4) \qquad \det(cA) = c^n \det(A)$$

$$(5) \qquad (cA)^{-1} = \frac{1}{c}A^{-1}$$

$$(6) \qquad adj(cA) = c^{n-1}adj(A)$$

Then we see

$$\begin{aligned} (7)\ \ adj^{\circ 2}(A) &= adj\left[adj(A)\right] \\ &= \det\left[adj(A)\right]\left[adj(A)\right]^{-1} \text{ by (1)} \\ &= \left[\det(A)\right]^{n-1}\frac{1}{\det(A)}A \text{ by (3) and (2)} \\ &= \left[\det(A)\right]^{n-2}A. \end{aligned}$$

Continuing with our calculations, we have

$$\begin{aligned} (8)\ \ adj^{\circ 3}(A) &= adj\left[adj^{\circ 2}(A)\right] \\ &= adj\left[\left[\det(A)\right]^{n-2}A\right] \text{ by (7)} \\ &= \left\{\left[\det(A)\right]^{n-2}\right\}^{n-1} adj(A) \text{ by (6)} \\ &= \left[\det(A)\right]^{(n-1)(n-2)}\det(A)A^{-1}\text{by (1)} \\ &= \left[\det(A)\right]^{n^2-3n+3}A^{-1} \end{aligned}$$

We observe that repeated applications of adj will produce terms of the form $[\det(A)]^{[p_k(n)}\,A^{(-1)k}$, where $p_i(n)$ is a polynomial of degree $k-1$ in n.

Specifically, for $k = 1, 2, 3, \ldots, n-1$, we have

$$
\begin{aligned}
(9)\quad adj^{\circ(k+1)}(A) &= adj\left[adj^{\circ k}(A)\right] \\
&= adj\left[[\det(A)]^{p_k(n)}\,A^{(-1)^k}\right] \text{ by induction} \\
&= \left\{[\det(A)]^{p_k(n)}\right\}^{n-1} adj\left(A^{(-1)^k}\right) \text{ by (6)} \\
&= [\det(A)]^{(n-1)p_k(n)} \det\left(A^{(-1)^k}\right)\left[A^{(-1)^k}\right]^{-1} \text{ by (1)} \\
&= [\det(A)]^{(n-1)p_k(n)+(-1)^k}\,A^{(-1)^{k+1}}
\end{aligned}
$$

Therefore, we can recursively compute the polynomials which give the exponent on $\det(A)$ and obtain a concrete formula for $adj(A) : p_{k+1}(n) = (n-1)p_k(n) + (-1)^k$.

By (1) $adj(A) = \det(A)A^{-1}$, so $p_1(n) = 1$.

By (7) $adj^{\circ 2}(A) = [\det(A)]^{n-2}A$, so $p_2(n) = n-2$.

Then $p_3(n) = (n-1)p_2(n) + (-1)^2 = (n-1)(n-2) + 1 = n^2 - 3n + 3$, agreeing with (8).

Continuing, we find that

$p_4(n) = n^3 - 4n^2 + 6n - 4$ and

$p_5(n) = n^4 - 5n^3 + 10n^2 - 10n + 5$.

The appearance of the binomial coefficients is unmistakable. We deduce that, for $k = 1, 2, 3, \ldots, n$,

$p_k(n) = \dfrac{(n-1)^k + (-1)^{k-1}}{n}$, a polynomial of degree $k-1$.

The capstone of this sequence of polynomials: $p_n(n) = \dfrac{(n-1)^n + (-1)^{n-1}}{n}$, allows us to calculate $adj^{\circ n}(A)$ as:

$$
(10)\qquad adj^{\circ n}(A) = [\det(A)]^{\dfrac{(n-1)^n + (-1)^{n-1}}{n}}A^{(-1)^n}.
$$

Therefore, $A_n \in M_{n\times n}(\mathbf{C})$,

$$
\begin{aligned}
\det\left(adj^{\circ n}(A_n)\right) &= \det\left\{[\det(A)]^{\dfrac{(n-1)^n + (-1)^{n-1}}{n}}\,A^{(-1)^n}\right\} \text{ by(10)} \\
&= \left([\det(A)]^{\dfrac{(n-1)^n + (-1)^{n-1}}{n}}\right)^n \det\left[A^{(-1)^n}\right] \text{ by (4)} \\
&= [\det(A)]^{(n-1)^n + (-1)^{n-1}} \det\left[A^{(-1)^n}\right]
\end{aligned}
$$

$$= [\det(A)]^{(n-1)^n + (-1)^{n-1} + (-1)^n}$$

$$= [\det(A)]^{(n-1)^n}.$$

Thus,

$$\ln\left(\left|\det\left(adj^{\circ n}\left(A_n\right)\right)\right|\right) = \ln\left|[\det(A_n)]^{(n-1)^n}\right| = (n-1)^n \ln|\det(A_n)|,$$

so, for $n \geq 2$,

$$\frac{n^n \ln\left(|\det(A_n)|\right)}{\ln\left(\left|\det\left(adj^{\circ n}(A_n)\right)\right|\right)} = \frac{n^n \ln\left(|\det(A_n)|\right)}{(n-1)^n \ln\left(|\det(A_n)\right)|} = \frac{n^n}{(n-1)^n} = \left(\frac{n}{n-1}\right)^n.$$

That is, the individual A_n has disappeared and our complex fraction has become very simple.

Now it is easy to show by calculus that the limit is e.

- **5151:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Find the value of

$$\prod_{n=1}^{\infty}\left(\sqrt{\frac{\pi}{2}} \cdot \frac{(2n-1)!!\sqrt{2n+1}}{2^n n!}\right)^{(-1)^n}.$$

More generally, if $x \neq n\pi$ is a real number, find the value of

$$\prod_{n=1}^{\infty}\left(\frac{x}{\sin x}\left(1-\frac{x^2}{\pi^2}\right)\cdots\left(1-\frac{x^2}{(n\pi)^2}\right)\right)^{(-1)^n}.$$

Solution by the proposer

The first product equals $\sqrt{\dfrac{2\sqrt{2}}{\pi}}$ and the second one equals $\dfrac{2\sin\frac{x}{2}}{x}$. Recall the infinite product representation for the sine function

$$\sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right).$$

Since the first product can be obtained from the second one, when $x = \pi/2$, we concentrate on the calculation of the second product. Let

$$\begin{aligned}
S_{2n} &= \sum_{k=1}^{2n}(-1)^k\left(\ln\left(1-\frac{x^2}{\pi^2}\right)+\cdots+\ln\left(1-\frac{x^2}{k^2\pi^2}\right)+\ln\frac{x}{\sin x}\right) \\
&= -\left(\ln\left(1-\frac{x^2}{\pi^2}\right)+\ln\frac{x}{\sin x}\right)+\left(\ln\left(1-\frac{x^2}{\pi^2}\right)+\ln\left(1-\frac{x^2}{2^2\pi^2}\right)+\ln\frac{x}{\sin x}\right) \\
&\dots
\end{aligned}$$

$$
\begin{aligned}
& -\left(\ln\left(1-\frac{x^2}{\pi^2}\right)+\ln\left(1-\frac{x^2}{2^2\pi^2}\right)+\cdots+\ln\left(1-\frac{x^2}{(2n-1)^2\pi^2}\right)+\ln\frac{x}{\sin x}\right) \\
& +\left(\ln\left(1-\frac{x^2}{\pi^2}\right)+\ln\left(1-\frac{x^2}{2^2\pi^2}\right)+\cdots+\ln\left(1-\frac{x^2}{(2n-1)^2\pi^2}\right)+\ln\left(1-\frac{x^2}{(2n)^2\pi^2}\right)+\ln\frac{x}{\sin x}\right) \\
= & \ln\left(\left(1-\frac{x^2}{(2\pi)^2}\right)\left(1-\frac{x^2}{(4\pi)^2}\right)\cdots\left(1-\frac{x^2}{(2n\pi)^2}\right)\right) \\
= & \ln\left(\left(1-\frac{(x/2)^2}{\pi^2}\right)\left(1-\frac{(x/2)^2}{(2\pi)^2}\right)\cdots\left(1-\frac{(x/2)^2}{(n\pi)^2}\right)\right).
\end{aligned}
$$

Letting n tend to infinity in the preceding equality we get that $\lim_{n\to\infty} S_{2n} = \ln\dfrac{2\sin(x/2)}{x}$, and the problem is solved

Also solved by Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy.