Problems

Ted Eisenberg, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before September 15, 2017

- **5451:** *Proposed by Kenneth Korbin, New York, NY*

  Given triangle ABC with sides $a = 8, b = 19$ and $c = 22$. The triangle has an interior point P where $\overline{AP}$, $\overline{BP}$, and $\overline{CP}$ each have positive integer length. Find $\overline{AP}$ and $\overline{BP}$, if $\overline{CP} = 4$.

- **5452:** *Proposed by Roger Izard, Dallas, TX*

  Let point O be the orthocenter of a given triangle ABC. In triangle ABC let the altitude from B intersect line segment AC at E, and the altitude from C intersect line segment AB at D. If AC and AB are unequal, derive a formula which gives the square of BC in terms of AC, AB, EO, and OD.

- **5453:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

  If $a, b, c \in (0,1)$ or $a, b, c \in (1, \infty)$ and m, n are positive real numbers, then prove that

$$\frac{\log_a b + \log_b c}{m + n\log_a c} + \frac{\log_b c + \log_c a}{m + n\log_b a} + \frac{\log_c a + \log_a b}{m + n\log_c b} \geq \frac{6}{m+n}$$

- **5454:** *Proposed by Arkady Alt, San Jose, CA*

  Prove that for integers k and l, and for any $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$, the following inequality holds:

$$k^2 \tan\alpha + l^2 \tan\beta \geq \frac{2kl}{\sin(\alpha+\beta)} - \left(k^2 + l^2\right)\cot(\alpha+\beta).$$

- **5455:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Find all real solutions to the following system of equations:

$$\begin{aligned} \frac{1}{a} + \frac{1}{b} + \frac{1}{c} &= \frac{1}{abc} \\ a + b + c &= abc + \frac{8}{27}\left(a+b+c\right)^3 \end{aligned}$$

- **5456:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let k be a positive integer. Calculate

$$\lim_{x\to\infty} \mathrm{e}^{-x} \sum_{n=k}^{\infty} (-1)^n \binom{n}{k} \left(\mathrm{e}^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} \right).$$

Solutions

- **5433:** *Proposed by Kenneth Korbin, New York, NY*

  Solve the equation: $\sqrt[4]{x+x^2} = \sqrt[4]{x} + \sqrt[4]{x-x^2}$, with $x > 0$.

Solution 1 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

Let $f(x) = \sqrt[4]{x} + \sqrt[4]{x-x^2} - \sqrt[4]{x+x^2}$. Then $f(x)$ is continuous on $[0,1]$. We have $f(1/2) > 0$ and $f(1) < 0$. By the Intermediate Value Theorem our original equation has at least one solution with $x > 0$.

Now consider

$$\begin{aligned}
\sqrt[4]{x+x^2} = \sqrt[4]{x} + \sqrt[4]{x-x^2} \quad &\Longrightarrow \quad \sqrt[4]{1+x} = 1 + \sqrt[4]{1-x} \\
&\Longrightarrow \quad \sqrt[4]{1+x} - \sqrt[4]{1-x} = 1 \\
&\Longrightarrow \quad \sqrt{1+x} - 2\sqrt[4]{1-x^2} + \sqrt{1-x} = 1 \\
&\Longrightarrow \quad \sqrt{1+x} + \sqrt{1-x} = 1 + 2\sqrt[4]{1-x^2} \\
&\Longrightarrow \quad 1 + x + 2\sqrt{1-x^2} + 1 - x = 1 + 4\sqrt[4]{1-x^2} + 4\sqrt{1-x^2} \\
&\Longrightarrow \quad 1 - 2\sqrt{1-x^2} = 4\sqrt[4]{1-x^2} \\
&\Longrightarrow \quad 1 - 4\sqrt{1-x^2} + 4(1-x^2) = 16\sqrt{1-x^2} \\
&\Longrightarrow \quad 5 - 4x^2 = 20\sqrt{1-x^2} \\
&\Longrightarrow \quad 25 - 40x^2 + 16x^4 = 400(1-x^2) \\
&\Longrightarrow \quad 16x^4 + 360x^2 - 375 = 0
\end{aligned}$$

As a quadratic in x^2 the roots of this polynomial are

$$x^2 = \frac{-360 \pm 160\sqrt{6}}{32} = \frac{-45 \pm 20\sqrt{6}}{4}$$

and so

$$x = \pm \frac{\sqrt{-45 \pm 20\sqrt{6}}}{2}$$

This is a positive real number only if we choose both signs positive. Thus our original equation has at most one positive real solution.

Our last two paragraphs show that

$$x = \frac{\sqrt{20\sqrt{6} - 45}}{2}.$$

is the unique positive real solution to our original equation.

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

Since $x>0$, we lose no solutions if we divide by $\sqrt[4]{x}$ to obtain

$$\sqrt[4]{1+x}=1+\sqrt[4]{1-x}.$$

If we let $X=\sqrt[4]{1+x}$ and $Y=\sqrt[4]{1-x}$, then $X^4+Y^4=2$ and we can solve for XY in the following steps:

$$\begin{aligned}
X-Y &= 1\\
(X-Y)^4 &= 1\\
X^4-4X^3Y+6X^2Y^2-4XY^3+Y^4 &= 1\\
X^4+Y^4-2XY\left(2X^2-3XY+2Y^2\right) &= 1\\
-2XY\left[2\left(X-Y\right)^2+XY\right] &= -1\\
2XY\left(XY+2\right) &= 1\\
2X^2Y^2+4XY-1 &= 0\\
XY &= \frac{-2\pm\sqrt{6}}{2}.
\end{aligned}$$

The condition $XY=\sqrt[4]{1-x^2}\geq 0$ implies that

$$\begin{aligned}
\sqrt[4]{1-x^2} &= \frac{\sqrt{6}-2}{2}\\
1-x^2 &= \left(\frac{\sqrt{6}-2}{2}\right)^4=\frac{49-20\sqrt{6}}{4}\\
x^2 &= 1-\frac{49-20\sqrt{6}}{4}=\frac{20\sqrt{6}-45}{4}.
\end{aligned}$$

Because $x>0$, our solution is

$$x=\frac{\sqrt{20\sqrt{6}-45}}{2}.$$

Solution 3 by Brian D. Beasley, Presbyterian College, Clinton, SC

Solution. Since $x>0$, we may divide the given equation by $\sqrt[4]{x}$ to produce

$$\sqrt[4]{1+x}=1+\sqrt[4]{1-x}.$$

Squaring both sides then yields $\sqrt{1+x}=1+2\sqrt[4]{1-x}+\sqrt{1-x}$, or $\sqrt{1+x}-\sqrt{1-x}-1=2\sqrt[4]{1-x}$. Squaring yet again produces

$$(1+x)+(1-x)+1-2\sqrt{1+x}+2\sqrt{1-x}-2\sqrt{1-x^2}=4\sqrt{1-x},$$

or $3-2\sqrt{1-x^2}=2\sqrt{1+x}+2\sqrt{1-x}$. We square once more to obtain

$$9-12\sqrt{1-x^2}+4(1-x^2)=4(1+x)+4(1-x)+8\sqrt{1-x^2}$$

and thus $5-4x^2 = 20\sqrt{1-x^2}$. Squaring for the last time yields
$25-40x^2+16x^4 = 400(1-x^2)$ and hence $16x^4+360x^2-375 = 0$. Finally, the only real positive solution of this equation is

$$x = \sqrt{-\frac{45}{4}+5\sqrt{6}} = \frac{\sqrt{-45+20\sqrt{6}}}{2}.$$

Addendum. It is interesting to note that this solution is approximately 0.99872354, very close to 1. In particular, this implies that $49/4$ is a good rational approximation of $5\sqrt{6}$, which also means that $7/2$ is a good rational approximation of $\sqrt[4]{150}$.

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Jeremiah Bartz, University of North Dakota, Grand Forks, ND; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Aykut Ismailov, Shumen, Bulgaria; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY at Oneonta, Oneonta, NY; Boris Rays, Brooklyn, NY; Brandon Richardson (student), Auburn University at Montgomery, AL; Toshihiro Shimizu, Kawasaki, Japan; Trey Smith, Angelo State University, San Angelo, TX; Albert Stadler, Herrliberg, Switzerland; Anna V. Tomova (three solutions), Varna, Bulgaria, and the proposer.

- **5434:** *Proposed by Titu Zvonaru, Comnesti, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania*

Calculate, without using a calculator or log tables, the number of digits in the base 10 expansion of 2^{96}.

Solution 1 by Ed Gray, Highland Beach, FL

$$\left(2^{12}\right)^8 = 2^{96} > (4\cdot 10^3)^8 = 4^8\cdot 10^{24} > 6\cdot 10^4\cdot 10^{24} = 6\cdot 10^{28}.$$

Also

$$\left(2^8\right)^{12} = 2^{96} < \left(3\cdot 10^2\right)^{12} = 3^{12}\cdot 10^{24} < \left(6\cdot 10^5\right)\cdot 10^{24} = 6\cdot 10^{29}.$$

Therefore, $6\cdot 10^{28} < 2^{96} < 6\cdot 10^{29}$. So $n = 29$.

Solution 2 by Paul M. Harms, North Newton, KS

We see that

$$4(10^3) < 2^{12} = 4096 < 4.1(10^3).$$

Then

$$16(10^6) < 2^{24} < 16.81(10^6) < 17(10^6).$$

Taking the fourth power of the appropriate terms we obtain,

$$16^4(10^{24}) = 65536(10^{24}) = 0.65536(10^{29}) < 2^{96} < 17^4(10^{24}) = 83521(10^{24}) = 0.83521(10^{29}).$$

Since 2^{96} is bounded by integers who have 29 digits in the base 10 expansion, the integer 2^{96} must also have 29 digits in its base 10 expansion.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

The required number of digits is 29 because, as we shall show, $10^{28} \le 2^{96} < 10^{29}$. More exactly, we shall prove that $1 < \dfrac{2^{96}}{10^{28}} < 10$. Since

$$\frac{2^{96}}{10^{28}} = \left(\frac{2^{24}}{10^7}\right)^4 = \left(\frac{\left(2^{12}\right)^2}{10^7}\right)^4 = \left(\frac{4096^2}{10^7}\right)^4 = \left(\frac{1,6777216 \cdot 10^7}{10^7}\right)^4 = (1,6777216)^4\,,$$

we obtain that

$1^4 < \dfrac{2^{96}}{10^{28}} < 1,68)^4$, that is $1 < \dfrac{2^{96}}{10^{28}} < (2.8224)^2$ and, hence, $1 < \dfrac{2^{96}}{10^{28}} < 3^2 < 10$.

Note: another way to show that $10^{28} < 2^{96}$ is, for example:

$$\left.\begin{matrix} 5^2 < 2^5 \\ 5 < 2^3 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} 5^2 < 2^5 \\ 5^3 < 2^9 \end{matrix}\right\} \Rightarrow 5^5 < 2^5 \cdot 5^3 < 2^{12} \Rightarrow \left.\begin{matrix} 5^5 < 2^{12} \\ 5^2 < 2^5 \end{matrix}\right\} \Rightarrow 5^7 < 2^5 \cdot 5^5 < 2^{17} \Rightarrow$$

$$\Rightarrow 2^7 \cdot 5^7 < 2^{24} \Rightarrow$$

$$\Rightarrow \left(10^7\right)^4 < \left(2^{24}\right)^4 \Rightarrow$$

$$\Rightarrow 10^{28} < 2^{96}.$$

Solution 4 by Toshihiro Shimizu, Kawasaki, Japan

Since $10^3 < 2^{10} = 1024 < 1.03 \times 10^3$ and $2^{96} = \left(2^{10}\right)^9 \times 2^6 = \left(2^{10}\right)^9 \times 10 \times 6.4$ we have

$$6.4 \times 10 \times 10^{3\times 9} < 2^{96} < 6.4 \times 10 \times 10^{3\times 9} \times (1.03)^9\,.$$

We evaluate 1.03^9. We have $1.03 \times 1.03 \times 1.03 = 1.0609 \times 1.03 = 1.092727 < 1.1$ and $1.1 \times 1.1 \times 1.1 = 1.331 < 1.4$ (I never use calculator.) Therefore, we have

$$10^{28} < 6.4 \times 10^{28} < 2^{96} < 6.4 \times 1.4 \times 10^{28} = 8.96 \times 10^{28} < 10^{29}.$$

Therefore, the number of digits in 2^{96} is 29.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposers.

- **5435:** *Proposed by Valcho Milchev, Petko Rachov Slaveikov Seconday School, Bulgaria*

Find all positive integers a and b for which $\dfrac{a^4 + 3a^2 + 1}{ab - 1}$ is a positive integer.

Solution 1 by Moti Levy, Rehovot, Israel

This solution is based on similar problem and solution which appeared in [1].

$\dfrac{a^4+3a^2+1}{ab-1}$ may be replaced by equivalent expression with *symmetric* polynomial in the numerator.

Indeed,

$$\frac{a^4+3a^2+1}{ab-1}=\frac{a^2\left(a^2+b^2+3\right)-(ab-1)(ab+1)}{ab-1}.$$

Now, a and $ab-1$ satisfy the equation $b*a+(-1)*(ab-1)=1$, which implies that a and $ab-1$ are relatively prime and clearly a^2 and $ab-1$ are also relatively prime. Thus, $\dfrac{a^4+3a^2+1}{ab-1}$ is a positive integer if and only if $\dfrac{a^2+b^2+3}{ab-1}$ is a positive integer. We call the ordered pair (a,b) a *solution* if

$$\frac{a^2+b^2+3}{ab-1}=m, \tag{1}$$

where m is a positive integer. The set of solutions is not empty since $(1,2)$ is a solution. We exclude (a,a) from the set of solutions since $\dfrac{2a^2+3}{a^2-1}=2+\dfrac{5}{a^2-1}\notin N$ for all $a>0$. Equation (1) is re-written as follows

$$a^2-mab+b^2=-(m+3). \tag{2}$$

It is easily verified (see (3)) that if (a,b) is a solution then $(ma-b,a)$ is a solution as well.

$$(ma-b)^2-m(ma-b)a+a^2=a^2-mab+b^2, \tag{3}$$

Let (a_0,b_0) be the "smallest" solution in the sense that $a_0+b_0\le a+b$, where (a,b) is any solution.

$$a_0+b_0\le(ma_0-b_0)+a_0,$$

or

$$\frac{2b_0}{a_0}\le m. \tag{4}$$

$$\frac{2b_0}{a_0}\le\frac{a_0^2+b_0^2+3}{a_0b_0-1}$$

$$0\le-2a_0b_0^2+2b_0+a_0^3+3a_0 \tag{5}$$

Let (a_0,a_0+k) be a solution. Then substituting in (5) gives,

$$\begin{aligned}0&\le-2a_0(a_0+k)^2+2(a_0+k)+a_0^3+3a_0\\&=-2k^2a_0-4ka_0^2+2k-a_0^3+5a_0.\end{aligned}$$

Solving $-2k^2a_0-4ka_0^2+2k-a_0^3+5a_0\ge0$, we get

$$\frac{1}{2a_0}\left(1-2a_0^2-\sqrt{6a_0^2+2a_0^4+1}\right)\le k\le\frac{1}{2a_0}\left(1-2a_0^2+\sqrt{6a_0^2+2a_0^4+1}\right),$$

hence, k will have positive values only if

$$\sqrt{6a_0^2+2a_0^4+1}+1\ge2a_0^2.$$

This inequality holds for $a_0=1$ and $a_0=2$. For $a_0=1$, possible values for k are 1 or 2; for $a_0=2$, possible value for k is 1.

Thus we have to check the following set of potential solutions: $\{(1,2),(1,3),(2,1)\}$. Clearly $(1,2)$ and $(2,1)$ are solutions, but $(1,3)$ is not.

For $(1,2)$ and $(2,1)$ the value of m is 8. We conclude that the sole value of m is 8.

It follows from (3) that the pairs (a_n, b_n) (and by symmetry (b_n, a_n)), which satisfy condition (1) are expressed by the recurrence formulas

$$\begin{aligned} a_{n+1} &= 8a_n - b_n, \\ b_{n+1} &= a_n, \end{aligned}$$

which are equivalent to the recurrence formulas

$$\begin{aligned} a_{n+2} &= 8a_{n+1} - a_n, \\ b_{n+2} &= 8b_{n+1} - b_n. \end{aligned} \tag{6}$$

We have two sets of initial conditions:

1) $a_0 = 1$, $a_1 = 6, b_0 = 2, b_1 = 1$; the pairs resulting from these initial conditions are $(1,2)$, $(6,1)$, $(47,6)$, $(370,47)$,....

$$\begin{aligned} a_n &= \left(\frac{1}{2} - \frac{1}{\sqrt{15}}\right)\left(4-\sqrt{15}\right)^n + \left(\frac{1}{2} + \frac{1}{\sqrt{15}}\right)\left(4+\sqrt{15}\right)^n, \\ b_n &= \left(1 + \frac{7}{2\sqrt{15}}\right)\left(4-\sqrt{15}\right)^n + \left(1 - \frac{7}{2\sqrt{15}}\right)\left(4+\sqrt{15}\right)^n. \end{aligned}$$

2) $a_0 = 2$, $a_1 = 15, b_0 = 1, b_1 = 2$; the pairs resulting from these initial conditions are $(2,1)$, $(15,2)$, $(118,15)$, $(929,118)$, ...

$$\begin{aligned} a_n &= \left(1 - \frac{7}{2\sqrt{15}}\right)\left(4-\sqrt{15}\right)^n + \left(1 + \frac{7}{2\sqrt{15}}\right)\left(4+\sqrt{15}\right)^n, \\ b_n &= \left(\frac{1}{2} + \frac{1}{\sqrt{15}}\right)\left(4-\sqrt{15}\right)^n + \left(\frac{1}{2} - \frac{1}{\sqrt{15}}\right)\left(4+\sqrt{15}\right)^n. \end{aligned}$$

Reference:

[1] La Gaceta de la RSME, Vol. 18 (2015), No. 1, "*Solution to Problem 241, by Roberto de la Cruz Moreno*".

Solution 2 by Anthony Bevelacqua, University of North Dakota, Grand Forks, ND

1) There are no solutions to our problem with $a = b$. We have $a^4 + 3a^2 + 1 \equiv 5 \bmod (a^2 - 1)$. Assume there is a solution with $a = b$. Then $a^2 - 1$ divides $a^4 + 3a^2 + 1$ so $a^4 + 3a^2 + 1 \equiv 0 \bmod (a^2 - 1)$. Thus $5 \equiv 0 \bmod (a^2 - 1)$ and so $a^2 - 1$ divides 5. But then $a^2 = 2$ or $a^2 = 6$, a contradiction in either case.

2) The only solutions with $a \leq 4$ are $(a, b) = (1,2)$, $(2,1)$, $(1,6)$ and $(2,15)$. Suppose (a, b) is a solution to our problem. If $a = 1$ then $b - 1$ divides 5 so $b - 1 = 1$ or $b - 1 = 5$. Both $(1,2)$ and $(1,6)$ are solutions. If $a = 2$ then $2b - 1$ divides 29 so $2b - 1 = 1$ or $2b - 1 = 29$. Both $(2,1)$ and $(2,15)$ are solutions. If $a = 3$ then $3b - 1$ divides 109 so $3b - 1 = 1$ or $3b - 1 = 109$, a contradiction. If $a = 4$ then $4b - 1$ divides $305 = 5 \cdot 61$ so $4b - 1 \in \{1, 5, 61, 305\}$, a contradiction.

3) $ab-1$ divides a^4+3a^2+1 if and only if $ab-1$ divides a^2+b^2+3.
We have

$$\begin{aligned}(ab-1)(a^3b+3ab+a^2+3) &= a^4b^2+3a^2b^2+a^3b+3ab-a^3b-3ab-a^2-3\\ &= a^4b^2+3a^2b^2-a^2-3\end{aligned}$$

and so

$$b^2(a^4+3a^2+1)-(ab-1)(a^3b+3ab+a^2+3)=a^2+b^2+3.$$

Thus if $ab-1$ divides a^4+3a^2+1 then $ab-1$ divides a^2+b^2+3. Conversely suppose $ab-1$ divides a^2+b^2+3. Then $ab-1$ divides $b^2(a^4+3a^2+1)$. Since $ab-1$ and b^2 are relatively prime we have that $ab-1$ divides a^4+3a^2+1.

Now if $k>0$ and (a,b) is a solution to $a^2+b^2+3=k(ab-1)$ then b is a root of the polynomial $a^2+x^2+3=k(ax-1)$ which can be rewritten as
$x^2-kax+(a^2+3+k)=0$. Thus if b' is the other root we have, by Vieta's formulas, $b+b'=ka$ and $bb'=a^2+3+k$. The first shows that b' is an integer and the second shows that $b'>0$. Thus (a,b') is another solution to $a^2+b^2+3=k(ab-1)$.

4) If $ab-1$ divides a^2+b^2+3 then $a^2+b^2+3=8(ab-1)$. Suppose there are positive integers a,b,k such that $a^2+b^2+3=k(ab-1)$. For this fixed k let S be the set of all positive integer pairs (a,b) such that $a^2+b^2+3=k(ab-1)$. Choose an $(a,b)\in S$ such that $a+b$ is minimal. Without loss of generality we have $a\le b$. Since $a\ne b$ by 1) we have $a<b$. Now (a,b') is another solution. Since $a+b$ is minimal we have $a+b\le a+b'$ and hence $b\le b'$. Thus

$$b^2\le bb'=a^2+3+k \implies k\ge b^2-a^2-3$$

and so

$$\begin{aligned}a^2+b^2+3 &= k(ab-1)\\ &\ge (b^2-a^2-3)(ab-1)\\ &= ab^3-b^2-a^3b+a^2-3ab+3.\end{aligned}$$

Hence

$$3ab+2b^2\ge ab^3-a^3b \implies 3a+2b\ge ab^2-a^3.$$

Since $a<b$ we have $3a+2b<5b$ and $ab^2-a^3=a(b+a)(b-a)>ab$. Thus $5b>ab$ and so $a<5$. By 2) the only possible (a,b) are then $(1,2)$, $(1,6)$, and $(2,15)$. Each of these gives $k=8$.

Thus 3) and 4) show that our original problem is equivalent to finding all positive integers a and b such that $a^2+b^2+3=8(ab-1)$. We could rewrite this as $(a-4b)^2-15b^2=-11$ and apply the theory of equations of the form $x^2-Dy^2=N$ as found in, say, section 58 of Nagell's *Number Theory.* Instead we will determine the solutions by "Vieta jumping" as in the proof of (4).

Let S be the set of all positive integers pairs (a,b) such that $a^2+b^2+3=8(ab-1)$. Clearly if $(a,b)\in S$ then $(b,a)\in S$, and, by 1) there are no $(a,b)\in S$ with $a=b$. Recall that if $(a,b)\in S$ then $(a,b')\in S$ where $b+b'=8a$ and $bb'=a^2+11$.

5) For any $(a,b)\in S$ define $\rho(a,b)=(b',a)$ and $\lambda(a,b)=(b,8b-a)$. Then $\rho(a,b)\in S$, $\lambda(a,b)\in S$, and $\lambda(\rho(a,b))=(a,b)$.

Let $(a,b) \in S$. We have $(a,b') \in S$ and hence $\rho(a,b) = (b',a) \in S$. Now

$$\begin{aligned} b^2 + (8b-a)^2 + 3 &= 64b^2 - 16ab + (a^2+b^2+3) \\ &= 64b^2 - 16ab + 8(ab-1) \\ &= 64b^2 - 8ab - 8 \\ &= 8(b(8b-a)-1) \end{aligned}$$

so $\lambda(a,b) = (b, 8b-a) \in S$. Finally,

$$\lambda(\rho(a,b)) = \lambda(b',a) = (a, 8a-b')$$

where

$$8a - b' = 8a - \frac{a^2+11}{b} = \frac{8ab - a^2 - 11}{b} = \frac{b^2}{b} = b.$$

6) The only $(a,b) \in S$ such that $a < b \leq 10$ are $(a,b) = (1,2)$ and $(1,6)$.

Since $a^2 + b^2 + 3 \equiv 0 \bmod 8$ we see that a and b must have opposite parity and neither can be divisible by 4. Moreover the only such solutions with a or b less than 4 are $(1,2)$ and $(1,6)$ by 2). This leaves only

$$(a,b) = (5,6), (6,7), (6,9), (5,10), (7,10), (9,10)$$

and none of these satisfy $a^2 + b^2 + 3 = 8(ab-1)$.

7) Let $(a,b) \in S$ such that $b \geq 11$. If $a < b$ then $b' < a$

Suppose first that $b' \leq 10$. Assume $a \leq b'$. Since $(a,b') \in S$ we have $a \neq b'$. Thus $a < b' \leq 10$. So, by 6), we must have $a = 1$. But if $a = 1$ we have $b = 1$ or $b = 6$, a contradiction with $b \geq 11$. Hence $b' < a$.

Suppose now that $b' \geq 11$. Again assume $a \leq b'$. Then, as in the last paragraph, $a < b'$. We have

$$bb' = a^2 + 11 < (b')^2 + 11 \implies b < b' + \frac{11}{b'} \leq b' + 1$$

and so $b \leq b'$. Now swapping b and b' we have

$$bb' = a^2 + 11 < b^2 + 11 \implies b' < b + \frac{11}{b} \leq b + 1$$

and so $b' \leq b$. Thus $b = b'$. Since $8a = b + b' = 2b$ we have $b = 4a$. But then

$$a^2 + 16a^2 + 3 = 8(4a^2 - 1) \implies 11 = 15a^2,$$

a contradiction. Hence $b' < a$.

Finally,

8) $(a,b) \in S$ if and only if $\{a,b\} = \{s_n, s_{n+1}\}$ or $\{a,b\} = \{t_n, t_{n+1}\}$ for $n \geq 0$ where

$$s_0 = 1,\ s_1 = 2,\ \text{and } s_n = 8s_{n-1} - s_{n-2} \text{ for } n \geq 2$$

and

$$t_0 = 1,\ t_1 = 6,\ \text{and } t_n = 8t_{n-1} - t_{n-2} \text{ for } n \geq 2.$$

Note that $\lambda^n(1,2) = (s_n, s_{n+1})$ and $\lambda^n(1,6) = (t_n, t_{n+1})$ for all $n \geq 0$.

Since $(1,2)$ and $(1,6) \in S$ we see that $(a,b) \in S$ for any $\{a,b\} = \{s_n, s_{n+1}\}$ or $\{a,b\} = \{t_n, t_{n+1}\}$ and $n \geq 0$ by (5).

Now suppose $(a,b) \in S$. Since $(b,a) \in S$ as well, we can suppose without loss of generality that $a < b$. By 5) and 7) there exists an integer $d \geq 0$ such that $\rho^d(a,b) = (a^*, b^*)$ with $a^* < b^* \leq 10$. By (6) we must have $\rho^d(a,b) = (1,2)$ or $\rho^d(a,b) = (1,6)$. Since $(a,b) = \lambda^d(\rho^d(a,b))$ we have $(a,b) = \lambda^d(1,2)$ or $(a,b) = \lambda^d(1,6)$.

Thus $ab - 1$ divides $a^4 + 3a^2 + 1$ if and only if a and b are consecutive elements of either of the sequences s_n or t_n given above. Since the first few terms of s_n are $1, 2, 15, 118, 929, 7314, 57583, \ldots$ and the first few terms of t_n are $1, 6, 47, 370, 2913, 22934, 180559, \ldots$ the first few solutions to our problem (with $a \leq b$) are

$$(a,b) = (1,2), (2,15), (15,118), (118,929), (929,7314), (7314,57583), \ldots$$

and

$$(a,b) = (1,6), (6,47), (47,370), (370,2913), (2913,22934), (22934,180559), \ldots$$

Also solved by Ed Gray, Highland Beach, FL; Kenneth Korbin, NewYork, NY; Toshihiro Shimizu, Kawasaki, Japan; Anna V. Tomova (three solutions), Varna, Bulgaria, and the proposer.

- **5436:** *Proposed by Arkady Alt, San Jose, CA*

Find all values of the parameter t for which the system of inequalities

$$\mathbf{A} = \begin{cases} \sqrt[4]{x+t} \geq 2y \\ \sqrt[4]{y+t} \geq 2z \\ \sqrt[4]{z+t} \geq 2x \end{cases}$$

a) has solutions;

b) has a unique solution.

Solution by the Proposer

a) Note that **(A)** $\iff \begin{cases} t \geq 16y^4 - x \\ t \geq 16z^4 - y \\ t \geq 16x^4 - z \end{cases} \implies 3t \geq 16y^4 - x + 16z^4 - y + 16x^4 - z =$

$\left(16x^4 - x\right) + \left(16y^4 - y\right) + \left(16z^4 - z\right) \geq 3 \min\limits_x \left(16x^4 - x\right) \implies t \geq \min\limits_x \left(16x^4 - x\right).$

For $x \in \left(0, \dfrac{1}{16}\right)$, using the AM-GM Inequality, we obtain

$x - 16x^4 = x\left(1 - 16x^3\right) = \sqrt[3]{x^3\left(1 - 16x^3\right)^3} = \sqrt[3]{\dfrac{\left(48x^3\right)\left(1 - 16x^3\right)^3}{48}} \leq$

$\sqrt[3]{\dfrac{1}{48} \cdot \left(\dfrac{48x^3 + 3 - 3 \cdot 16x^3}{4}\right)^4} = \sqrt[3]{\dfrac{1}{48} \cdot \left(\dfrac{3}{4}\right)^4} = \dfrac{3}{16}$. And since $x - 16x^4 \leq 0$ for

$x \notin \left(0, \dfrac{1}{16}\right)$, then for all x the inequality $x - 16x^4 \leq \dfrac{3}{16}$ holds. Since the upper bound is $\dfrac{3}{16}$ for values

$x - 16x^4$ is attainable when $x = \dfrac{1}{4}$, then $\max\left(x - 16x^4\right) = \dfrac{3}{16} \iff$

$\min\limits_x \left(16x^4 - x\right) = -\dfrac{3}{16}$.

Thus $t \geq -\dfrac{3}{16}$ is a necessary condition for the solvability of system **(A).**

Let's prove sufficiency.

Let $t \geq -\dfrac{3}{16}$. Since function $h(x)$ is continuous in R and $\min\limits_x \left(16x^4 - x\right) = -\dfrac{3}{16}$, then $\left[-\dfrac{3}{16}, \infty\right)$ is the range of $h(x)$. This means that for any $t \geq -\dfrac{3}{16}$ the equation

$16x^4 - x = t$

has solution in R and since for any u which is a solution of the equation $16x^4 - x = t$ the triple $(x, y, z) = (u, u, u,)$ is a solution of the system **(A)** then for such t system **(A)** solvable as well.

Remark.

Actually the latest reasoning about the solvability of system **(A)** if $t \geq -\dfrac{3}{16}$ is redundant for **(a)** because suffices to note that for such t the triple $(x, y, z) = \left(\dfrac{1}{4}, \dfrac{1}{4}, \dfrac{1}{4}\right)$ satisfies to **(A)**.

But for **(b)** criteria of solvability of equation $16x^4 - x = t$ in form of inequality $t \geq -\dfrac{3}{16}$ is

important.

b) Note that system **(A)** always have more the one solution if $t > -\dfrac{3}{16}$.

Indeed, let for any $t_1, t_2 \in \left(-\dfrac{3}{16}, t\right)$ such that $t_1 \neq t_2$ equation $16u^4 - u = t_i$ has solution $u_i, i = 1, 2$.

Then $u_1 \neq u_2$ and two distinct triples $(u_1, u_1, u_1), (u_2, u_2, u_2)$ satisfy to the system **(A).**

Let $t = -\dfrac{3}{16}$.Then $-\dfrac{3}{16} \geq 16y^4 - x \implies -\dfrac{3}{16} + x - y \geq 16y^4 - y \geq -\dfrac{3}{16}$.

Hereof $x - y \geq 0 \iff x \geq y$. Similarly $-\dfrac{3}{16} \geq 16z^4 - y$ and $-\dfrac{3}{16} \geq 16x^4 - z$ implies $y \geq z$ and $z \geq x$, respectively. Thus in that case $x = y = z$ and all solutions of the system **(A)** are represented by solutions of one equation $16x^4 - x = -\dfrac{3}{16} \iff$

$16x^4 - x + \dfrac{3}{16} = 0 \iff 256x^4 - 16x + 3 = 0$ which has only root $\dfrac{1}{4}$ because

$256x^4 - 16x + 3 = (4x - 1)^2 \left(16x^2 + 8x + 3\right)$.

Thus, system **(A)** has unique solution iff $t = \dfrac{1}{4}$.

Also solved by Ed Gray, Highland Beach,FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; David Stone and John Hawkins, Georgia

Southern University, Statesboro, GA, and Toshihiro Shimizu, Kawasaki, Japan.

- **5437:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $f : C - \{2\} \to C$ be the function defined by $f(z) = \dfrac{2-3z}{z-2}$. If $f^n(z) = (\underbrace{f \circ f \circ \ldots \circ f}_{n})(z)$, then compute $f^n(z)$ and $\lim\limits_{n\to+\infty} f^n(z)$.

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

Assume first that $z \neq 2$ and $f^n(z)$ exists for all $n \geq 1$. Then, direct computation yields

$$f^2(z) = \frac{10-11z}{5z-6} \quad \text{and} \quad f^3(z) = \frac{42-43z}{21z-22}. \tag{1}$$

When these are combined with the formula for $f(z)$, it appears that there is a sequence $\{x_n\}$ of positive integers such that

$$f^n(z) = \frac{2x_n - (2x_n+1)z}{x_n z - (x_n+1)} \tag{2}$$

for all $n \geq 1$. Since $f(z) = \dfrac{2-3z}{z-2}$, we have $x_1 = 1$. Further, if (2) holds for some $n \geq 1$, then

$$\begin{aligned}
f^{n+1}(z) &= f(f^n(z)) \\
&= \frac{2-3f^n(z)}{f^n(z)-2} \\
&= \frac{2-3\left[\dfrac{2x_n-(2x_n+1)z}{x_n z-(x_n+1)}\right]}{\left[\dfrac{2x_n-(2x_n+1)z}{x_n z-(x_n+1)}\right]-2} \\
&= \frac{2[x_n z-(x_n+1)] - 3[2x_n-(2x_n+1)z]}{[2x_n-(2x_n+1)z] - 2[x_n z-(x_n+1)]} \\
&= \frac{(8x_n+2)-(8x_n+3)z}{(4x_n+1)z-(4x_n+2)}.
\end{aligned}$$

This suggests that $x_{n+1} = 4x_n + 1$ for $n \geq 1$. These conditions on $\{x_n\}$ are consistent with the formula for $f(z)$ and property (2). Note finally that

$$x_1 = 1 = \frac{3}{3} = \frac{4-1}{3}, \quad x_2 = 5 = \frac{15}{3} = \frac{4^2-1}{3}, \quad \text{and} \quad x_3 = 21 = \frac{63}{3} = \frac{4^3-1}{3}.$$

This leads us to conjecture that $x_n = \dfrac{4^n - 1}{3}$ and hence,

$$f^n(z) = \frac{2\left(\dfrac{4^n-1}{3}\right) - \left[2\left(\dfrac{4^n-1}{3}\right)+1\right]z}{\left(\dfrac{4^n-1}{3}\right)z - \left[\left(\dfrac{4^n-1}{3}\right)+1\right]} = \frac{2(4^n-1)-(2\cdot 4^n+1)z}{(4^n-1)z-(4^n+2)}$$

for all $n \geq 1$.

If $f^n(z)$ exists for all $n \geq 1$, let $P(n)$ be the statement

$$f^n(z) = \frac{2(4^n-1)-(2\cdot 4^n+1)z}{(4^n-1)z-(4^n+2)}. \tag{3}$$

If $n = 1$,

$$\frac{2(4-1)-(2\cdot 4+1)z}{(4-1)z-(4+2)} = \frac{6-9z}{3z-6}$$

$$= \frac{2-3z}{z-2}$$

and thus, $P(1)$ is true. Assume that $P(n)$ is true, i.e.,

$$f^n(z) = \frac{2(4^n-1)-(2\cdot 4^n+1)z}{(4^n-1)z-(4^n+2)}$$

for some $n \geq 1$. Then,

$$f^{n+1}(z) = f(f^n(z))$$

$$= \frac{2-3\left[\dfrac{2(4^n-1)-(2\cdot 4^n+1)z}{(4^n-1)z-(4^n+2)}\right]}{\left[\dfrac{2(4^n-1)-(2\cdot 4^n+1)z}{(4^n-1)z-(4^n+2)}\right]-2}$$

$$= \frac{2[(4^n-1)z-(4^n+2)]-3[2(4^n-1)-(2\cdot 4^n+1)z]}{[2(4^n-1)-(2\cdot 4^n+1)z]-2[(4^n-1)z-(4^n+2)]}$$

$$= \frac{[2(4^n-1)+3(2\cdot 4^n+1)]z-[2(4^n+2)+6(4^n-1)]}{[2(4^n-1)+2(4^n+2)]-[2\cdot 4^n+1+2(4^n-1)]z}$$

$$= \frac{\left(2\cdot 4^{n+1}+1\right)z-2\left(4^{n+1}-1\right)}{(4^{n+1}+2)-(4^{n+1}-1)z}$$

$$= \frac{2\left(4^{n+1}-1\right)-\left(2\cdot 4^{n+1}+1\right)z}{(4^{n+1}-1)z-(4^{n+1}+2)}$$

and therefore, $P(n+1)$ is also true. By Mathematical Induction, $P(n)$ is true for all $n \geq 1$.

Because formula (3) required the assumption that $f^n(z)$ exists for all $n \geq 1$, we need to determine if there are points $z \in C \setminus \{2\}$ for which there is a positive integer m such that

$f^n(z)$ does not exist for $n > m$. The existence of $f^n(z)$ requires that z, $f(z)$, $\ldots$, $f^{n-1}(z) \neq 2$. Therefore, we have to find all points z for which $f^m(z) = 2$ for some $m \geq 1$. One way to do this is to consider the inverse function

$$f^{-1}(z) = \frac{2z+2}{z+3}$$

and describe

$$f^{-m}(z) = \Big(\underbrace{f^{-1} \circ f^{-1} \circ \ldots \circ f^{-1}}_{m}\Big)(z)$$

in a manner similar to that used to find formula (3). If we do so, we see that for $z \neq -3$,

$$f^{-m}(z) = \frac{(4^m+2)\,z + 2\,(4^m-1)}{(4^m-1)\,z + 2\cdot 4^m + 1}.$$

In particular,

$$f^{-m}(2) = \frac{(4^m+2)\cdot 2 + 2\,(4^m-1)}{(4^m-1)\cdot 2 + 2\cdot 4^m + 1} = \frac{4^{m+1}+2}{4^{m+1}-1}.$$

If $z_m = \dfrac{4^{m+1}+2}{4^{m+1}-1}$ for some $m \geq 1$, then it follows that $f^m(z_m) = 2$ and hence, $f^n(z_m)$ is undefined for $n > m$. Therefore, $\lim\limits_{n\to+\infty} f^n(z_m)$ does not exist for these points.

Let

$$S = \{2\} \cup \left\{\frac{4^{m+1}+2}{4^{m+1}-1} : m \in N\right\}.$$

For $z \notin S$, $f^n(z)$ exists for all $n \geq 1$. If $z = 1$, then $z \notin S$ and (3) implies that

$$\begin{aligned} f^n(1) &= \frac{2\,(4^n-1) - (2\cdot 4^n+1)}{(4^n-1) - (4^n+2)} \\ &= \frac{-3}{-3} \\ &= 1 \end{aligned}$$

for all $n \geq 1$. Hence, $\lim\limits_{n\to+\infty} f^n(1) = 1$. For all other values of $z \notin S$,

$$\begin{aligned} \lim_{n\to+\infty} f^n(z) &= \lim_{n\to+\infty} \frac{2\,(4^n-1) - (2\cdot 4^n+1)\,z}{(4^n-1)\,z - (4^n+2)} \\ &= \lim_{n\to+\infty} \frac{2\,(1-4^{-n}) - (2+4^{-n})\,z}{(1-4^{-n})\,z - (1+2\cdot 4^{-n})} \\ &= \frac{2-2z}{z-1} = -2. \end{aligned}$$

Therefore, for $z \notin S$,

$$\lim_{n\to+\infty} f^n(z) = \begin{cases} 1 & \text{if } z = 1 \\ -2 & \text{otherwise} \end{cases}$$

Solution 2 by Henry Ricardo, Westchester Math Circle, NY

We take advantage of the well-known homomorphism between 2×2 matrices and Möbius transformations: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow f(z) = \dfrac{az+b}{cz+d}$. In this relation, the n-fold composition $f^n(z)$ corresponds to the nth power of A. Here we are dealing with powers of the matrix $A = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}$.

Now we invoke a known result that is a consequence of the Cayley-Hamilton theorem: If $A \in M_2(C)$ and the eigenvalues λ_1, λ_2 of A are not equal, then for all $n \geq 1$ we have

$$A^n = \lambda_1^n B + \lambda_2^n C, \text{ where } B = \frac{1}{\lambda_1 - \lambda_2}\left(A - \lambda_2 I_2\right) \text{ and } C = \frac{1}{\lambda_2 - \lambda_1}\left(A - \lambda_1 I_2\right). \; (*)$$

(See, for example, Theorem 2.25(a) in *Essential Linear Algebra with Applications* by T. Andreescu, Birkhäuser, 2014.)

The eigenvalues of the given matrix A are -1 and -4, so we apply $(*)$ to get

$$\begin{aligned} A^n &= \frac{(-1)^n}{3}(A + 4I_2) - \frac{(-4)^n}{3}(A + I_2) \\ &= \left(\frac{(-1)^n - (-4)^n}{3}\right) A + \left(\frac{4 \cdot (-1)^n - (-4)^n}{3}\right) I_2 \\ &= \begin{pmatrix} \frac{1}{3}(-1)^n(1 + 2 \cdot 4^n) & \frac{2}{3}(-1)^n + \frac{2}{3}(-1)^{n+1}4^n \\ \frac{1}{3}(-1)^n + \frac{1}{3}(-1)^{n+1}4^n & \frac{1}{3}(-1)^n(2 + 4^n) \end{pmatrix}. \end{aligned}$$

After some simplification, we see that

$$f^n(z) = \frac{(2 \cdot 4^n + 1)z - 2(4^n - 1)}{(1 - 4^n)z + (4^n + 2)}.$$

Finally, we note that $f^n(1) = 3/3 = 1$; and, for $z \neq 1$, we have

$$\lim_{n \to +\infty} f^n(z) = \lim_{n \to +\infty} \frac{(2 \cdot 4^n + 1)z - 2(4^n - 1)}{(1 - 4^n)z + (4^n + 2)} = \frac{2(z-1)}{1-z} = -2.$$

Therefore,

$$\lim_{n \to +\infty} f^n(z) = \begin{cases} 1 & \text{if } z = 1, \\ -2 & \text{if } z \neq 1 \end{cases}.$$

Solution 3 by David E. Manes, Oneonta, NY

We will show by induction that

$$f^{(n)}(z) = \frac{2 - \dfrac{2a_n + 1}{a_n} z}{z - \dfrac{a_n + 1}{a_n}}$$

where $a_n = \dfrac{4^n - 1}{3}$. If $n = 1$, then $a_1 = 1$ and $f^{(1)}(z) = \dfrac{(2 - 3z)}{(z - 2)} = f(z)$. Therefore, the result is true for $n = 1$. Assume the positive integer $n \geq 1$ and the given formula is valid for $f^{(n)}(z)$. Then

$$\begin{aligned}
f^{(n+1)}(z) &= f(f^{(n)}(z) = \frac{2-3\left(\dfrac{2-\dfrac{2a_n+1}{a_n}z}{z-\dfrac{a_n+1}{a_n}}\right)}{\left(\dfrac{2-\dfrac{2a_n+1}{a_n}z}{z-\dfrac{a_n+1}{a_n}}\right)-2} = \frac{2z-2\left(\dfrac{a_n+1}{a_n}\right)-6+3\left(\dfrac{2a_n+1}{a_n}\right)z}{2-\dfrac{2a_n+1}{a_n}z-2z+2\left(\dfrac{a_n+1}{a_n}\right)} \\
&= \frac{2a_nz-2a_n-2-6a_n+6a_nz+3z}{2a_n-2a_nz-z-2a_nz+2a_n+2} = \frac{-2-8a_n+(8a_n+3)z}{-(4a_n+1)z+(4n+2)} \\
&= \frac{2+8a_n-(8a_n+3)z}{(4a_n+1)z-(4n+2)} = \frac{2+8\left(\dfrac{4^n-1}{3}\right)-\left(8\left(\dfrac{4^n-1}{3}\right)+3\right)z}{\left(4\left(\dfrac{4^n-1}{3}\right)+1\right)z-\left(4\left(\dfrac{4^n-1}{3}\right)+2\right)} \\
&= \frac{(-2+2\cdot 4^{n+1})-(1+2\cdot 4^{n+1})z}{(4^{n+1}-1)z-(4^{n+1}+2)} \\
&= \frac{2-\left(\dfrac{2\cdot 4^{n+1}+1}{4^{n+1}-1}\right)z}{z-\left(\dfrac{4^{n+1}+2}{4^{n+1}-1}\right)} = \frac{2-\left(\dfrac{\frac{2\cdot 4^{n+1}+1}{3}}{\frac{4^{n+1}-1}{3}}\right)z}{z-\left(\dfrac{\frac{4^{n+1}+2}{3}}{\frac{4^{n+1}-1}{3}}\right)} \\
&= \frac{2-\left(\dfrac{2a_{n+1}+1}{a_{n+1}}\right)z}{z-\left(\dfrac{a_{n+1}+1}{a_{n+1}}\right)}
\end{aligned}$$

where $a_{n+1} = \dfrac{(4^{n+1}-1)}{3}$. Note that $\dfrac{4^{n+1}+2}{3} = \dfrac{4^{n+1}-1}{3}+1 = a_{n+1}+1$ and

$$\frac{2\cdot 4^{n+1}+1}{3} = \frac{2\cdot 4^{n+1}-2}{3}+1 = 2\Big(\frac{4^{n+1}-1}{3}\Big)+1 = 2a_{n+1}+1.$$

Hence, the result is true for the integer $n+1$ so that by the principle of mathematical induction the result is valid for all positive integers n.

For the limit question, note that if $f(z) = z$, then $z = 1$ or $z = -2$. Therefore, one of the fixed points of f is $z = 1$ so that $f^{(n)}(1) = 1$ for each positive integer n and $\lim_{n\to+\infty} f^{(n)}(1) = 1$. Moreover, observe that

$$\lim_{n\to+\infty}\frac{1}{a_n} = \lim_{n\to+\infty}\frac{3}{4^n-1} = 0.$$

Therefore, if $z \neq 1$, then

$$\lim_{n\to+\infty} f^{(n)}(z) = \lim_{n\to+\infty}\left(\frac{2-\dfrac{2a_n+1}{a_n}z}{z-\dfrac{a_n+1}{a_n}}\right) = \frac{\left(2-\lim\limits_{n\to+\infty}\left(2+\dfrac{1}{a_n}\right)z\right)}{\left(z-\lim\limits_{n\to+\infty}\left(1+\dfrac{1}{a_n}\right)\right)} = \frac{2-2z}{z-1} = -2.$$

Hence,

$$\lim_{n\to+\infty} f^{(n)}(z) = \begin{cases} 1, & \text{if z} = 1, \\ -2, & \text{if } z \neq 1. \end{cases}$$

Solution 4 by Jeremiah Bartz, University of North Dakota, Grand Forks, ND

Recall the map $f(z) = \dfrac{az+b}{cz+d} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ gives a group isomorphism between group of fractional linear transformations

$$\left\{ f : f(z) = \frac{az+b}{cz+d} \text{ where } a,b,c,d \in C \text{ and } ad-bc \neq 0 \right\}$$

under function composition and the group

$$GL(2,C) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in C \text{ and } ad-bc \neq 0 \right\}$$

under matrix multiplication.
To compute $f^n(z)$, let $M = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}$. Using induction, we show

$$M^n = \frac{(-1)^n}{3}\begin{bmatrix} 2^{2n+1}+1 & -2^{2n+1}+2 \\ -4^n+1 & 4^n+2 \end{bmatrix}.$$

Observe $M^1 = \dfrac{-1}{3}\begin{bmatrix} 2^3+1 & -2^3+2 \\ -3 & 6 \end{bmatrix} = \dfrac{-1}{3}\begin{bmatrix} 9 & -6 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}$.
Assume

$$M^n = \frac{(-1)^n}{3}\begin{bmatrix} 2^{2n+1}+1 & -2^{2n+1}+2 \\ -4^n+1 & 4^n+2 \end{bmatrix}$$

and observe

$$\begin{aligned} M^{n+1} &= M^n M \\ &= \frac{(-1)^n}{3}\begin{bmatrix} 2^{2n+1}+1 & -2^{2n+1}+2 \\ -4^n+1 & 4^n+2 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} \\ &= \frac{(-1)^n}{3}\begin{bmatrix} -3(2^{2n+1}+1)+(-2^{2n+1}+2) & 2(2^{2n+1}+1)-2(-2^{2n+1}+2) \\ -3(-4^n+1)+(4^n+2) & 2(-4^n+1)-2(4^n+2) \end{bmatrix} \\ &= \frac{(-1)^{n+1}}{3}\begin{bmatrix} 2^{2(n+1)+1}+1 & -2^{2(n+1)+1}+2 \\ -4^{n+1}+1 & 4^{n+1}+2 \end{bmatrix}. \end{aligned}$$

Using the aforementioned group isomorphism and simplifying, we conclude

$$f^n(z) = \frac{(2^{2n+1}+1)z-2^{2n+1}+2}{(-4^n+1)z+4^n+2} = \frac{(2\cdot 4^n+1)z+(2-2\cdot 4^n)}{(1-4^n)z+(2+4^n)}.$$

Notice that the map $f^n(z)$ is undefined for $z = \dfrac{4^k+2}{4^k-1}$ where $1 \le k \le n$. Consequently $\lim\limits_{n\to+\infty} f(z)$ does not exist for these values of z. Furthermore,

$$\begin{aligned}
\lim_{n\to+\infty} f^n(z) &= \lim_{n\to+\infty} \frac{(2\cdot 4^n+1)z+(2-2\cdot 4^n)}{(1-4^n)z+(2+4^n)} \\
&= \lim_{n\to+\infty} \frac{\left(2+\frac{1}{4^n}\right)z+\left(\frac{2}{4^n}-2\right)}{\left(\dfrac{1}{4^n}-1\right)z+\left(\dfrac{2}{4^n}+1\right)} \\
&= \frac{2z-2}{-z+1} \\
&= -2\left(\frac{1-z}{1-z}\right).
\end{aligned}$$

Note $f(1) = 1$ so $f^n(1) = 1$ for all $n \ge 1$. It follows that

$$\lim_{n\to+\infty} f(z) = \begin{cases} \text{DNE} & \text{if } z = \dfrac{4^n+2}{4^n-1} \text{ where } n \in Z_{>0} \\ 1 & \text{if } z = 1 \\ -2 & \text{otherwise.} \end{cases}$$

(DNE = does not exist)

Comment by Editor : **David Stone and John Hawkins of Georgia Southern University** stated the following in their solution: "The appearance of so many sums of powers of 4 prompts us to offer a candidate for the cutest representation of $f^{(n)}(z)$:

$$f^{(n)}(z) = \frac{(2\cdot 111\ldots\ldots 1_4+1)\,z-2\cdot 111\ldots 1_4}{-111\ldots\ldots 1_4 z+(111\ldots 1_4+1)},$$

where each of the base 4 repunits has $n-1$ digits."

Solution 5 by Toshihiro Shimizu, Kawasaki, Japan

Let $f^n(z) = \dfrac{a_n z+b_n}{c_n z+d_n}$. Then, we have

$$\begin{aligned}
\frac{a_{n+1}z+b_{n+1}}{c_{n+1}z+d_{n+1}} &= f^{n+1}(z) \\
&= f^n\left(\frac{2-3z}{z-2}\right) \\
&= \frac{(b_n-3a_n)\,z+2\,(a_n-b_n)}{(d_n-3c_n)\,z+2\,(c_n-d_n)}
\end{aligned}$$

Therefore, we have $a_{n+1} = b_n - 3a_n$, $b_{n+1} = 2a_n - 2b_n$ and $c_{n+1} = d_n - 3c_n$, $d_{n+1} = 2c_n - 2d_n$. Since $f^0(z) = z$, $a_0 = 1, b_0 = c_0 = 0$ and $d_0 = 1$. Since $b_n = a_{n+1} + 3a_n$, we have

$$\begin{aligned}
a_{n+2}+3a_{n+1} &= 2a_n - 2\,(a_{n+1}+3a_n) \\
a_{n+2}+5a_{n+1}+4a_n &= 0
\end{aligned}$$

and $a_1 = b_0 - 3a_0 = -3$. Thus, we have

$$\begin{aligned} a_n &= \frac{1}{3}(-1)^n + \frac{2}{3}(-4)^n \\ b_n &= a_{n+1} + 3a_n \\ &= \frac{1}{3}(-1)^{n+1} + \frac{2}{3}(-4)^{n+1} + (-1)^n + 2(-4)^n \\ &= \frac{2}{3}(-1)^n - \frac{2}{3}(-4)^n. \end{aligned}$$

Similarly, we have $c_{n+2} + 5c_{n+1} + 4c_n = 0$ and $c_1 = d_0 - 3c_0 = 1$. Thus, we have

$$\begin{aligned} c_n &= \frac{1}{3}(-1)^n - \frac{1}{3}(-4)^n \\ d_n &= c_{n+1} + 3c_n \\ &= \frac{2}{3}(-1)^n + \frac{1}{3}(-4)^n \end{aligned}$$

Therefore,

$$f^n(z) = \frac{((-1)^n + 2(-4)^n)z + (2(-1)^n - 2(-4)^n)}{((-1)^n - (-4)^n)z + (2(-1)^n + (-4)^n)}.$$

If $z \neq 1$, we have

$$\begin{aligned} f^n(z) &= \frac{\left(\left(\frac{1}{4}\right)^n + 2\right)z + \left(2\left(\frac{1}{4}\right)^n - 2\right)}{\left(\left(\frac{1}{4}\right)^n - 1\right)z + \left(2\left(\frac{1}{4}\right)^n + 1\right)} \\ &\to \frac{2z-2}{-z+1} \\ &= -2 \quad (n \to +\infty). \end{aligned}$$

If $z = 1$, the value of $f^n(z)$ is always 1 and its limit is also 1.

Solution 6 by Kee-Wai Lau, Hong Kong, China

It can easily be proved by induction that

$$f^n(z) = \frac{2(2^{2n}-1) - (2^{2n+1}+1)z}{(2^{2n}-1)z - 2(2^{2n-1}+1)},$$

whenever $z \notin S_n$, where $S_n = \{2\} \cup \left\{\dfrac{2(2^{2k-1}+1)}{2^{2k}-1} : k = 1, 2, 3, \cdots, n\right\}$.

Clearly, $\lim\limits_{n\to\infty} f^n(1) = 1$ and if $z \notin \mathbf{T}$, where $\mathbf{T} = \{1, 2\} \cup \left\{\dfrac{2(2^{\mathrm{2k-1}}+1}{2^{\mathrm{2k}}-1}, \mathrm{k} = 1, 2, 3\cdots\right\}$, then $\lim\limits_{n\to\infty} f^n(z) = -2$.

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Brian D. Beasley, Presbyterian College, Clinton, SC; Brian Bradie, Christopher Newport University, Newport News,VA; Bruno Salgueiro Fanego Viveiro, Spain; Ed Gray, Highland Beach, FL; Moti Levy (two solutions), Rehovot, Israel; Francisco Perdomo and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Trey Smith, Angelo State University, San Angelo, TX; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5438: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $k \geq 0$ be an integer and let $\alpha > 0$ be a real number. Prove that

$$\frac{x^{2k}}{(1-x^2)^\alpha} + \frac{y^{2k}}{(1-y^2)^\alpha} + \frac{z^{2k}}{(1-z^2)^\alpha} \geq \frac{x^k y^k}{(1-xy)^\alpha} + \frac{y^k z^k}{(1-yz)^\alpha} + \frac{x^k z^k}{(1-xz)^\alpha},$$

for $x, y, z \in (-1, 1)$.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We note that by the Binomial theorem,

$$\frac{t^{2k}}{(1-t^2)^\alpha} = t^{2k} \sum_{j=0}^{\infty} \binom{-\alpha}{j} \left(-t^2\right)^j = \sum_{j=0}^{\infty} \binom{-\alpha}{j} t^{2k+2j}, \ -1 < t < 1,$$

where $(-1)^j \binom{-\alpha}{j} = \dfrac{\alpha(\alpha+1)\cdots(\alpha+j-1)}{j!} > 0$ for all indices $j \geq 0$.

Therefore, by the AM$-$GM inequality,

$$\begin{aligned}
\frac{x^{2k}}{(1-x^2)^\alpha} + \frac{y^{2k}}{(1-y^2)^\alpha} + \frac{z^{2k}}{(1-z^2)^\alpha} &= \frac{1}{2} \sum_{cycl} \left(\frac{x^{2k}}{(1-x^2)^\alpha} + \frac{y^{2k}}{(1-y^2)^\alpha} \right) \\
&= \frac{1}{2} \sum_{cycl} \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha}{j} \left(x^{2k+2j} + y^{2k+2j} \right) \\
&\geq \sum_{cycl} \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha}{j} |xy|^{k+y} \\
&\geq \sum_{cycl} \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha}{j} (xy)^{k+y} \\
&= \sum_{cycl} \frac{(xy)^k}{(1-xy)^\alpha}, \quad \text{as claimed.}
\end{aligned}$$

Solution 2 by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC

It is well known that for any real numbers a, b, c

$$a^2 + b^2 + c^2 \geq ab + bc + ca. \qquad (1)$$

We show that $a, b \in (-1, 1)$

$$\sqrt{(1-a^2)(1-b^2)} \leq 1 - ab. \qquad (2)$$

Suppose that to the contrary $\sqrt{(1-a^2)(1-b^2)} > 1 - ab$, by squaring both sides of the inequality, we get $1 - a^2 - b^2 + a^2b^2 > 1 - 2ab + a^2b^2$, which implies that

$-a^2-b^2+2ab=-(a-b)^2>0$, which is impossible, that is, (2) is proved. From (2), we can conclude that

$$\frac{1}{\sqrt{(1-a^2)(1-b^2)}} \geq \frac{1}{1-ab}. \qquad (3)$$

Now, using (1) and (3), we write

$$\begin{aligned}
&\frac{x^{2k}}{(1-x^2)^\alpha}+\frac{y^{2k}}{(1-y^2)^\alpha}+\frac{z^{2k}}{(1-z^2)^\alpha} \\
\geq\ & \frac{x^k y^k}{((1-x^2)(1-y^2))^{\frac{\alpha}{2}}}+\frac{y^k z^k}{((1-y^2)(1-z^2))^{\frac{\alpha}{2}}}+\frac{z^k x^k}{((1-z^2)(1-x^2))^{\frac{\alpha}{2}}} \\
=\ & \frac{x^k y^k}{\left(\sqrt{(1-x^2)(1-y^2)}\right)^\alpha}+\frac{y^k z^k}{\left(\sqrt{(1-y^2)(1-z^2)}\right)^\alpha}+\frac{z^k x^k}{\left(\sqrt{(1-z^2)(1-x^2)}\right)^\alpha} \\
\geq\ & \frac{x^k y^k}{(1-xy)^\alpha}+\frac{y^k z^k}{(1-yz)^\alpha}+\frac{z^k x^k}{(1-zx)^\alpha}.
\end{aligned}$$

Solution 3 by Moti Levy, Rehovot, Israel

Since $\dfrac{|a|^k}{(1-|a|)^\alpha} \geq \dfrac{a^k}{(1-a)^\alpha},\ \ a\in(-1,1)$ then

$$\frac{|x|^k\,|y|^k}{(1-|x|\,|y|)^\alpha}+\frac{|y|^k\,|z|^k}{(1-|y|\,|z|)^\alpha}+\frac{|z|^k\,|x|^k}{(1-|z|\,|x|)^\alpha} \geq \frac{x^k y^k}{(1-xy)^\alpha}+\frac{y^k z^k}{(1-yz)^\alpha}+\frac{z^k x^k}{(1-zx)^\alpha}.$$

Therefore, we can assume that $x,y,z\in(0,1)$. Using the generalized binomial theorem,

$$\frac{1}{(1-u)^\alpha}=\sum_{n=0}^{\infty}\binom{n+\alpha-1}{n}u^n=\sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma(\alpha)}u^n, \qquad |u|<1.$$

$$\begin{aligned}
\frac{x^{2k}}{(1-x^2)^\alpha} &= \sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma(\alpha)}x^{2(n+k)} \\
\frac{x^k y^k}{(1-xy)^\alpha} &= \sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma(\alpha)}(xy)^{n+k}
\end{aligned}$$

By the inequality $a^2+b^2+c^2\geq ab+bc+ca, \quad a,b,c\geq 0$,

$$\left(x^{n+k}\right)^2+\left(y^{n+k}\right)^2+\left(z^{n+k}\right)^2 \geq x^{n+k}y^{n+k}+y^{n+k}z^{n+k}+z^{n+k}k^{n+k}.$$

$$
\begin{aligned}
&\frac{x^{2k}}{(1-x^2)^{\alpha}}+\frac{y^{2k}}{(1-y^2)^{\alpha}}+\frac{z^{2k}}{(1-z^2)^{\alpha}}\\
&=\sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma(\alpha)}x^{2(n+k)}+\sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma(\alpha)}y^{2(n+k)}+\ \sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma(\alpha)}z^{2(n+k)}\\
&=\sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma(\alpha)}\left(x^{2(n+k)}+y^{2(n+k)}+z^{2(n+k)}\right)\\
&\geq\sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma(\alpha)}\left(x^{n+k}y^{n+k}+y^{n+k}z^{n+k}+z^{n+k}k^{n+k}\right)\\
&=\sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma(\alpha)}xy^{(n+k)}y^{(n+k)}+\sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma(\alpha)}y^{(n+k)}z^{(n+k)}+\sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma(\alpha)}z^{(n+k)}k^{(n+k)}\\
&=\frac{x^ky^k}{(1-xy)^{\alpha}}+\frac{y^kz^k}{(1-yz)^{\alpha}}+\frac{z^kx^k}{(1-zx)^{\alpha}}.
\end{aligned}
$$

Solution 4 by Kee-Wai Lau, Hong Kong, China

We first note that

$$0<(1-x^2)(1-y^2)=(1-xy)^2-(x-y)^2\leq(1-x)^2.$$

Hence by the AM-GM inequality, we have

$$\frac{x^{2k}}{(1-x^2)^{\alpha}}+\frac{y^{2k}}{(1-y^2)^{\alpha}}\geq\frac{2|x^ky^k|}{\sqrt{(1-x^2)^{\alpha}(1-y^2)^{\alpha}}}\geq\frac{2|x^ky^k|}{(1-xy)^{\alpha}}.$$

Similarly,

$$
\begin{aligned}
\frac{y^{2k}}{(1-y^2)^{\alpha}}+\frac{z^{2k}}{(1-z^2)^{\alpha}} &\geq \frac{2|y^kz^k|}{(1-yz)^{\alpha}} \quad\text{and}\\
\frac{z^{2k}}{(1-z^2)^{\alpha}}+\frac{x^{2k}}{(1-x^2)^{\alpha}} &\geq \frac{2|z^kx^k|}{(1-zx)^{\alpha}}.
\end{aligned}
$$

Adding these inequalities, we easily deduce the inequality of the problem.

Also solved by Ed Gray, Highland Beach, FL; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania; Toshihiro Shimizu, Kawasaki, Japan, and the proposer.

Mea Culpa

For a variety of reasons, mostly caused by sloppy bookkeeping, those listed below were not credited for having solved the following problems, but should have been.

5427: Paul M. Harms, North Newton, KS.

5428: Ed Gray, Highland Beach, FL;
David Stone and John Hawkins, Georgia Southern University, Statesboro, GA.

5429: Brian D. Beasley, Presbyterian College, Clinton, SC.

5431: Albert Stadler, Herrliberg, Switzerland.