Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
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5547: *Proposed by Kenneth Korbin, New York, NY*

Given Heronian Triangle ABC with $\overline{AC} = 10201$ and $\overline{BC} = 10301$. Observe that the sum of the digits of $\overline{AC}$ is 4 and the sum of the digits of BC is 5. Find $\overline{AB}$ if the sum of its digits is 3.

(An Heronian Triangle is one whose side lengths and area are integers.)

5548: *Proposed by Michel Bataille, Reoun, France*

Given nonzero real numbers p and q, solve the system

$$\begin{cases} 2p^2x^3 - 2pqxy^2 - (2p-1)x = y \\ \\ 2q^2y^3 - 2pqx^2y + (2q+1)y = x \end{cases}$$

5549: *Proposed by Arkady Alt, San Jose, CA*

Let P be an arbitrary point in $\triangle\ ABC$ that has side lengths a, b, and c .

a) Find minimal value of

$$F(P) := \frac{a^2}{d_a(P)} + \frac{b^2}{d_b(P)} + \frac{c^2}{d_c(P)};$$

b) Prove the inequality $\dfrac{a^2}{d_a(P)} + \dfrac{b^2}{d_b(P)} + \dfrac{c^2}{d_c(P)} \geq 36r$, where r is the inradius.

5550: *Proposed by Ángel Plaza, University of the Las Palmas de Gran Canaria, Spain*

Prove that

$$\sum_{n=4}^{\infty}\sum_{k=2}^{n-2}\frac{1}{k\binom{n}{k}} = \frac{1}{2}.$$

5551: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $n \geq 2$ be positive real numbers. Prove that the following inequality holds:

$$1 + \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \frac{\left(\sqrt{\alpha_i \alpha_{j+1}} - \sqrt{\alpha_j \alpha_{i+1}}\right)^2}{\alpha_i \alpha_j} \leq \left(\frac{1}{n} \sum_{k=1}^{n} \left(\frac{\alpha_{k+1}}{\alpha_k} \right)^2 \right)^{1/2}$$

(Here the subscripts are taken modulo n.)

5552: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Find all differentiable functions $f : \Re \to \Re$ such that $f'(x) - f(-x) = e^x, \forall x \in \Re$, with $f(0) = 0$.

Solutions

5529: *Proposed by Kenneth Korbin, New York, NY*

Convex cyclic quadrilateral $ABCD$ has integer length sides and integer area. The distance from the incenter to the circumcenter is 91. Find the length of the sides.

Solution 1 by David E. Manes, Oneonta, NY

Let $ABCD$ be a bicentric quadrilateral with inradius r and circumradius R and side lengths $AB = a$, $BC = b$, $CD = c$ and $DA = d$. Then $a + c = b + d$ since the quadrilateral has an inscribed circle. Denote the diagonals $AC = p$ and $BD = q$. Finally, let $D = 2R$ represent the diameter of the circumscribed circle. If $x = 91$ denotes the distance between the incenter and the circumcenter, then Fuss' theorem gives a relation between r, R and $x = 91$; namely;

$$\frac{1}{(R-x)^2} + \frac{1}{(R+x)^2} = \frac{1}{r^2}.$$

Solving this equation for r, one obtains

$$r = \frac{R^2 - x^2}{\sqrt{2(R^2 + x^2)}} = \frac{R^2 - 91^2}{\sqrt{2(R^2 + 91^2)}}.$$

Substituting values for $R > 91$ in this equation, one quickly finds that if $R = 221$, then $r = 120$. Therefore,
$pq = 2r\left(r + \sqrt{4R^2 + r^2}\right) = 2(120)\left(120 + \sqrt{4 \cdot 221^2 + 120^2}\right) = 138720.$
Consider the quadrilateral with side lengths $a = AB = 170$, $b = BC = 408$, $c = CD = 408$ and $d = DA = 170$. Then $a + c = b + d = 578 = s$, the semi-perimeter of $ABCD$. Moreover,

$$a^2 + c^2 = 170^2 + 408^2 = b^2 + d^2 = D^2 = 442^2 = p^2;$$

hence, the quadrilateral is a kite. It consists of two congruent right triangles with a common hypotenuse, the diameter D of the circumscribed circle which is also the diagonal $p = AC$. For the given side lengths, note that the circumradius R is given by

$$R = \frac{1}{2}\sqrt{a^2 + c^2} = \frac{1}{2}\sqrt{170^2 + 408^2} = \frac{1}{2}\sqrt{b^2 + d^2} = 221.$$

and the inradius r is given by

$$r = \frac{pq}{2\sqrt{pq+4R^2}} = \frac{138720}{2\sqrt{138720+4(221)^2}} = 120.$$

Since the quadrilateral $ABCD$ is a kite, the two diagonals $p = AC$ and $q = BD$ are perpendicular so that $\sin\theta = 1$, where θ is the angle between p and q. Therefore, the following formulas for the area K of $ABCD$ all agree:

$$\begin{aligned} K &= \sqrt{(s-a)^2(s-c)^2} = \sqrt{(578-170)^2(578-408)^2} \\ &= \sqrt{abcd} = \sqrt{(ab)^2} = ab = 170 \cdot 408 \\ &= r\left(r + \sqrt{4R^2+r^2}\right)\sin\theta = 120\left(120 + \sqrt{4\cdot 221^2 + 120^2}\right) \\ &= 69360. \end{aligned}$$

Finally, the four sides a, b, c, d of a bicentric quadrilateral with inradius $r = 120$, circumradius $R = 221$ and semi-perimeter $s = 578$ are the four roots of the quartic equation

$$y^4 - 2sy^3 + \left(s^2 + 2r^2 + 2r\sqrt{4R^2+r^2}\right)y^2 - 2rs\left(\sqrt{4R^2+r^2}+r\right)y + r^2s^2 = 0.$$

Therefore,

$$y^4 - 1156y^3 + 472804y^2 - 80180160y + 4810809600 = 0,$$

$$(y-408)^2(y-170)^2 = 0.$$

Hence, the roots are 170 and 408, each of multiplicity two. This completes the solution.

Solution 2 by Ed Gray, Highland Beach, FL

We start with Fuss' Theorem which says: Given $R =$ circumradius, $r =$ inradius,, $x =$ distance between the incenter and the circumcenter, then:

1. $$\frac{1}{(R+x)^2} + \frac{1}{(R-x)^2} = \frac{1}{r^2}$$

2. $$\frac{(R-x)^2 + (R+x)^2}{(R+x)^2\cdot(R-x)^2} = \frac{1}{r^2}$$

3. $$\frac{R^2 - 2Rx + x^2 + R^2 + 2Rx + x^2}{(R^2-x^2)^2} = \frac{1}{r^2}$$

4. $$2r^2\cdot(R^2+x^2) = (R^2-x^2)^2$$

5. $$2\cdot r^2\cdot R^2 + 2\cdot r^2\cdot x^2 = R^4 - 2\cdot R^2\cdot x^2 + x^4$$

   Writing (5) as a quadratic in R^2,

6. $R^4 - (2\cdot r^2 + 2\cdot x^2)R^2 + x^4 - 2\cdot r^2\cdot x^2 = 0,$ with solution

7. $2R^2 = 2r^2 + 2x^2 + \sqrt{4r^4 + 8 \cdot r^2 \cdot x^2 + 4x^4 - 4(x^4 - 2 \cdot r^2 \cdot x^2)}$

The + sign is used to ensure$R \geq r\sqrt{2}$.

8. $2R^2 = 2(r^2 + x^2) + \sqrt{4r^4 + 16 \cdot r^2 \cdot x^2}$, and

9. $R^2 = r^2 + x^2 + r\sqrt{r^2 + 4x^2}$

Letting $x = 91$, consider the discriminant:

10. $D^2 = r^2 + 33124$

11. $(D - r)(D + r) = 2^2 \cdot 7^2 \cdot 13^2$

$(D - r)$ and $(D + r)$ must have the same parity since their sum is even. Since their product is even, each factor is even. $D - r$ must be less than $2 \cdot 7 \cdot 13, D + r$ must be greater than $2 \cdot 7 \cdot 13$. We try for a solution assuming that r is an integer. The possible values for $D - r$ are, $2, 14, 26, 98$. Since $x = 91$, the disparity between r and R cannot be exceedingly large. Accordingly, we start with the largest value for $D - r$.

12. $D - r = 98$

13. $D + r = 338$

14. $2D = 436, D = 218, r = 120$. Substituting these values into (9),

15. $R^2 = 14400 + 8281 + 120 \cdot 218 = 48841$. Then:

16. $R = 221$.

To get an idea of the character of the sides, we co-ordinate the quantities in a Cartesian coordinate system. For convenience, we put the circumcenter, O, at the origin, $(0, 0)$. The incenter, I, will be on the positive $y-$axis and have coordinates $(0, 91)$. With $r = 120$, notice that the incircle has its extreme point on the $y-$ axis with coordinates $(0, 211)$. The upper extreme for the circumcenter is $(0, 221)$, so that they only differ by 10. The lower extreme for the incircle has coordinates $(0, -29)$. Clearly, picturing the sides shows the quadrilateral must have two long sides for the lower two, and two much shorter sides for the upper two, suggesting a kite-like shape for the quadrilateral. In fact, we will pursue this concept, placing vertex A at $(0, 221)$, vertex C at $(0, -221)$. Vertex B will have $x > 0, y > 0$, Vertex D will have $x < 0, y > 0$. We have $AB = AD, BC + DC$, and, of course, as in all bi-centric quadrilaterals, $AB + CD = BD + CB$.

Now consider the side AB. It is tangent to the incircle at point T, so that IT is perpendicular to AB. Triangle AIT is a right triangle, with hypotenuse $AI = 130$, leg $IT = r = 120$.
So that $AT = 50$. Let $\angle TAI = t$. We note that $\cos(t) = 5/13, \sin(t) = 12/13$. The equation of side AB is $y = mx + b$, where
$m = \tan(t - 90) = \dfrac{\sin(t - 90)}{\cos(t - 90)} = \dfrac{(\sin(t) \cdot \cos(90) - \cos(t) \cdot \sin(90)}{\cos(t) \cdot \cos(90) + \sin(t) \cdot \sin(90)} = \dfrac{(-5/13)}{(12/13)} = -\dfrac{5}{12}$.
When $x = 0, y = 221$, so the equation of the chord AB is: 17. $y = 2215x/12$. The equation of the circumcircle is: 18. $x^2 + y^2 = 48841$.

The coordinates of vertex B can be found by solving (17), (18) simultaneously.
19. $x^2 + (215x/12)^2 = 48841$
20. $x^2 + 488412210 \cdot x/12 + (25x^2)/144 = 48841$
21. $x^2(1 + 25/144) = 2210x/12$
22. $(169/144)x = 2210/12$
23. $x = (2210/12) \cdot (144/169) = (12) \cdot (13.07692308) = 156.9230769$
24. $y = 2215(156.9230769)/12 = 22165.38461538 = 155.6153846$
25. The coordinates of vertex $B = (156.9230769, 155.6153846)$
Using the distance formula, we can compute the length of side AB.
26. $AB = \sqrt{(156.92307690)^2 + (221155.6153846)^2}$
27. $AB = \sqrt{24624.85206 + 4275.147931}$
28. $AB = \sqrt{28900)} = 170.$
29. We can now compute the length of chord BC by the Law of Cosines, using $\triangle ABC$.
We have: $(BC)^2 = 170^2 + 442^2 2 \cdot 170 \cdot 442 \cdot (5/13)$ 30.
$(BC)^2 = 28900 + 19536457800 = 166464$
31. $BC = 408.$
The sides appear to be $170, 170, 408, 408$. As noted, integer area did not come into play, Explicitly. We show that, indeed, the area is an integer by using Brahmaguptas formula:
32. $A = \sqrt{(s-a)(s-b)(s-c)(s-d)} = \sqrt{408 \cdot 408 \cdot 170 \cdot 170} = 408 \cdot 170 = 69,360.$
33. As a check, $r = A/s = 69,369/578 = 120.$

Solution 3 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

For clarity we label the lengths of the sides of quadrilateral $ABCD$ so AB has length a, BC has length b, CD has length c, and DA has length d.

We will show that a quadrilateral with sides $a = d = 408$ and $b = c = 170$ is a cyclic quadrilateral with integer area and distance from its incenter to its circumcenter is 91.

We were not able to show that this is the only such quadrilateral.

Because our quadrilateral is given to have an incenter, there must be an inscribed circle, tangent to all four sides (hence, known as a tangential quadrilateral).

Such a cyclic, tangential, quadrilateral is termed a bicentric quadrilateral. (Wikipedia: *https://en.wikipedia.org/wiki/Bicentric_quadrilateral*).
FromWikipedia(URL *https : //en.wikipedia.org/wiki/Pitot_theorem*), we find the following:

The **Pitot theorem,** named after the French engineer Henri Pitot, states that in a tangential quadrilateral the two sums of lengths of opposite sides are the same. Both sums of lengths equal the semiperimeter of the quadrilateral.

A convex quadrilateral $ABCD$ with sides a, b, c, d is bicentric if and only if opposite sides satisfy Pitot's theorem for tangential quadrilaterals and the cyclic quadrilateral property that opposite angles are supplementary; that is,
opposite sides equal: $a + c = b + d$
opposite angles are supplementary: $A + B = C + D = \pi$.

For a bicentric quadrilateral, Fuss' Theorem gives a relation between the inradius r, the circumradius R and the distance x between the incenter and the circumcenter:

$$\frac{1}{(R-x)^2} + \frac{1}{(R+x)^2} = \frac{1}{r^2}$$

Some relevant facts about a bicentric quadrilateral:
(1) the area is given by $A = \sqrt{abcd}$
(2) the inradius is given by $r = \frac{A}{s} = \frac{\sqrt{abcd}}{s} = \frac{\sqrt{abcd}}{a+c} = \frac{\sqrt{abcd}}{b+d}$.

By (2), the inradius of our quadrilateral must be rational. There are no *a priori* restrictions on the circumradius R.

However, well first look for integer values for r and R.

Substituting our known value, $x = 91$, into Fuss' Theorem and solving for r yields
$r = \frac{R^2 - 91^2}{\sqrt{(R+91)^2 + (R-91)^2}} = \frac{R^2 - 91^2}{\sqrt{2(R^2 + 91^2)}}$.

At the worst, the quantity inside the radical must be the square of a rational; well impose the condition that it be the square of an integer:
$2(R^2 + 91^2) = z^2$ so $r = \frac{R^2 - 91^2}{z}$.

Thus, z must be even; say $z = 2w$:
$2(R^2 + 91^2) = (2w)^2 = 4w^2$.
$\mathrm{R}^2 + 91^2 = 2w^2$
(3) $R^2 - 2w^2 = -91^2$.

This is a Pell-like equation. With some initial assistance from Excel, we can find infinitely many solutions in integers. Because R must be larger than 91, the smallest valid solution is $R = 221, w = 169, soz = 338$.
This yields an integer value for $r : r = 120$.

Now the fun begins we must find values for a, b, c, d.

We want
$120 = \frac{\sqrt{abcd}}{a+c} = \frac{abcd}{b+d}$,
$120^2(a+c)^2 = abcd$ and $120^2(b+d)^2 = abcd$.

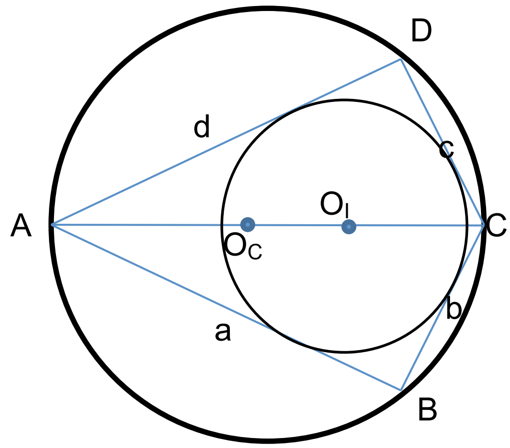
Using the prime factorization of 120 and applying some ingenuity, we find that the values $a = d = 408$ and $b = c = 170$. satisfy the conditions.
This would make our quadrilateral a (convex) kite, which is automatically a tangential quadrilateral.

However, the lengths of the sides by themselves do not completely specify a quadrilateral. We must proscribe its shape.

Noting that
$a = d = 408 = 34 \cdot 12$
and
$b = c = 170 = 34 \cdot 5$, we build the quadrilateral so the principal diagonal
$AC = 442 = 34 \cdot 13$.

This forces ABC and ADC to be right triangles (scaled-up copies of the 5-12-13 triangle), with AC being a diameter of the circumcircle.

Hence, our quadrilateral is inscribed in a circle, hence is cyclic and bicentric, as required. The difference x between the incenter and the circumcenter must equal 91 by our derivation of R, r.



In fact, the only kite which is cyclic is one formed by two congruent right triangles joined along the hypotenuse (= the diameter). Its sometimes known as a *right kite.*

Comment: We make no claim that our solution is unique. For instance, even after R and r were determined, the conditions $r^2(a+c)^2 = abcd$ and $r^2(b+d)^2 = abcd$ could admit other solutions (although a computer search found none).

Moreover, the Pell equation $R^2 - 2w^2 = -91^2$ has infinitely many solutions.
Using $\begin{pmatrix} R_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} 221 \\ 169 \end{pmatrix}$ or $\begin{pmatrix} 299 \\ 221 \end{pmatrix}$ or $\begin{pmatrix} 637 \\ 455 \end{pmatrix}$ as a base, we can generate infinitely many more solutions by the recursive scheme
$\begin{pmatrix} R_{k+1} \\ w_{k+1} \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} R_k \\ w_k \end{pmatrix}$.

Of the solutions that we have checked, each produces a rational, non-integer value for the inradius r (which is acceptable but makes it much more difficult to find a, b, c, d). So there could be many other solutions to the problem.

Also solved by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece and the proposer.

5530: *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

Polygon $ABCD$ is an 11 by 12 rectangle ($AB > AD$). Points P, Q, R, and S are on sides AB, BC, CD, and DA, respectively, such that PR and SQ are parallel to AD and AB, respectively. Moreover, $X = PR \cap QS$. If the perimeter of rectangle $PBQX$ is $5/7$ the perimeter of rectangle $SAPX$, and the perimeter of rectangle $RCQX$ is $9/10$ the perimeter of rectangle $PBQX$, find the area of rectangle $SDRX$.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

Let $u = SD$ and $v = DR$. Then $AS = AD - SD = 11 - u$ and $RC = DC - DR = 12 - v$. Since $BQ = AS$, $PB = RC$, and $QC = SD$, the perimeter of rectangles $PBQX$, $SAPX$, and $RCQX$ are, respectively,

$2(PB+BQ)=2(12-v+11-u),\ 2(AS+AP)=2(11-u+v)$, and
$2(RC+QC)=2(12-v+u)$.

Hence,

$$2(12-v+11-u)=\left(\frac{5}{7}\right)2(11-u+v),\ \text{and } 2(12-v+u)=\left(\frac{9}{10}\right)2(12-v+11-u),$$

which implies $(u,v)=(5,8)$, so the area of rectangle $SDRX$ is $SD\cdot DR=uv=40$.

Solution 2 by David A. Huckaby, Angelo State University, San Angelo, TX

From the given dimensions of rectangle $ABCD$, we have $PX+RX=11$ and $QX+SX=12$. Since the perimeter of rectangle $PBQX$ is $\frac{5}{7}$ the perimeter of rectangle $SAPX$, we have $BQ+PX+BP+QX=\frac{5}{7}(AP+SX+AS+PX)$, that is, $2PX+2QX=\frac{5}{7}(2SX+2PX)$ or $PX+QX=\frac{5}{7}(SX+PX)$. Similarly, since the perimeter of rectangle $RCQX$ is $\frac{9}{10}$ the perimeter of rectangle $PBQX$, we have $QX+RX=\frac{9}{10}(QX+PX)$.

So we have the following system of four equations in four unknowns:

$$\begin{cases} PX & & +RX & & =11 \\ & QX & & +SX & =12 \\ \frac{2}{7}PX & +QX & & -\frac{5}{7}SX & =0 \\ -\frac{9}{10}PX & +\frac{1}{10}QX & +RX & & =0 \end{cases}$$

Solving this systems yields $PX=6$, $QX=4$, $RX=5$, and $SX=8$, whence the area of rectangle $SDRX$ is $(RX)(SX)=(5)(8)=40$.

Also solved by Ashland University Undergraduate Problem Solving Group, Ashland, Ohio; Michel Bataille, Rouen, France; Ed Gray, Highland Beach, FL; David E. Manes, Oneonta, NY; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.)

5531: *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Drobeta Turnu-Severin, Mehedinti, Romania*

For real numbers x,y,z prove that if $x,y,z>1$ and $xyz=2\sqrt{2}$, then

$$x^y+y^z+z^x+y^x+z^y+x^z>9.$$

Solution 1 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

If $f(t)=t\ln\left(\dfrac{2\sqrt{2}}{t}\right)$, with $t>1$, the $f'(t)=\ln\left(\dfrac{2\sqrt{2}}{t}\right)-1$, and for $f'(t_k)=0$, we have $t_k=\dfrac{2\sqrt{2}}{e}>1$. So, we have: $f(t)\ge f\left(\dfrac{2\sqrt{2}}{e}\right)=\dfrac{2\sqrt{2}}{e}\ln\left(2\sqrt{2}\cdot\dfrac{e}{2\sqrt{2}}\right)=\dfrac{2\sqrt{2}}{e}$.
Furthermore, we have:

$$x^y \geq 1 + y\ln x, \quad y^x \geq 1 + x\ln y, \quad y^x \geq 1 + z\ln y,$$

$$z^y \geq 1 + y\ln z, \quad z^x \geq 1 + x\ln z, \quad x^z \geq 1 + z\ln z,$$

So, we have:

$$\begin{aligned} x^y + y^z + z^x + y^x + z^y + x^z &\geq 6 + x\ln(yz) + y\ln(xz) + z\ln(xy) \\ &= 6 + x\ln\left(\frac{2\sqrt{2}}{x}\right) + y\ln\left(\frac{2\sqrt{2}}{y}\right) + z\ln\left(\frac{2\sqrt{2}}{z}\right) \\ &\geq 6 + 3\cdot\frac{2\sqrt{2}}{e} > 9. \end{aligned}$$

Solution 2 by Adrian Naco, Polytechnic University of Tirana, Albania

Since, $x > 1, y > 1$, and using the Bernoulli inequality, we have that

$$x^y = [1 + (x-1)]^y > 1 + y(x-1). \qquad (2)$$

Acting analogously it implies that,

$$x^y + y^z + z^x + y^x + z^y + x^z > 6 + 2(xy + yz + zx) - 2(x + y + z). \qquad (3)$$

To prove the given inequality (1), it is enough to prove the following equivalent inequalities,

$$6 + 2(xy + yz + zx) - 2(x + y + z) > 9 \quad \text{or equivalently} \quad (xy + yz + zx) - (x + y + z) > \frac{3}{2}$$

Let

$$f(x,y,z) = (xy + yz + zx) - (x + y + z) - \frac{3}{2}. \qquad g(x,y,z) = xyz - 2\sqrt{2}$$

and using Langrange Multipliers method, we have that,

$$F(x,y,z) = f(x,y,z) - \lambda g(x,y,z) = (xy + yz + zx) - (x + y + z) - \frac{3}{2} - \lambda(xyz - 2\sqrt{2}).$$

$$F_x = y + z - 1 - \lambda yz = 0$$

$$F_y = x + z - 1 - \lambda xz = 0$$

$$F_z = x + y - 1 - \lambda xy = 0$$

$$F_\lambda = -xyz + 2\sqrt{2} = 0$$

Subtracting side by side, each couple of the last three first equations, we get the following:

$$(z-1)(x-y) = 0$$

$$(y-1)(x-z) = 0$$

$$(x-1)(z-y) = 0$$

$$xyz = 2\sqrt{2}$$

So, $x = y = z = \sqrt{2}$, is the only solution (since $x > 1, y > 1, z > 1$). Finally,

$$\min f(x,y,z) = f(\sqrt{2};\sqrt{2};\sqrt{2}) = 2+2+2-3\sqrt{2}-\frac{3}{2} = \frac{3}{2}(3-2\sqrt{2}) > 0.$$

Note. Even if we consider the case when $x = 1$, we have that,

$$f(1,y,z) = y+yz+z-1-z-y-\frac{3}{2} = 2\sqrt{2})-\frac{5}{2} > f(\sqrt{2};\sqrt{2};\sqrt{2}) = \frac{3}{2}(3-2\sqrt{2}) > 0.$$

Solution 3 by Moti Levy, Rehovot, Israel

Let $f(u,v) := u^v + v^u, \quad u,v > 1$. By verifying that the Hessian of $f(u,v)$ is positive semi-definite, it becomes evident that $f(u,v)$ is convex function in the domain $u,v > 1$.

$$Hess\left(u^v+v^u\right) = \begin{bmatrix} u^{v-2}(v-1)v + v^u \ln^2 v & u^{v-1}+v^{u-1}+vu^{v-1}\ln u + uv^{u-1}\ln v \\ u^{v-1}+v^{u-1}+vu^{v-1}\ln u + uv^{u-1}\ln v & (u-1)uv^{u-2}+u^v\ln^2 v \end{bmatrix}$$

Then by Jensen's inequality

$$\begin{aligned} & x^y + y^z + z^x + y^x + z^y + x^z \qquad (1) \\ & = f(x,y) + f(y,z) + f(z,x) \geq 3f\left(\frac{x+y+z}{3}, \frac{y+z+x}{3}\right) \end{aligned}$$

By AM-GM inequality,

$$xyz = 2\sqrt{2} \quad \Longrightarrow \quad \frac{x+y+z}{3} \geq \sqrt[3]{2\sqrt{2}} = \sqrt{2}. \qquad (2)$$

Inequalities (1) and (2) imply the required result,

$$x^y + y^z + z^x + y^x + z^y + x^z \geq 3f\left(\sqrt{2},\sqrt{2}\right) = 6\left(\sqrt{2}\right)^{\sqrt{2}} > 9.$$

Also solved by Khaled Abd Imouti, Zaki Al Arzousi School, Damascus, Syria, (communicated to SSM by Daniel Sitaru of Romania); Michael Brozinsky, Central Islip, NY; Ed Gray, Highland Beach, FL; Tran Hong (student), Cao Lang School, Dong Thap, Vietnam (communicated to SSM by Daniel Sitaru of Romania) and the proposer.

5532: *Proposed by Arkady Alt, San Jose, CA*

Let a, b, c be positive real numbers and let $a_n = \dfrac{an+b}{an+c}, n \in N$. For any natural number m find $\displaystyle\lim_{n\to\infty} \prod_{k=n}^{nm} a_k$.

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA

For large n,

$$a_n = \frac{an+b}{an+c} = \frac{1+\frac{b}{an}}{1+\frac{c}{an}} \sim 1 + \frac{b-c}{an}.$$

Thus,

$$\ln \prod_{k=n}^{mn} a_k = \sum_{k=n}^{mn} \ln a_k \sim \sum_{k=n}^{mn} \frac{b-c}{an} = \frac{b-c}{a}\left(H_{mn} - H_{n-1}\right),$$

where H_n denotes the nth Harmonic number. Now,

$$H_n \sim \ln n + \gamma,$$

where γ is the Euler-Mascheroni constant, so

$$H_{mn} - H_{n-1} \sim \ln \frac{mn}{n-1}$$

and

$$\ln \prod_{k=n}^{mn} a_k \sim \frac{b-c}{a} \ln \frac{mn}{n-1}.$$

Thus,

$$\lim_{n\to\infty} \ln \prod_{k=n}^{mn} a_k = \frac{b-c}{a} \ln m,$$

and

$$\lim_{n\to\infty} \prod_{k=n}^{mn} a_k = \exp\left(\frac{b-c}{a}\ln m\right) = m^{(b-c)/a}.$$

Solution 2 by Moti Levy, Rehovot, Israel

We rewrite the product as

$$\prod_{k=n}^{mn} a_k = \prod_{k=n}^{mn}\left(1 + \frac{\alpha}{k+\beta}\right), \qquad \alpha = \frac{b-c}{a}, \quad \beta = \frac{c}{a}.$$

$$\ln \prod_{k=n}^{mn} a_k = \sum_{k=n}^{mn} \ln\left(1 + \frac{\alpha}{k+\beta}\right) = \sum_{k=n}^{mn}\left(\frac{\alpha}{k+\beta} + O\left(\frac{1}{k^2}\right)\right) = \sum_{k=n}^{mn}\left(\frac{\alpha}{k} + O\left(\frac{1}{k^2}\right)\right)$$

$$\lim_{n\to\infty} \ln \prod_{k=n}^{mn} a_k = \lim_{n\to\infty} \sum_{k=n}^{mn} \frac{\alpha}{k} = \alpha \lim_{n\to\infty} \sum_{k=n}^{mn} \frac{1}{k}$$

$$\sum_{k=n}^{mn} \frac{1}{k} = \frac{1}{n} \sum_{k=0}^{(m-1)n} \frac{1}{1+\frac{k}{n}}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{(m-1)n} \frac{1}{1+\frac{k}{n}} = \int_0^{m-1} \frac{1}{1+x}dx = \ln m.$$

$$\lim_{n\to\infty} \ln \prod_{k=n}^{mn} a_k = \alpha \ln m,$$

hence

$$\prod_{k=n}^{mn} a_k = m^{\alpha} = m^{\frac{b-c}{a}}.$$

Solution 3 by Albert Stadler, Herrliberg, Switzerland

The Euler gamma function $\Gamma(x)$ satisfies the functional equation $\Gamma(x+1) = x\Gamma(x)$. Therefore

$$\prod_{k=n}^{mn} a_k = \prod_{k=n}^{mn} \frac{k+\frac{b}{a}}{k+\frac{c}{a}} = \frac{\Gamma\left(mn+1+\frac{b}{a}\right)}{\Gamma\left(n+\frac{b}{a}\right)} \cdot \frac{\Gamma\left(n+\frac{b}{a}\right)}{\Gamma\left(mn+1+\frac{c}{a}\right).}$$

Stirling's asymptotic formula for the Euler gamma function states that $\Gamma(x) = \sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x\left(1+O\left(\frac{1}{x}\right)\right)$, as $x \to \infty$. So,

$$\prod_{k=n}^{mn} a_k \sim \frac{\sqrt{\dfrac{2\pi}{mn+1+\dfrac{b}{a}}}\left(\dfrac{mn+1+\dfrac{b}{a}}{e}\right)^{mn+1+\frac{b}{a}}}{\sqrt{\dfrac{2\pi}{n+\dfrac{b}{a}}}\left(\dfrac{n+\dfrac{b}{a}}{e}\right)^{n+\frac{b}{a}}} \cdot \frac{\sqrt{\dfrac{2\pi}{n+\dfrac{c}{a}}}\left(\dfrac{n+\dfrac{c}{a}}{e}\right)^{n+\frac{c}{a}}}{\sqrt{\dfrac{2\pi}{mn+1+\dfrac{c}{a}}}\left(\dfrac{mn+1+\dfrac{c}{a}}{e}\right)^{mn+1+\frac{c}{a}}} \sim$$

$$\sim \frac{\left(mn+1+\dfrac{b}{a}\right)^{\frac{b}{a}}}{\left(n+\dfrac{b}{a}\right)^{\frac{b}{a}}} \cdot \frac{\left(n+1+\dfrac{c}{a}\right)^{\frac{c}{a}}}{\left(mn+1+\dfrac{c}{a}\right)^{\frac{c}{a}}} \sim m^{\frac{b-c}{a}} \text{ as } n \to \infty.$$

Solution 4 by Michel Bataille, Rouen, France

We show that the required limit is $m^{(b-c)/a}$.

We shall use the following well-known result about the Gamma function: if s is a positive real number, then

$$\lim_{n\to\infty} \frac{n! \cdot n^s}{s(s+1)(s+2)\cdots(s+n)} = \Gamma(s).$$

For $n \geq 2$, we have

$$\prod_{k=n}^{nm}(ak+b) = a^{nm-n+1}\prod_{k=n}^{nm}\left(\frac{b}{a}+k\right) = a^{nm-n+1} \cdot \frac{\prod_{k=0}^{nm}\left(\frac{b}{a}+k\right)}{\prod_{k=0}^{n-1}\left(\frac{b}{a}+k\right)}$$

so that, as $n \to \infty$,

$$\prod_{k=n}^{nm}(ak+b) \ \sim \ a^{nm-n+1} \cdot \frac{(nm)!(nm)^{b/a}}{\Gamma(b/a)} \cdot \frac{\Gamma(b/a)}{(n-1)!(n-1)^{b/a}} = K_{m,n} \cdot \left(\frac{nm}{n-1}\right)^{b/a}$$

where $K_{m,n} = a^{nm-n+1} \cdot \frac{(nm)!}{(n-1)!}$.

Similarly, $\prod_{k=n}^{nm}(ak+c) \ \sim \ K_{m,n} \cdot \left(\frac{nm}{n-1}\right)^{c/a}$ and it follows that

$$\prod_{k=n}^{nm} a_k \ \sim \ \left(\frac{nm}{n-1}\right)^{(b-c)/a}.$$

Since $\lim\limits_{n\to\infty} \frac{nm}{n-1} = m$, we obtain that $\lim\limits_{n\to\infty} \prod\limits_{k=n}^{nm} a_k = m^{(b-c)/a}$.

Solution 5 by Kee-Wai Lau, Hong Kong, China

We have $\ln a_k = \ln\left(1 + \dfrac{b-c}{ak+c}\right) = \dfrac{b-c}{ak+c} + O\left(\dfrac{1}{k^2}\right)$ as $k \to \infty$, where the constant implied by O depends at most on a, b, c. Hence

$$\sum_{k=n}^{mn} \ln a_k = (b-c)\sum_{k=n}^{nm} \frac{1}{ak+c} + O\left(\frac{1}{n}\right).$$

For $x > 0$, let $f(x)$ be the decreasing function $\dfrac{1}{ax+c}$ so that

$$\frac{1}{a}\ln\left(\frac{anm+c}{an+c}\right) = \int_n^{nm} \frac{dx}{ax+c} < \sum_{k=n}^{nm} \frac{1}{ak+c} < \int_{n-1}^{nm} \frac{dx}{ax+c} = \frac{1}{a}\ln\left(\frac{anm+c}{an+c-a}\right).$$

It follows that $\lim\limits_{n\to\infty} \sum\limits_{k=n}^{nm} \dfrac{1}{ak+c} = \dfrac{\ln m}{a}$. Thus

$$\lim_{n\to\infty} \prod_{ki=n}^{nm} a_k = e^{\lim\limits_{n\to\infty} \sum\limits_{k=n}^{nm} \frac{1}{ak+c}} = m^{(b-c)/a}.$$

Also solved by Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News,VA; and the proposer.

5533: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Find the value of the sum

$$\sum_{n=1}^{+\infty} \frac{n^2\alpha^n}{(n-1)!}$$

for any real number $\alpha > 0$. (Here, $0! = 1! = 1$).

Solution 1 by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith, Angelo State University, San Angelo, TX

The solution is

$$\left(\alpha^3 + 3\alpha^2 + \alpha\right)e^{\alpha} = \sum_{n=1}^{\infty} \frac{n^2\alpha^n}{(n-1)!}$$

for all real α. To avoid encountering the disputed expression 0^0 in our work, we note first that for $\alpha = 0$,

$$\left(\alpha^3 + 3\alpha^2 + \alpha\right)e^{\alpha} = 0 = \sum_{n=1}^{\infty} \frac{n^2\alpha^n}{(n-1)!}.$$

For $\alpha \neq 0$, we proceed as follows. Since

$$e^{\alpha} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!},$$

we have

$$\alpha e^{\alpha} = \sum_{n=0}^{\infty} \frac{\alpha^{n+1}}{n!} = \sum_{n=1}^{\infty} \frac{\alpha^{n}}{(n-1)!}.$$

Then, if we differentiate with respect to α, we obtain

$$(\alpha+1)\, e^{\alpha} = \sum_{n=1}^{\infty} \frac{n\alpha^{n-1}}{(n-1)!}$$

and hence,

$$\left(\alpha^2+\alpha\right) e^{\alpha} = \sum_{n=1}^{\infty} \frac{n\alpha^{n}}{(n-1)!}.$$

Differentiate again with respect to α to get

$$\left(\alpha^2+3\alpha+1\right) e^{\alpha} = \sum_{n=1}^{\infty} \frac{n^2\alpha^{n-1}}{(n-1)!}$$

and therefore,

$$\left(\alpha^3+3\alpha^2+\alpha\right) e^{\alpha} = \sum_{n=1}^{\infty} \frac{n^2\alpha^{n}}{(n-1)!}.$$

Comment: Once we know the answer, we can verify this result directly as follows. As noted above, when $\alpha = 0$,

$$\left(\alpha^3+3\alpha^2+\alpha\right) e^{\alpha} = 0 = \sum_{n=1}^{\infty} \frac{n^2\alpha^{n}}{(n-1)!}.$$

For $\alpha \neq 0$,

$$\begin{aligned}
\left(\alpha^3+3\alpha^2+\alpha\right) e^{\alpha} &= \left(\alpha^3+3\alpha^2+\alpha\right) \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{(n-1)!} \\
&= \sum_{n=1}^{\infty} \frac{\alpha^{n+2}}{(n-1)!} + \sum_{n=1}^{\infty} \frac{3\alpha^{n+1}}{(n-1)!} + \sum_{n=1}^{\infty} \frac{\alpha^{n}}{(n-1)!} \\
&= \sum_{n=3}^{\infty} \frac{\alpha^{n}}{(n-3)!} + \sum_{n=2}^{\infty} \frac{3\alpha^{n}}{(n-2)!} + \sum_{n=1}^{\infty} \frac{\alpha^{n}}{(n-1)!} \\
&= 3\alpha^2 + \left(\alpha+\alpha^2\right) + \sum_{n=3}^{\infty} \left[\frac{1}{(n-3)!} + \frac{3}{(n-2)!} + \frac{1}{(n-1)!}\right] \alpha^n \\
&= \alpha + 4\alpha^2 + \sum_{n=3}^{\infty} \frac{(n-2)\,(n-1) + 3\,(n-1) + 1}{(n-1)!} \alpha^n \\
&= \alpha + 4\alpha^2 + \sum_{n=3}^{\infty} \frac{n^2\alpha^n}{(n-1)!} \\
&= \sum_{n=1}^{\infty} \frac{n^2\alpha^n}{(n-1)!}.
\end{aligned}$$

Solution 2 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

We have:

$$n^2 = (n-1)(n-2) + 3(n-1) + 1,$$

for $n \in N$ with $n \geq 1$. So we have:

$$\frac{n^2\alpha^n}{(n-1)!} = \frac{\alpha^n}{(n-3)!} + \frac{3\alpha^n}{(n-2)!} + \frac{\alpha^n}{(n-1)!},$$

and

$$\begin{aligned}\sum_{n=1}^{+\infty} \frac{n^2\alpha^n}{(n-1)!} &= \alpha^3 \sum_{n=1}^{+\infty} \frac{\alpha^{n-3}}{(n-3)!} + 3\alpha^2 \sum_{n=1}^{+\infty} \frac{\alpha^{n-2}}{(n-2)!} + \alpha \sum_{n=1}^{+\infty} \frac{\alpha^{n-1}}{(n-1)!} \\ &= \alpha^3 e^\alpha + 3\alpha^2 e^\alpha + \alpha e^\alpha = (\alpha^3 + 3\alpha^2 + \alpha)e^\alpha.\end{aligned}$$

Solution 3 by Moti Levy, Rehovot, Israel

Let $F(z)$ be the generating function of the sequence $\left(\frac{n^2}{(n-1)!}\right)_{n=1}^{\infty}$,

$$F(z) := \sum_{n=1}^{\infty} \frac{n^2}{(n-1)!} z^n.$$

Then by two repeated integrations, one may write,

$$\int_0^z \frac{1}{v} \int_0^v \frac{1}{u} F(u)\, du = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^n = ze^z.$$

Now we can express $F(z)$ by

$$F(z) = z \frac{d\left(z \frac{d(ze^z)}{dz}\right)}{dz} = z\left(z^2 + 3z + 1\right)e^z.$$

We conclude that

$$\sum_{n=1}^{\infty} \frac{n^2\alpha^n}{(n-1)!} = \alpha\left(\alpha^2 + 3\alpha + 1\right)e^\alpha, \qquad for \;\; \alpha \in C.$$

Remark: the value of the sum holds true for any complex number α. There is no reason to restrict to positive real numbers.

Solution 4 by Henry Ricardo, Westchester Area Math Circle, NY

We start with the power series expansion $e^z = \sum_{n=0}^{\infty} z^n/n!$, convergent for all complex numbers z and note that the series may be differentiated term-by-term.

Then

$$\frac{d}{dz}(e^z) = \sum_{n=0}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!}, \qquad z\frac{d}{dz}(e^z) = \sum_{n=1}^{\infty} \frac{z^n}{(n-1)!},$$

$$\frac{d}{dz}\left\{z\frac{d}{dz}(e^z)\right\} = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{(n-1)!}, \qquad z\frac{d}{dz}\left\{z\frac{d}{dz}(e^z)\right\} = \sum_{n=1}^{\infty} \frac{nz^n}{(n-1)!},$$

$$\frac{d}{dz}\left[z\frac{d}{dz}\left\{z\frac{d}{dz}(e^z)\right\}\right] = \sum_{n=1}^{\infty} \frac{n^2 z^{n-1}}{(n-1)!},$$

and, finally,

$$z\,\frac{d}{dz}\left[z\,\frac{d}{dz}\left\{z\frac{d}{dz}(e^z)\right\}\right]=\sum_{n=1}^{\infty}\frac{n^2z^n}{(n-1)!}. \qquad (*)$$

After some tedious but simple differentiations and multiplications, the left-hand side of $(*)$ becomes $ze^z(z^2+3z+1)$. Letting $z=\alpha\in C$ in $(*)$ gives us

$$\sum_{n=1}^{\infty}\frac{n^2\alpha^n}{(n-1)!} \;=\; \alpha e^{\alpha}(\alpha^2+3\alpha+1).$$

Solution 5 by Kee-Wai Lau, Hong Kong, China

Since $\dfrac{n^2}{(n-1)!}=\dfrac{1}{(n-3)!}+\dfrac{3}{(n-2)!}+\dfrac{1}{(n-1)!}$ for $n\geq 3$, so

$$\begin{aligned}\sum_{n=1}^{+\infty}\frac{n^2\alpha^n}{(n-1)!} &= \alpha+4\alpha^2+\sum_{n=3}^{+\infty}\frac{\alpha^n}{(n-3)!}+3\sum_{n=3}^{+\infty}\frac{\alpha^n}{(n-2)!}+\sum_{n=3}^{+\infty}\frac{\alpha^n}{(n-1)!}\\ &= \alpha+4\alpha^2+\alpha^3e^{\alpha}+3\alpha^2(e^{\alpha}-1)+\alpha(e^{\alpha}-1-\alpha)\\ &= \alpha e^{\alpha}\left(\alpha^2+3\alpha+1\right).\end{aligned}$$

Solution 6 by Arkady Alt, San Jose, CA

Since $e^x=\sum\limits_{n=1}^{+\infty}\dfrac{x^{n-1}}{(n-1)!}$ then $(xe^x)'=\left(\sum\limits_{n=1}^{+\infty}\dfrac{x^n}{(n-1)!}\right)' \iff e^x+xe^x=\sum\limits_{n=1}^{+\infty}\dfrac{nx^{n-1}}{(n-1)!}$

and, therefore, $\left(xe^x+x^2e^x\right)'=\left(\sum\limits_{n=1}^{+\infty}\dfrac{nx^n}{(n-1)!}\right)' \iff e^x\left(x^2+3x+1\right)=\sum\limits_{n=1}^{+\infty}\dfrac{n^2x^{n-1}}{(n-1)!}$.

Hence, $\sum\limits_{n=1}^{+\infty}\dfrac{n^2\alpha^n}{(n-1)!}=\alpha e^{\alpha}\left(\alpha^2+3\alpha+1\right)$

Editor's Comment: **David Stone and John Hawkins of Georgia Southern University in Statesboro, GA,** generalized the procedure used in (4) and (6) above, and showed that $\sum\limits_{n=1}^{\infty}\dfrac{n^3\alpha^n}{(n-1)!}=(\alpha^4+6\alpha^3+7\alpha^2+\alpha)e^n$.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Michel Bataille, Rouen, France; Naren Bhandari, Bajura School, Nepal, India; Brian Bradie, Christopher Newport University, Newport, News, VA; Michael Brozinsky, Central Islip, NY; Bruno Salgueiro Fanego,Viveiro, Spain; Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News,VA; David E. Manes, Oneonta, NY; Adrian Naco, Polytechnic University of Tirana, Albania; Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Ravi Prakash, Oxford University Press,

New Delhi, India; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins of Georgia Southern University, Statesboro, GA, and the proposer.

5534: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate $\int_0^1 \int_0^1 (x+y)\ln(x-xy+y)dxdy.$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We note that

$$\int_0^1 \int_0^1 (x+y)\ln(x-xy+y)dxdy = \int_0^1 \int_0^1 (x+y)\ln(1-(1-x)(1-y))dxdy =$$

$$= \quad -\sum_{k=1}^{\infty} \frac{1}{k}\int_0^1 \int_0^1 (x+y)(1-x)^k(1-y)^k dxdy = -2\sum_{k=1}^{\infty} \frac{1}{k}\frac{k!}{(k+2)!}\frac{1}{(k+1)} =$$

$$= \quad -2\sum_{k=1}^{\infty} \frac{1}{k(k+1)^2(k+2)} = -2\sum_{k=1}^{\infty}\left(\frac{1}{2k} - \frac{1}{(k+2)^2} - \frac{1}{2(k+2)}\right) = 2\left(\frac{1}{2}+\frac{1}{4}-\frac{\pi^1}{6}+1\right) =$$

$$= \frac{\pi^2}{3} - \frac{7}{2}$$

where we have used that for natural numbers m and n,
$\int_0^1 x^m(1-x)^n dx = \frac{m!n!}{(m+n+1)!}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$

Solution 2 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

We prove that $\int_0^1 \int_0^1 (x+y)\ln(x-xy+y)dxdy = \frac{\pi^2}{3} - \frac{7}{2}$. By symmetry we have:

$$I = \int_0^1 \int_0^1 (x+y)\ln(x-xy+y)dxdy. = \int_0^1 2\int_0^1 \ln(x-xy+y)dxdy.$$

and integration by parts we have:

$$\int_0^1 \ln(x-xy+y)dxdy = -\int_0^1 \frac{y(1-x)}{(1-x)y+x}dy = -1 + \int_0^1 \frac{dy}{y+\frac{x}{1-x}} = -1 - \frac{x\ln x}{1-x}.$$

So we have;

$$I = -2\int_0^1 x\left(1+\frac{x\ln x}{1-x}\right)dx = -1 - 2\int_0^1 \frac{x^2\ln x}{1-x}dx,$$

and if $x = e^t$, then:

$$I = -1 + 2\int_0^{+\infty} \frac{te^{-3t}}{1-e^{-t}}dt \quad = \quad -1 + 2\int_0^{+\infty} te^{-3t}\sum_{n\geq 0} e^{-nt}dt$$

$$
\begin{aligned}
&= -1+2\sum_{n\geq 0}\int_0^{+\infty} te^{-(n+3)t}dt = -1+2\sum_{n\geq 0}\frac{1}{(n+3)^2}\\
&= -1+2\left(\sum_{n\geq 0}\frac{1}{n^2}-1-\frac{1}{4}\right)\\
&= -1+2\left(\frac{\pi^2}{6}-\frac{5}{4}\right)=\frac{\pi^2}{3}-\frac{7}{2}.
\end{aligned}
$$

Also solved by Michel Bataille, Rouen, France; Brian Bradie, Christopher Newport University, Newport News, VA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News,VA; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel, and the proposer.

Mea Culpa

Received from Ed Gray, Highland Beach, FL.

"I have been reviewing my solution to 5523 which you published in the last column. I regret to say that the case for P = 2 is not correct. The problem is that the formula for the circumscribed circle, R, is not satisfied. R = abc/4A. If you recall, we got excited about discovering more than 1 solution, later found to be incorrect. You sent a note asking if there could be three solutions? P=2, area = 420, sides (25,39,56), diameter 65. And if so, are there still others? The answer is that there is only 1 solution, the one you sent. I would be most happy if you printed my error in the next column."

Arkady Alt of San Jose, CA should have been credited with having solved problem 5525. Mea Culpa.