Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before January 15, 2012

- **5176:** *Proposed by Kenneth Korbin, New York, NY*

  Solve:
$$\begin{cases} x^2+xy+y^2=3^2 \\ y^2+yz+z^2=4^2 \\ z^2+xz+x^2=5^2. \end{cases}$$

- **5177:** *Proposed by Kenneth Korbin, New York, NY*

  A regular nonagon $ABCDEFGHI$ has side 1.

  Find the area of $\triangle ACF$.

- **5178:** *Proposed by Neculai Stanciu, Buzău, Romania*

  Prove: If x, y and z are positive real numbers such that $xyz \geq 7+5\sqrt{2}$, then
$$x^2+y^2+z^2-2(x+y+z) \geq 3.$$

- **5179:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Find all positive real solutions $(x_1, x_2, \ldots, x_n)$ of the system
$$\begin{cases} x_1+\sqrt{x_2+11}=\sqrt{x_2+76}, \\ x_2+\sqrt{x_3+11}=\sqrt{x_3+76}, \\ \cdots\cdots\cdots \\ x_{n-1}+\sqrt{x_n+11}=\sqrt{x_n+76}, \\ x_n+\sqrt{x_1+11}=\sqrt{x_1+76}. \end{cases}$$

- **5180:** *Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy*

  Let a, b and c be positive real numbers such that $a+b+c=1$. Prove that
$$\frac{1+a}{bc}+\frac{1+b}{ac}+\frac{1+c}{ab} \geq \frac{4}{\sqrt{a^2+b^2-ab}}+\frac{4}{\sqrt{b^2+c^2-bc}}+\frac{4}{\sqrt{a^2+c^2-ac}}.$$

- **5181:** *Proposed by Ovidiu Furdui, Cluj, Romania*

  Calculate:
  $$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n\cdot m}{(n+m)!}.$$

Solutions

- **5158:** *Proposed by Kenneth Korbin, New York, NY*

  Given convex cyclic quadrilateral $ABCD$ with integer length sides $\overline{AB}=\overline{BC}=x$, and $\overline{CD}=\overline{DA}=x+1$.

  Find the distance between the incenter and the circumcenter.

  Solution by Michael Brozinsky, Central Islip, NY

  Since the perpendicular bisector of the base of an isosceles triangle passes through the vertex angle and the circumcenter of that triangle, it follows (by considering isosceles triangles CBA and CDA) that the line segment joining B and D is a diameter of the circumcircle, and thus the inscribed angles A and C are right angles.

  The circumcenter **E** is also the circumcenter of right triangle BAC, and thus it is the midpoint of the hypotenuse BD, and so it is $\dfrac{\sqrt{x^2+(x+1)^2}}{2}$ from B. The incenter **F** (being equidistant from BA and BD) is on the angle bisector of angle A and also on BD (by symmetry as triangles ABD and CBD are congruent), and so $\dfrac{BF}{FD}=\dfrac{x}{x+1}$ (since an angle bisector of a triangle divides the opposite side into segments proportional to the adjacent sides).

  Since $BF+FD=\sqrt{x^2+(x+1)^2}$ we have $BF=\dfrac{x}{2x+1}\cdot\sqrt{x^2+(x+1)^2}$ and hence the distance between **E** and **F** is
  $$\sqrt{x^2+(x+1)^2}\cdot\left(\frac{1}{2}-\frac{x}{2x+1}\right)=\frac{\sqrt{2x^2+2x+1}}{4x+2}.$$

  Comments: Most of the solvers realized that there is no need to restrict x to being an integer; x can be any positive real number. **David Stone and John Hawkins** also mentioned in their solution that even though the inradius (ρ) and the circumradius (r) grow large with x, as does the difference $r-\rho$, the distance d between the centers has the limiting value of $\dfrac{\sqrt{2}}{4}\approx 0.35355339$. So for large x, the incircle and the circumcircle are relatively concentric.

  Also solved by Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Boris Rays of Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5159:** *Proposed by Kenneth Korbin, New York, NY*

  Given square $ABCD$ with point P on diagonal $\overline{AC}$ and with point Q at the midpoint of side $\overline{AB}$.

  Find the perimeter of cyclic quadrilateral $ADPQ$ if its area is one unit less than the area of square $ABCD$.

  Solution by Trey Smith, San Angelo, TX

  Fix a point E on $\overline{AB}$ such that $\overline{PE}$ is perpendicular to $\overline{AB}$. Similarly, fix a point F on $\overline{AD}$ such that $\overline{PF}$ is perpendicular to $\overline{AD}$. Let k =length($\overline{AB}$). $AQPD$ is a cyclic quadrilateral, so it must be the case that $\angle QPD$ is a right angle, since it and $\angle DAQ$ are supplementary. Now $\angle FPE$ is also a right angle which forces $\angle QPE \cong \angle DPF$. And since $\triangle EPQ$ and $\triangle FPD$ are both right triangles with $\overline{PE} \cong \overline{PF}$, it is the case that $\triangle FPD \cong \triangle EPQ$. Finally, observing that $\overline{EQ} \cong \overline{FD} \cong \overline{EB}$ we have that E is the midpoint of $\overline{QB}$ and so the length of $\overline{AE}$ is $\dfrac{3k}{4}$.

  Since $\triangle FPD \cong \triangle EPQ$, it is easy to see that the area of $AQPD$ is the same as the area of square $AEPF$. Thus, the area of $AQPD$ is $\dfrac{9k^2}{16}$ and so the difference in the area of $ABCD$ and $AQPD$ is $\dfrac{7k^2}{16}$. Setting this equal to 1 and solving, we obtain $k = \dfrac{4}{\sqrt{7}}$. Now

$$\begin{aligned}
\text{length}(\overline{AQ}) = \frac{k}{2} &= \frac{2}{\sqrt{7}}, \\
\text{length}(\overline{QP}) = \text{length}(\overline{PD}) &= \sqrt{\left(\frac{k}{4}\right)^2 + \left(\frac{3k}{4}\right)^2} = \frac{k\sqrt{10}}{4} = \frac{\sqrt{10}}{\sqrt{7}}, \text{and} \\
\text{length}(\overline{DA}) &= k = \frac{4}{\sqrt{7}}.
\end{aligned}$$

  Summing these, we obtain the perimeter $\dfrac{6+2\sqrt{10}}{\sqrt{7}} = \dfrac{6\sqrt{7}+2\sqrt{70}}{7}$.

  Comment by David Stone and John Hawkins. It is easy to show that the point P must be located $\dfrac{3}{4}$ of the way from A to C along the diagonal AC in order to make $ADPQ$ a cyclic quadrilateral.

  Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Tania Moreno García, UHO, Cuba jointly with Jose P. Suárez, ULPGC, Spain; Paul M. Harms, North Newton, KS; Caleb Hemmick, Kaleb Davis, Logan Belgrave and Brianna Leever (jointly, students at Taylor University), Upland, IN; Kee-Wai Lau, Hong Kong, China; Sugie Lee, Jon Patton, and Matthew Fox (jointly, students at Taylor University), Upland, IN; David E. Manes, Oneonta, NY; Aaron Milauksas, Daniel Perrine, Kari Webster (jointly, students at Taylor University), Upland, IN; Tom Peller, Stephen Chou and Tal Knighton (jointly, students at Taylor University), Upland, IN; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5160:** *Proposed by Michael Brozinsky, Central Islip, NY*

  In Cartesianland, where immortal ants live, there are n (where $n \geq 2$) roads $\{l_i\}$ whose equations are

$$l_i: \ x\cos\left(\frac{2\pi i}{n}\right) + y\sin\left(\frac{2\pi i}{n}\right) = i, \text{where } i = 1, 2, 3, \ldots, n.$$

Any anthill must be located so that the sum of the squares of its distances to these n lines is $\dfrac{n(n+1)(2n+1)}{6}$. Two queen ants are (im)mortal enemies and have their anthills as far apart as possible. If the distance between these queens' anthills is 4 units, find n.

Solution by Kee-Wai Lau, Hong Kong, China

We show that the anthills are $2\csc\left(\dfrac{\pi}{n}\right)$ units apart for $n \geq 3$. In the present case that they are 4 units apart, we see that $n = 6$. If $n = 2$, then the anthill can be located anywhere on the $y-$axis, so that the distance between them can be as large as possible.

For simplicity, we denote π/n by m. Let the coordinates of an anthill be $(r\cos\theta, r\sin\theta)$, where $r \geq 0$ and $0 \leq \theta \leq 2\pi$. Its distance to l_i is given by

$$|r\cos\theta\cos(2mi) + r\sin\theta\sin(2mi) - i| = |r\cos(2mi - \theta) - i|.$$

Since $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, so according to the rule for location , we have $\displaystyle\sum_{i=1}^{n}(r\cos(2mi-\theta) - i)^2 = \sum_{i=1}^{n} i^2$. Clearly the origin satisfies the rule. If $r \neq 0$, then

$$\sum_{i=1}^{n}\Big(r\cos^2(2mi-\theta) - 2i\cos(2mi-\theta)\Big) = 0. \qquad (1)$$

As $\cos^2(2mi-\theta) = \dfrac{1}{2}(1+\cos(4mi-2\theta))$, so (1) is equivalent to

$$\begin{aligned} rn + r\cos 2\theta\sum_{i=1}^{n}\cos(4mi) \quad &+ \quad r\sin 2\theta\sum_{i=1}^{n}(4mi) \\ -4\cos\theta\sum_{i=1}^{n} i\cos(2mi) \quad &- \quad 4\sin\theta\sum_{i=1}^{n} i\sin(2mi) = 0. \qquad (2) \end{aligned}$$

For $\sin(x/2) \neq 0$ and positive integers k, we have the following known results,

$$\sum_{i=1}^{k}\cos(ix) = \frac{\sin(kx/2)\cos(k+1)x/2}{\sin(x/2)}, \qquad \sum_{i=1}^{k}\sin(ix) = \frac{\sin(kx/2)\sin(k+1)x/2}{\sin(x/2)},$$

$$\sum_{i=1}^{k} i\cos(ix) = \frac{(k+1)\cos(kx)k\cos(k+1)x - 1}{4\sin^2(x/2)}, \qquad \sum_{i=1}^{k} i\sin(ix) = \frac{(k+1)\sin(kx) - k\sin(k+1)x}{4\sin^2(x/2)},$$

which can be proved readily by induction on k. Thus for $n \geq 3$, we have

$$\sum_{i=1}^{n} \cos(4mi) = \sum_{i=1}^{n} \sin(4mi) = 0, \qquad \sum_{i=1}^{n} i\cos(2mi) = \frac{n}{2}, \qquad \sum_{i=1}^{n} \sin(2mi) = \frac{-n\cot(m)}{2},$$

and from (2) we deduce that for $m - \pi < \theta < m$,

$$r = \frac{2\sin(m-\theta)}{\sin m}. \qquad (3)$$

Together with the origin, (3) represents the locus of a circle. In rectangular coordinates the equation of the circle is $(x-1)^2\,(y+\cot m)^2 = \csc^2 m$. Thus the distance between the anthills equals the diameter $2\csc m$ of the circle and this completes the solution.

Also solved by Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5161:** *Proposed by Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy*

It is well known that for any function $f : \Re \to \Re$, continuous or not, the set of points on the y-axis where it attains a maximum or a minimum can be at most denumerable. Prove that any function can have at most a denumerable set of inflection points, or give a counterexample.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

Let C be the Cantor ternary set, defined by

$$C = [0,1] - \bigcup_{m=1}^{\infty} \bigcup_{k=0}^{3^{m-1}-1} \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$$

(see [1] as a reference). It is well known that C is uncountable and has Lebesgue measure zero. (Therefore C does not contain any interval).

For any point $x \in [0,1]$ define the distance from x to C by $d(x,C) = \inf\limits_{y\in C} |x-y|$. If $x \in [0,1]$ there is (at least) one point $y(x) \in C$ such that $d(x,C) = |x - y(x)|$, since C is closed. Furthermore $d(x,C) = 0$ if and only if $x \in C$.

Define a real function $f : [0,1] \longrightarrow R$ by $f(x) = (x - y(x))^3$, if the point $y(x) \in C$ that is closest to x is unique, and put $f(x) = 0$ if there is not a unique closest point to x. Extend f to a 1-periodic function to the whole of the real line. f is a piecewise cubic polynomial (and therefore f is piecewise differentiable). Any point of the form $n + y$ where n is an integer and $y \in C$ is an inflection point.

We produce a second counterexample and show that there is even a continuously differentiable function with uncountably many inflection points by "lifting" the previous example to a continuously differentiable function. We put $C^* = \{n + y | n$ an integer, $y \in C\}$ and define

$$f(x) = \int_0^x d^2\,(t, C^*)\,dt.$$

f is differentiable and $f'(x) = d^2(x, C^*)$ where $f'(x$ is continuous, since $d(x, C*)$ is. The derivative is zero if and only if $x \in C^*$. The points of zero derivatives are uncountable, since C^* is uncountable, and every point of C^* is an inflection point.

Reference: [1] <http://en.wikipedia.org/wiki/Cantor set>

Solution 2 by proposer

We propose the counter example.

Let $f(x) = \int_0^x \rho(t, C)dt$ where $0 \le x \le 1$, C is a Cantor set (ternary for example) and let $\rho(t, C) = \inf_{t' \in C} |t - t'| = \min_{t' \in C} |t - t'|$ (the equality due to the closeness of $\overline{C} = C$).

We know that C is non-denumerable and nowhere dense. The nowhere density means that for any $t \in C$, $t \in C$, there exists an open interval $I = (a, b)$ such that $t < a < b < t'$ and $I \cap C = \emptyset$.

Now we observe that:

1) $f(x)$ is differentiable since $\rho(t, C)$ is continuous, and

2) $f'(x) = 0$ if $x \in C$ and $f'(x) > 0$ if $x \notin C$, (this is due to the closeness of C).

The non-denumerability of C implies the non-denumerability of set of points x where $f'(x) = 0$ and moreover they are inflection points because $f'(x) > 0$ if $x \notin C$.

The nowhere density of C together with $\rho(t, C) > 0$ imply that the ordinates of two different points are necessarily different so getting the non-denumerability of the ordinates of these inflection points.

Editor's comment: Several readers stated that at most there can be a denumerable number of inflection points. **Michael Fried of Kibbutz Revivim in Israel** was one them, but upon seeing Paolo's proof he wrote:

Yes, Paolo is right. The mistake in my objection was to assume implicitly that the inflection points corresponded to *distinct* maximum/minimum values of the derivative function. This would indeed imply that the distinct ordinates of the inflection points were as numerous as the those of the maximum/minimum, and, therefore, at most denumerable.

But think about what this function $\rho(t, C) = \inf\limits_{t' \in C} |t - t'|$ looks like.

The set of all points at which the function ρ has a minimum is precisely the Cantor set, as Paolo claimed, so that set is non-denumerable. All its minimum *values,* which occur at every point of the Cantor set, however, are all equal to zero. As for its maximum values, there is one for each step in the process producing the ternary Cantor set (i.e. one maximum value for each "removal of the middle third"), so that the ordinates of the maximum values of ρ are denumerable. There is no contradiction, then, of the fact that the maximum/minimum values of the functions can be at most denumerable.

Hence, we have the following situation:

1) ρ is the derivative function of $f(x) = \int_0^x \rho(t, C)dt$ where $0 \le x \le 1$. Therefore, f has a non-denumerable set of infection points.

2) Since f is defined as an integral, it is an increasing function. Therefore, the value of f at each inflection point is unique.

Very interesting!

Each of the other solvers came up with the opposite conclusion, namely that the number of inflection points must be at most denumerable. Their reasoning is reflected in Michael Brozinsky's argument. He stated: "If a function $f(x)$ has an inflection point at $x = x_0$ then there is an open interval $a_{x_0} < x < b_{x_0}$ containing x_0 such that the concavity on (a_{x_0}, x_0) and (x_0, b_{x_0}) is different and thus (a_{x_0}, b_{x_0}) cannot contain another inflection point of $f(x)$. Thus the inflection points of $f(x)$ are isolated points and hence at most denumerable. (We can, without loss of generality, take a_{x_0} and b_{x_0} to be rational since the rational numbers are dense and then associate to x_0 the midpoint of the aforementioned interval, i.e., the rational number $\dfrac{a_{x_0} + b_{x_0}}{2}$. Since the rationals are denumerable, the inflection points of $f(x)$ are at most denumerable."

David Stone and John Hawkins were in correspondence with me about this problem because I took issue with their solution, which was in the spirit of Michael Brozinsky's. I sent them Paolo's proof and Michael Fried's comment about it, and they responded as follows:

John and I looked at Paolo's counterexample and Michael's comment and now the reason for the confusion is clear. We're using the standard calculus notion – an inflection point is a place where the concavity changes. Moreover, concavity is defined over an interval, not at a point. You can see our meaning in the proof we sent you.... But Paolo and Michael seem to be using a different definition, more like "an inflection point is a place where the derivative achieves a max or min". Their work never mentions "concavity" – not in their mind at all. Wikipedia mostly agrees with this notion. (It's not an issue here, but what if the derivative didn't exist? What would be meant by "inflection point"?) I think we are all correct, subject to the differing definitions (and the problem statement proscribed no particular meaning of the term "inflection point").
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As editor of this column, I agree with them, that both solutions are correct, depending upon which definition of inflection point is used. But using the change in concavity definition of an inflection point makes this problem much less challenging than using the extremities of the first derivative definition. Here is what Albert (proof #1) wrote about his initial thoughts on the problem.

I have given problem 5161 a few thoughts. It is clear that the number of inflection points is countable if the function $f(x)$ is sufficiently smooth, let's say two times continuously differentiable. The inflection points are then the extrema of the function $f'(x)$, and the set of (local) extrema of a function is countable. So if we want to find a counterexample we should concentrate on more "exotic" functions. I have in mind to construct a counterexample that is based on the Cantor set. We start from the function $f(x) = x$ defined on the interval $[0, 1]$ and replace linear segments by cubics in the following sense: if $0 < a < b < 1$ then we replace the function $f(x) = x$ by $g(x) = a + 3\dfrac{(x-a)^2}{(b-a)} - 2\dfrac{(x-a)^3}{(b-a)^2}$. Then $g(a) = a, g(b) = b, g'(a) = g'(b) = 0$. In the first iteration we take $a = 1/3, b = 2/3$ and do the replacement for the third in the middle. We continue the Cantor construction and do similar replacements for the first and third third. Continuing this way we get a continuous function that is piecewise differentiable (multiple times). We now have to analyze in more detail what happens at the points of the Cantor set, and see whether all these points are inflection points. Kind regards - Albert

Also solved by: Michael Brozionsky, Central Islip, NY; Michael N. Fried, Kibbtuz Revivim, Israel, and David Stone and John Hawkins (jointly), Statesboro, GA,

- **5162:** *Proposed by José Luis Díaz-Barrero and José Gibergans-Báguena, Barcelona, Spain*

Let a, b, c be the lengths of the sides of an acute triangle ABC. Prove that

$$\sqrt{\frac{b^2+c^2-a^2}{a^2+2bc}}+\sqrt{\frac{c^2+a^2-b^2}{b^2+2ca}}+\sqrt{\frac{a^2+b^2-c^2}{c^2+2ab}}\leq\sqrt{3}.$$

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

Using the Law of Cosines and the Arithmetic - Geometric Mean Inequality, we get

$$b^2+c^2-a^2=2bc\cos A$$

and

$$a^2+2bc=b^2+c^2-2bc\cos A+2bc\geq 4bc-2bc\cos A=2bc\,(2-\cos A)\,.$$

Since $0<A<\dfrac{\pi}{2}$, we have

$$\frac{b^2+c^2-a^2}{a^2+2bc}\leq\frac{2bc\cos A}{2bc\,(2-\cos A)}=\frac{\cos A}{2-\cos A}=\frac{1}{2\sec A-1}$$

and hence,

$$\sqrt{\frac{b^2+c^2-a^2}{a^2+2bc}}\leq\frac{1}{\sqrt{2\sec A-1}}.$$

Further, equality is attained if and only if $b=c$.

Similar steps show that

$$\sqrt{\frac{c^2+a^2-b^2}{b^2+2ca}}\leq\frac{1}{\sqrt{2\sec B-1}}\quad\text{and}\quad\sqrt{\frac{a^2+b^2-c^2}{c^2+2ab}}\leq\frac{1}{\sqrt{2\sec C-1}},$$

with equality if and only if $a=b=c$.

Consider the function $f(x)=\dfrac{1}{\sqrt{2\sec x-1}}$ on $\left(0,\dfrac{\pi}{2}\right)$. Since

$$\begin{aligned}f''(x)&=\frac{-\sec x\left(\sec^3 x-2\sec^2 x+\sec x+1\right)}{(2\sec x-1)^{\frac{5}{2}}}\\&=\frac{-\sec x\left[\sec x\,(\sec x-1)^2+1\right]}{(2\sec x-1)^{\frac{5}{2}}}\\&<0\end{aligned}$$

on $\left(0, \frac{\pi}{2}\right)$, it follows that $f(x)$ is concave down on $\left(0, \frac{\pi}{2}\right)$. Then, by Jensen's Theorem and our comments above,

$$\begin{aligned}
\sqrt{\frac{b^2+c^2-a^2}{a^2+2bc}} + \sqrt{\frac{c^2+a^2-b^2}{b^2+2ca}} + \sqrt{\frac{a^2+b^2-c^2}{c^2+2ab}} &\leq f(A) + f(B) + f(C) \\
&\leq 3f\left(\frac{A+B+C}{3}\right) \\
&= 3f\left(\frac{\pi}{3}\right) \\
&= 3 \cdot \frac{1}{\sqrt{3}} \\
&= \sqrt{3},
\end{aligned}$$

with equality if and only if $a = b = c$. That is, if and only if $\triangle ABC$ is equilateral.

Also solved by Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH, and the proposers.

- **5163:** *Proposed by Pedro H. O. Pantoja, Lisbon, Portugal*

Prove that for all $n \in N$

$$\int_0^\infty \frac{x^n}{2}\left(\coth\frac{x}{2} - 1\right)dx = \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \frac{1}{k_1 \cdots k_n (k_1 + \cdots + k_n)}.$$

Solution 1 by G. C. Greubel, Newport News, VA

It can be seen that

$$\coth\frac{x}{2} - 1 = \frac{2}{e^x - 1}.$$

With this the integral in question becomes

$$\begin{aligned}
I &= \int_0^\infty \frac{x^n}{2}\left(\coth\frac{x}{2} - 1\right)dx \\
&= \int_0^\infty \frac{x^n}{e^x - 1}dx \\
I &= \Gamma(n+1)\zeta(n+1).
\end{aligned}$$

Now we have to show that the $n-$ sums are equal to the same value. This can be done by considering the integral

$$\int_0^\infty e^{-ax}dx = \frac{1}{a}.$$

Using this we then have

$$S = \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \frac{1}{k_1 \cdots k_n (k_1 + \cdots + k_n)}$$

$$
\begin{aligned}
&= \int_0^\infty \sum_{k_1=1}^\infty \cdots \sum_{k_n=1}^\infty \frac{1}{k_1 \cdots k_n} e^{-(k_1+\cdots+k_n)x} dx \\
&= \int_0^\infty \left(\sum_{k_1=1}^\infty \frac{e^{-k_1 x}}{k_1} \right) \cdots \left(\sum_{k_n=1}^\infty \frac{e^{-k_n x}}{k_n} \right) dx \\
&= \int_0^\infty \left(\sum_{k_1=1}^\infty \frac{e^{-kx}}{k_1} \right)^n dx \\
S &= \int_0^\infty \left(-\ln\left(1-e^{-x}\right) \right)^n dx.
\end{aligned}
$$

By making the substitution $t = -\ln(1-e^{-x})$ we then have

$$
S = \int_0^\infty \frac{t^n}{e^t - 1} dt = \Gamma(n+1)\zeta(n+1).
$$

We have shown that $\displaystyle\int_0^\infty \frac{x^n}{2}\left(\coth\frac{x}{2} - 1\right) dx$ and $\displaystyle\sum_{k_1=1}^\infty \cdots \sum_{k_n=1}^\infty \frac{1}{k_1 \cdots k_n \left(k_1 + \cdots + k_n\right)}$ is each equal to $\Gamma(n+1)\zeta(n+1)$, thus they are equal to each other.

Solution 2 by Paolo Perfetto, Department of Mathematics, "Tor Vergatta" University, Rome, Italy

Proof: We write

$$
\frac{1}{k_1 + \cdots + k_n} = \int_0^1 t^{k_1+\cdots+k_n-1} dt
$$

and then

$$
\begin{aligned}
\sum_{k_1,\ldots,k_n=1}^\infty \frac{1}{k_1 k_2 \cdots k_n} \int_0^1 t^{k_1+\cdots+k_n-1} dt &= \int_0^1 t^{-1} \sum_{k_1,\ldots,k_n=1}^\infty \frac{t^{k_1+\cdots+k_n}}{k_1 k_2 \cdots k_n} dt \\
&= \int_0^1 t^{-1} (-1)^n (\ln(1-t))^n dt \\
&= (-1)^n \int_0^1 (1-t)^{-1} (\ln t)^n dt.
\end{aligned}
$$

Now we change variables letting $\ln t = -x$. Therefore,

$$
(-1)^n \int_0^\infty \frac{(-x)^n}{1-e^{-x}} dx = \int_0^\infty \frac{x^n}{1-e^{-x}} dx.
$$

The proof concludes by observing that

$$
\coth\frac{x}{2} - 1 = \frac{e^{\frac{x}{2}} + e^{\frac{-x}{2}}}{e^{\frac{x}{2}} - e^{\frac{-x}{2}}} - 1 = \frac{2e^{\frac{-x}{2}}}{e^{\frac{x}{2}} - e^{\frac{-x}{2}}} = \frac{2}{1-e^{-x}}.
$$

Comment by Paolo: Apart from $p = 0$ the series in the statement is the same as in problem #174 in the **Missouri Journal of Mathematical Sciences,** 22(1); downloadable at $< \mathrm{http://www.math-cs.ucmo.edu/mjms/2010.1/Prob7.pdf} >$

Solution 3 by Albert Stadler, Herrliberg, Switzerland

We find that:

$$\int_0^\infty \frac{x^n}{2}\left(\coth\frac{x}{2}-1\right)dx = \int_0^\infty \frac{x^n}{2}\left(\frac{e^{\frac{x}{2}}+e^{\frac{-x}{2}}}{e^{\frac{x}{2}}-e^{\frac{-x}{2}}}-1\right)dx = \int_0^\infty \frac{x^n e^{-x}}{1-e^{-x}}dx. \qquad (1)$$

We perform a change of variables: $y = 1 - e^{-x},\ dy = e^{-x}dx.$ So

$$\begin{aligned}
\int_0^\infty \frac{x^n e^{-x}}{1-e^{-x}}dx = \int_0^1 \frac{(-\log(1-y))^n}{y}dy \quad &= \quad \int_0^1 \frac{\left(\sum\limits_{k=1}^{\infty}\frac{y^k}{k}\right)^n}{y}dy \\
&= \quad \int_0^1 \sum_{k_1\ge1,k_2\ge1\ldots,k_n\ge1} \frac{y^{k_1+k_2+\ldots+k_n-1}}{k_1\cdot k_2\cdots k_n}dy \\
&= \quad \sum_{k_1\ge1,k_2\ge1\ldots,k_n\ge1} \frac{y^{k_1+k_2+\ldots+k_n-1}}{k_1\cdot k_2\cdots k_n}\int_0^1 y^{k_1+k_2+\ldots+k_n-1}dy \\
&= \quad \sum_{k_1\ge1,k_2\ge1\ldots,k_n\ge1} \frac{1}{k_1\cdot k_2\cdots k_n\,(k_1+k_2+\ldots+k_n)}.
\end{aligned}$$

The interchange of summation and integration is allowed becasue of absolute convergence (all involved terms are positive).

It is noteworthy that the integral (1) can be explicitly evaluated in terms of the Riemann zeta function:

$$\int_0^\infty \frac{x^n e^{-x}}{1-e^{-x}}dx = \sum_{k=1}^{\infty}\int_0^\infty x^n e^{-kx}dx = \sum_{k=1}^{\infty}\frac{1}{k^{n+1}}\int_0^\infty x^n e^{-x}dx = n!\sum_{k=1}^{\infty}\frac{1}{k^{n+1}} = n!\zeta(n+1).$$

It is well known that $\zeta(n+1)$ is a rational multiple of π^{n+1}, if n is odd.