Problems

Ted Eisenberg, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before January 15, 2013

- **5224:** *Proposed by Kenneth Korbin, New York, NY*

  Let $T_1 = T_2 = 1, T_3 = 2$, and $\mathrm{T}_n = T_{n-1} + T_{n-2} + T_{n-3}$. Find the value of
  $$\sum_{n=1}^{\infty} \frac{T_n}{\pi^n}.$$

- **5225:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  Find infinitely many integer squares x that are each the sum of a square and a cube and a fourth power of positive integers a, b, c. That is, $x = a^2 + b^3 + c^4$.

- **5226:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

  Calculate:
  $$\int_a^b \frac{\sqrt[n]{x-a}\left(1+\sqrt[n]{b-x}\right)}{\sqrt[n]{x-a}+2\sqrt[n]{-x^2+(a+b)x-ab}+\sqrt[n]{b-x}}dx,$$
  where $0 < a < b$ and $n > 0$.

- **5227:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

  Compute
  $$\lim_{n\to\infty} \prod_{k=1}^{n} \left(\frac{(n+1)+\sqrt{nk}}{n+\sqrt{nk}} \right).$$

- **5228:** *Proposed by Mohsen Soltanifar, University of Saskatchewan, Saskatoon, Canada*

  Given a random variable X with non-negative integer values. Assume the n^{th} moment of X is given by
  $$E\left(X^n\right) = \sum_{k=1}^{\infty} f_n(k) P\left(X \geq k\right) \qquad n = 1, 2, 3, \cdots,$$

where f_n is a non-negative function defined on N. Find a closed form expression for f_n.

- **5229:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $\beta > 0$ and let $(x_n)_{n \in N}$ be the sequence defined by the recurrence relation

  $$x_1 = a > 0,\ x_{n+1} = x_n + \frac{n^{2\beta}}{x_1 + x_2 \cdots + x_n}.$$

  1) Prove that $\lim\limits_{n \to \infty} x_n = \infty$.

  2) Calculate $\lim\limits_{n \to \infty} \dfrac{x_n}{n^\beta}$.

Solutions

- **5206:** *Proposed by Kenneth Korbin, New York, NY*

  The distances from the vertices of an equilateral triangle to an interior point P are $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$ respectively, where a, b, and c are positive integers.

  Find the minimum and the maximum possible values of the sum $a + b + c$ if the side of the triangle is 13.

Solution 1 by Brian D. Beasley, Presbyterian College, Clinton ,SC

We show that $a + b + c$ has a minimum value of 170 and a maximum value of 296. We

model the given triangle using vertices $A(0, 0)$, $B(13, 0)$, and $C(13/2, 13\sqrt{3}/2)$. Then the centroid of triangle ABC is $G(13/2, 13\sqrt{3}/6)$. Let $P(x, y)$ be a point interior to $\triangle ABC$. We denote $AP = \sqrt{a}$, $BP = \sqrt{b}$, and $CP = \sqrt{c}$ for positive integers a, b, and c; due to the symmetry of the equilateral triangle, we may assume without loss of generality that $a \le b \le c$. It is then straightforward to verify that

$$a + b + c = AG^2 + BG^2 + CG^2 + 3PG^2 = 169 + 3PG^2.$$

Since $AG^2 = 169/3$ is not an integer, we know $P \ne G$, so the minimum value of $a + b + c$ is greater than 169 and thus must be at least 170. In fact, taking P to be $(6, 2\sqrt{3})$ achieves this minimum value of 170, with $(a, b, c) = (48, 61, 61)$.

Next, we note that $x^2 + y^2 = a$ and $(13 - x)^2 + y^2 = b$, so $x = (a - b + 169)/26$. If $a = 1$, then P lies on the circle $x^2 + y^2 = 1$, so $1/2 < x < 1$ and hence $14 \le a - b + 169 \le 25$. A quick check produces $(a, b, c) = (1, 147, 148)$ for $x = 23/26$ and $y = 7\sqrt{3}/26$, so the maximum value of $a + b + c$ is no smaller than 296. If $a \ge 2$, then PG is less than the distance from G to either $(\sqrt{2}/2, \sqrt{6}/2)$ or $(\sqrt{2}, 0)$, the intersections of the circle $x^2 + y^2 = 2$ with the triangle. This yields

$$PG^2 < \frac{175}{3} - 13\sqrt{2} < 40,$$

so $a + b + c < 169 + 3(40) = 289$. Hence the maximum value of $a + b + c$ is 296.

Solution 2 by Paul M. Harms, North Newton, KS

Put the equilateral triangle of the problem on a coordinate system with

$$A(-6.5,0),\ B(0,6.5\sqrt{3}),\ C(6.5,0) \text{ with } P(x,y).$$

Then

$$\begin{aligned} a &= (x+6.5)^2+y^2, \\ b &= x^2+(y-6.5\sqrt{3})^2, \\ c &= (x-6.5)^2+y^2. \end{aligned}$$

Let $L=a+b+c$ and, temporarily, consider the domain of L to be the triangle and its interior. Using partial derivatives we find that L has a minimum of 169 at $x=0$, and $y=\dfrac{13\sqrt{3}}{6}$. At this point $a=b=c=\dfrac{169}{3}$. Other extremes may occur along the boundary of the domain. Checking for extremes along AC, we find an absolute maximum of 338 at each vertex and a minimum of 211.25 when $x=0$. The absolute minimum is then 169 and occurs at the one point $\left(0,\dfrac{13\sqrt{3}}{6}\right)$. At this point for a minimum L, the numbers a,b,and c are not integers. Then to satisfy the problem L must be at least 170. Also, the absolute maximum found above occurs at the vertices, and not at a point interior to the triangle, so this maximum will not satisfy the problem.

Consider L along $(0,y)$ where $0<y<6.5\sqrt{3}$. Here

$$\begin{aligned} a &= c=(6.5)^2+y^2 \\ b &= \left(y-6.5\sqrt{3}\right)^2. \end{aligned}$$

Then $y=6.5\sqrt{3}-\sqrt{b}$, so $a=c=4\,(6.5)^2+b^2-13\sqrt{3b}$ with $0<\sqrt{b}<6.5\sqrt{3}$.

We see that a,b and c will be integers when b is three times a perfect square. For these values of b, L is a minimum of 170 when $b=3(16)=48, a=c=61$. For these values of b, L is a maximum of 269, when $b=3(1)=3, a=c=133$. This minimum value of L satisfies the problem since the point is interior to the triangle with integer values for a,b, and c.

To check interior points for a maximum L, we check points close to a vertex, since for the general domain, the maximum occurs at a vertex.

Let us consider circles with radius $\sqrt{b}$ where b is an integer and the center of the circle is B.

For the problem, we only need to consider the portion of the circle interior to the triangle and in the first quadrant. Consider a first quadrant point P, interior to the triangle and on the circle with center at B and radius $\sqrt{b}$. Using the law of cosines for $\triangle ABP$ and $\triangle PBC$, we have

$$\begin{cases} a=13^2+b-2(13)\sqrt{b}\cos\theta \text{ and} \\ \\ c=13^2+b-2(13)\sqrt{b}\cos\left(60^\circ-\theta\right), \text{where } 0<\theta<60^\circ. \end{cases}$$

When $\theta=60^\circ$, we have integers for a,b, and c when b is a perfect square, but P is then on a side of the triangle and not interior to the triangle. When $b=1$, the possible integers for c are $145, 146$, and 147. We find that when $c = 147$, and $b = 1, a = 148$ with $L = 296$. For a fixed positive integer b and $30^\circ < \theta < 60^\circ$, the maximum $(a + c)$ occurs at 60°. Checking other values of b, we find that the maximum L is less than 296 for integers $b > 1$.

Thus for positive integers, a, b and c with P interior to the triangle, the minimum L is 170 and the maximum L is 296.

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that the minimum is 170 and the maximum is 296. The minimum occurs when $a = 61, b = 48$ and $c = 61$ and the maximum occurs when $a = 148, b = 1$ and $c = 147$.

Denote the triangle by ABC with $PA = \sqrt{a},\ PB = \sqrt{b},\ PC = \sqrt{c}$. Let $\angle PBA = \theta$ and $\angle$PBC=ϕ. Applying the cosine formula respectively to triangle PBA and PBC we obtain

$$\cos\theta = \frac{169 + b - a}{26\sqrt{b}} \text{ and } \cos\phi = \frac{169 + b - c}{26\sqrt{b}}.$$

Hence $\sin\theta = \dfrac{\sqrt{676b - (169 + b - a)^2}}{26\sqrt{b}}$ and $\sin\phi = \dfrac{\sqrt{676b - (169 + b - c)^2}}{26\sqrt{b}}$.

Since

$$\sin\theta\sin\phi = \cos\theta\cos\phi - \cos(\theta + \phi) = \cos\theta\cos\phi - \frac{1}{2}, \text{ so}$$

$$\left(\sqrt{676b - (169 + b - a)^2}\right)\left(\sqrt{676b - (169 + b - c)^2}\right) = (169 + b - a)(169 + b - c) - 338b.$$

Squaring both sides, expanding and simplifying, we obtain the equation

$$a^2 - (169 + b + c)a + b^2 + c^2 - bc - 169b - 169c + 28561 = 0. \text{ Hence}$$

$$a = \frac{1}{2}\left(169 + b + c \pm \sqrt{3}\sqrt{\left(\sqrt{b} + \sqrt{c} + 13\right)\left(\sqrt{b} + \sqrt{c} - 13\right)\left(\sqrt{b} - \sqrt{c} + 13\right)\left(\sqrt{c} - \sqrt{b} + 13\right)}\right).$$

By considering the special case $a = b = c = \dfrac{169}{3}$, we see that in fact

$$a = \frac{1}{2}\left(169 + b + c - \sqrt{3}\sqrt{\left(\sqrt{b} + \sqrt{c} + 13\right)\left(\sqrt{b} + \sqrt{c} - 13\right)\left(\sqrt{b} - \sqrt{c} + 13\right)\left(\sqrt{c} - \sqrt{b} + 13\right)}\right).$$

We now obtain the minimum and maximum values of $a + b + c$ stated above with the help of a computer. Here we impose the restrictions $1 \le b \le 168,\ b \le c \le a \le 168$ by symmetry, $\sqrt{a} + \sqrt{b} > 13,\ \sqrt{b} + \sqrt{c} > 13, \sqrt{c} + \sqrt{a} > 13$, and that a is a positive integer. This completes the solution.

Solution 4 by Albert Stadler of Herrliberg, Switzerland

Let $\alpha = \angle APB,\ \beta = \angle BPC, \gamma = \angle CPA$. Then by the law of cosines,

$$\cos\alpha = \frac{a + b - 169}{2\sqrt{ab}},\ \cos\beta = \frac{b + c - 169}{2\sqrt{bc}},\ \cos\gamma = \frac{c + a - 169}{2\sqrt{ca}}.$$

Obviously $\alpha + \beta + \gamma = 2\pi$. So

$$\cos\gamma = \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

$$(\cos\gamma - \cos\alpha\cos\beta)^2 = (1-\cos^2\alpha)(1-\cos^2\beta),$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma - 1 = 0,$$

$$\left(\frac{a+b-169}{2\sqrt{ab}}\right)^2 + \left(\frac{b+c-169}{2\sqrt{bc}}\right)^2 + \left(\frac{c+a-169}{2\sqrt{ca}}\right)^2 - 2\left(\frac{a+b-169}{2\sqrt{ab}}\right)\left(\frac{b+c-169}{2\sqrt{bc}}\right)\left(\frac{c+a-169}{2\sqrt{ca}}\right) = 1,$$

which is equivalent to

$$3\left(a^2+b^2+c^2+13^4\right) = \left(a+b+c+13^3\right)^2, \qquad (1)$$

as is seen when multiplying out.

A computer search on the set $\{(a,b,c)|1 \leq a,b,c \leq 169\}$ reveals that only the tuples of the table in the appendix satisfy (1) The minimal value of $a+b+c$ is 170 and the maximal value is 296.

Editor's note: Ken Korbin, proposer of the problem, also worked with the formula:

$$3\left(a^2+b^2+c^2+13^4\right) = \left(a+b+c+13^3\right)^2.$$

Albert presented a table listing all possible values satisfying the conditions of the problem. His appendix consisted of a table containing 258 rows for the various values of a, b and c; a few of rows are reproduced below.

David Stone and John Hawkins of Statesboro, GA *noted that it can be shown that the quantity $\sqrt{a}+\sqrt{b}+\sqrt{c}$, the sum of the distances from P to the three vertices, achieves its minimum of $\sqrt{3}s$ at the centroid of the triangle, and it achieves its maximum of $2s$ at any vertex (where "s" is a positive integer representing the side length of the equilateral triangle.)*

They also observed that because it is defined as the sum of the square of the distances to the vertices, the quantity $a+b+c$ can properly be called the **moment of inertia of the triangle about the point P.** *They showed that this moment of inertia of an equilateral triangle is minimized when P is the centroid and maximized at any vertex. The same conclusion holds for a square and, they hypothesize, for any regular polygon.*

Stadler's Table

a	b	c	$a+b+c$
48	61	61	170
49	57	64	170
49	64	57	170
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	147	148	296
147	1	148	296
157	1	144	302
157	144	1	302

Also solved by Farideh Firoozbakht and Jahangeer Kholdi (jointly), Isfahan, Iran; Adrian Naco, Polytechnic University, Tirana, Albania, David Stone and John Hawkins (jointly), Georgia Southern University, Statesboro GA, and the proposer.

- **5207:** *Proposed by Roger Izard, Dallas, TX*

Consider the following four algebraic terms:

$$\begin{aligned} T_1 &= a^2(b+c)+b^2(a+c)+c^2(a+b) \\ T_2 &= (a+b)(a+c)(b+c) \\ T_3 &= abc \\ T_4 &= \frac{b+c-a}{a}+\frac{a+c-b}{b}+\frac{a+b-c}{c} \end{aligned}$$

Suppose that $\dfrac{T_1\cdot T_2}{(T_3)^2}=\dfrac{616}{9}$. What values would then be possible for T_4?

Solution by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

We show the possible values of T_4 are $\dfrac{13}{3}$ and $-\dfrac{37}{3}$.

For convenience, let $T_5=\dfrac{a+b}{c}+\dfrac{b+c}{a}+\dfrac{a+c}{b}$.
Note that

$$\begin{aligned} T_4 &= \frac{b+c-a}{a}+\frac{a+c-b}{b}+\frac{a+b-c}{c} \\ &= \frac{b+c}{a}-1+\frac{a+c}{b}-1+\frac{a+b}{c}-1 \end{aligned}$$

$$= \quad T_5 - 3.$$

Now we expand and simplify:

$$\begin{aligned}
T_2 &= (a+b)(a+c)(b+c) = a^2b + abc + ab^2 + b^2c + a^2c + ac^2 + abc + bc^2 \\
&= \frac{a}{c}T_3 + 2T_3 + \frac{b}{c}T_3 + \frac{b}{a}T_3 + \frac{a}{b}T_3 + \frac{c}{b}T_3 + \frac{c}{a}T_3 \\
&= T_3\left(2 + \frac{a+b}{c} + \frac{b+c}{a} + \frac{a+c}{b}\right) \\
&= T_3\left(2 + T_5\right).
\end{aligned}$$

Therefore, $\dfrac{T_2}{T_3} = T_5 + 2.$

Similarly, $T_1 = T_3T_5$, so $\dfrac{T_1}{T_3} = T_5.$

Therefore, $\dfrac{616}{9} = \dfrac{T_1 \cdot T_2}{(T_3)^2} = \dfrac{T_1}{T_3}\dfrac{T_2}{T_3} = T_5\left(T_5 + 2\right).$

Hence,

$$\begin{aligned}
T_5^2 + 2T_5 - \frac{616}{9} &= 0 \\
\left(T_5 + \frac{28}{3}\right)\left(T_5 - \frac{22}{3}\right) &= 0. \text{ Thus,} \\
T_5 = -\frac{28}{3} \quad \text{or} &\quad T_5 = \frac{22}{3}, \text{ so,} \\
T_4 = -\frac{28}{3} - 3 = -\frac{37}{3} \quad \text{or} &\quad T_4 = \frac{22}{3} - 3 = \frac{13}{3}.
\end{aligned}$$

Comment: The question still unanswered—do there exist values of a, b, and c which make all of this happen?

Editor's remark: The above question was answered by **Albert Stadler of Herrliberg, Switzerland.** *In his solution to this problem he stated that both values obtained for T_4 are actually assumed: for instance for* $(a,b,c) = \left(1, 1, \dfrac{-17+\sqrt{253}}{6}\right)$ *and for* $(a,b,c) = \left(1, 1, \dfrac{4+\sqrt{7}}{3}\right)$.

Also solved by Arkady Alt, San Jose, CA; Brian D. Beasley, Presbyterian College, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX; Ben Carani, Jordan Melendez, Caleb Stevenson (students at Taylor University), Upland, IN; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney Australia and Elton Bojaxhiu, Kriftel, Germany; Samuel David Judge, Justin Wydra, and Karen Wydra (students at Taylor University), Upland, IN; Kee-Wai Lau, Hong, Kong, China; Adrian Naco,

Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics "Tor Vergata University," Rome, Italy; Jungmin Song, Nate Armstrong and Alex Senyshyn (students at Taylor University), Upland IN; Howard Sporn, Great Neck, NY, and the proposer.

- **5208:** *Proposed by D. M. Bătinetu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania*

Let the sequence of positive real numbers $\{a_n\}_{n\geq 1}$, $N \in Z^+$ be such that $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{n^2 \cdot a_n} = b$. Calculate:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{n+1} - \frac{\sqrt[n]{a_n}}{n}\right).$$

Solution 1 by Anastasios Kotronis, Athens, Greece

Setting $z_n := \frac{a_n}{n^{2n}}$, we have

$$\frac{z_{n+1}}{z_n} = \frac{a_{n+1}}{n^2 a_n}\left[\left(1+\frac{1}{n}\right)^n\right]^{-2}\left(1+\frac{1}{n}\right)^{-2} \to be^{-2}, \tag{1}$$

and by Cesàro Stolz:

$$\begin{aligned}
\lim_{n\to+\infty} z_n^{1/n} &= \exp\left(\lim_{n\to+\infty}\frac{\ln z_n}{n}\right)\\
&= \exp\left(\lim_{n\to+\infty}\ln\frac{z_{n+1}}{z_n}\right)\\
&= \exp\left(\ln\lim_{n\to+\infty}\frac{z_{n+1}}{z_n}\right)\\
&= be^{-2}.
\end{aligned}\tag{2}$$

On account of (1) and (2):

$$\left(\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}}\right)^n = \left(1+\frac{1}{n}\right)^n \frac{z_{n+1}}{z_n} z_{n+1}^{-\frac{1}{n+1}} \to e,$$

so

$$\frac{\sqrt[n+1]{a_{n+1}}}{n+1} - \frac{\sqrt[n]{a_n}}{n} = z_n^{1/n}\left(\frac{\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}} - 1}{\ln\left(\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}}\right)}\ln\left(\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}}\right)^n\right) \to be^{-2},$$

since

$$\lim_{n\to+\infty}\frac{\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}} - 1}{\ln\left(\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}}\right)} = \lim_{n\to+\infty}\frac{\exp\left(\ln\left(\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}}\right)\right) - 1}{\ln\left(\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}}\right)}$$

$$= \lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

Solution 2 by proposers

We have

(1) $\lim\limits_{n\to\infty} \dfrac{\sqrt[n]{a_n}}{n^2} = \lim\limits_{n\to\infty} \sqrt[n]{\dfrac{a_n}{n^{2n}}} = \lim\limits_{n\to\infty} \dfrac{a_{n+1}}{(n+1)^{2n+2}} \cdot \dfrac{n^{2n}}{a_n} = \lim\limits_{n\to\infty} \dfrac{a_{n+1}}{(n+1)^2 a_n} \cdot \dfrac{1}{e_n^2} = \dfrac{b}{e^2}$, where $e_n = \left(1 + \frac{1}{n}\right)^n$. (The second equality in the chain follows from the Cauchy-D'Alembert criteria.)

(2) Denote $u_n = \dfrac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} \cdot \dfrac{n}{n+1}, \forall n \geq 2$ and we deduce that

(3) $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \left(\dfrac{\sqrt[n+1]{a_{n+1}}}{(n+1)^2} \cdot \dfrac{n^2}{\sqrt[n]{a_n}} \cdot \dfrac{n+1}{n} \right) = \dfrac{b}{e^2} \cdot \dfrac{e^2}{b} \cdot 1 = 1$, respectively

(4) $\lim\limits_{n\to\infty} \dfrac{u_n - 1}{\ln u_n} = 1.$

(5) $\lim\limits_{n\to\infty} u_n^n = \lim\limits_{n\to\infty} \left(\dfrac{a_{n+1}}{a_n} \cdot \dfrac{1}{\sqrt[n+1]{a_{n+1}}} \cdot \left(\dfrac{n}{n+1}\right)^n \right) =$
$\lim\limits_{n\to\infty} \left(\dfrac{a_{n+1}}{n^2 a_n} \cdot \dfrac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} \cdot \dfrac{1}{e_n} \cdot \left(\dfrac{n}{n+1}\right)^2 \right) = b \cdot \dfrac{e^2}{b} \cdot \dfrac{1}{e} \cdot 1 = e.$

(6) Denote $x_n = \left(\dfrac{\sqrt[n+1]{a_{n+1}}}{n+1} - \dfrac{\sqrt[n]{a_n}}{n} \right) = \dfrac{\sqrt[n]{a_n}}{n} \cdot \left(\dfrac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} \cdot \dfrac{n}{n+1} - 1 \right) =$
$\dfrac{\sqrt[n]{a_n}}{n} (u_n - 1) = \dfrac{\sqrt[n]{a_n}}{n} \cdot \dfrac{u_n - 1}{\ln u_n} \cdot \ln u_n = \dfrac{\sqrt[n]{a_n}}{n^2} \cdot \dfrac{u_n - 1}{\ln u_n} \cdot \ln u_n^n.$

By (1), (4), (5) and (6) we obtain
(7) $L = \lim\limits_{n\to\infty} \left(\dfrac{\sqrt[n+1]{a_{n+1}}}{n+1} - \dfrac{\sqrt[n]{a_n}}{n} \right) = \lim\limits_{n\to\infty} x_n = \dfrac{b}{e^2} \cdot 1 \cdot \ln e = \dfrac{b}{e^2}.$

Also solved by Arkady Alt, San-Jose, CA; Kee-Wai Lau, Hong Kong, China; Adrian Naco, Polytechnic University, Tirana, Albania; and Albert Stadler, Herrliberg, Switzerland.

- **5209:** *Proposed by Tom Moore, Bridgewater, MA*

  We noticed that 27 is a cube and 28 is an even perfect number. Find all pairs of consecutive integers such that one is cube and the other is an even perfect number.

  Solution by Kee-Wai Lau, Hong Kong, China

  We show that 27 and 28 are the only consecutive integers such that one is cube and the other is an even perfect number.

  It is well known that every even perfect number is of the form $2^{p-1}(2^p - 1)$, where $2^p - 1$ is a prime. Suppose $2^{p-1}(2^p - 1) = a^3 + 1$, where a is an odd integer, then since

$a^3+1=(a+1)(a^2-a+1)$, we have $a+1=2^{p-1}$ and $a^2-a+1=2^p-1$. Hence $a^2-a+1=2a+1$ or $a=3$. This gives the pair 27 and 28.

Next we suppose that $2^{p-1}\left(2^p-1\right)=b^3-1$, where b is an odd integer, then since $b^3-1=(b-1)\left(b^2+b-1\right)$, we have $b-1=2^{p-1}$ and $b^2+b+1=2^p-1$. Hence $b^2+b+1=2b-3$ or $b^2-b+4=0$, which gives no real solutions.

This completes the solution.

Also solved by Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo Sate University, San Angelo TX; Brian D. Beasley, Presbyterian College, Clinton, SC; Farideh Firoozbakht and Jahangeer Kholdi (jointly), Isfahan, Iran; Paul M. Harms, North Newton, KS; David E. Manes, SUNY College at Oneonta, Oneonta, NY; David Stone and John Hawkins (jointly), Georgia Southern University, Statesboro, GA; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5210:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b, c, d be four positive real numbers. Prove that

$$1+\frac{1}{8}\left(\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\right)>\frac{2\sqrt{3}}{3}.$$

Solution by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Andrew Siefker, Angelo State University, San Angelo, TX

We will establish the slightly improved inequality

$$1+\frac{1}{8}\left(\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\right)>\frac{7}{6}.$$

This is a little better than the given result because

$$\frac{7}{6}-\frac{2\sqrt{3}}{3}=\frac{7-4\sqrt{3}}{6}=\frac{1}{6\left(7+4\sqrt{3}\right)}>0.$$

We begin with the following known inequality:
If $x_1, x_2, \ldots, x_n>0$, then

$$\left(x_1+x_2+\ldots+x_n\right)\left(\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}\right)\geq n^2. \qquad (1)$$

This follows from applying the Cauchy-Schwarz Inequality to the vectors $\mathbf{x}=\left(\sqrt{x_1}, \sqrt{x_2}, \ldots, \sqrt{x_n}\right)$ and $\mathbf{y}=\left(\frac{1}{\sqrt{x_1}}, \frac{1}{\sqrt{x_2}}, \ldots, \frac{1}{\sqrt{x_n}}\right)$.

If we let $x_1=a+b+c$, $x_2=b+c+d$, $x_3=c+d+a$, and $x_4=d+a+b$, then since $a, b, c, d>0$, statement (1) implies that

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}$$

$$\begin{aligned}
&> \frac{a}{b+c+d}+\frac{b}{c+d+a}+\frac{c}{d+a+b}+\frac{d}{a+b+c}\\
&= \left(\frac{a}{b+c+d}+1\right)+\left(\frac{b}{c+d+a}+1\right)+\left(\frac{c}{d+a+b}+1\right)+\left(\frac{d}{a+b+c}+1\right)-4\\
&= (a+b+c+d)\left(\frac{1}{a+b+c}+\frac{1}{b+c+d}+\frac{1}{c+d+a}+\frac{1}{d+a+b}\right)-4\\
&= \frac{1}{3}(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)-4\\
&\geq \frac{16}{3}-4\\
&= \frac{4}{3}.
\end{aligned}$$

Therefore,

$$1+\frac{1}{8}\left(\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\right)>1+\frac{1}{8}\left(\frac{4}{3}\right)=\frac{7}{6},$$

and our proof is complete.

Comments: **Kee-Wai Lau of Hong Kong, China** remarked that D.S. Mitrinović $\Big($Analytic Inequalities, Springer Verlag (1970; p. 132)$\Big)$ and L. J. Mordell $\Big($On the inequality $\sum x_r/(x_{r+}+x_{r+2})\geq\frac{1}{2}n$ Abh. Math. Sem. Univ. Hamburg 22, (1958; pp 229-241)$\Big)$ shown that

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\geq 2.$$

So the present problem can be sharpened to

$$1+\frac{1}{8}\left(\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\right)\geq\frac{5}{4}.$$

Albert Stadler of Herrliberg, Switzerland noted that the problem statement is a generalization of Nesbitt's inequality to four variables (see http://en.wikipedia.org/wiki/Nesbitt's_inequality). However this generalization is well known: see e.g., Pham Kim Hung's text "Secrets in Inequalities" (GIL Publishing House 2007.) Albert also noted that the inequality can be sharpened to ≥ 1.25, and he presented the proof in Kim Hung's text.

Prove that for all non-negative real numbers a,b,c,d,

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\geq 2.$$

Consider the following expressions

$$\begin{aligned}
S &= \frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b};\\
M &= \frac{b}{b+c}+\frac{c}{c+d}+\frac{d}{d+a}+\frac{a}{a+b};\\
N &= \frac{c}{b+c}+\frac{d}{c+d}+\frac{a}{d+a}+\frac{b}{a+b};
\end{aligned}$$

We have $M+N=4$. According to AM-GM, we get

$$\begin{aligned} M{+}S &= \frac{a+b}{b+c}+\frac{b+c}{c+d}+\frac{c+d}{d+a}+\frac{d+a}{a+b} \geq 4; \\ N{+}S &= \frac{a+c}{b+c}+\frac{b+d}{c+d}+\frac{a+c}{d+a}+\frac{b+d}{a+b} \\ &= \frac{a+c}{b+c}+\frac{a+c}{a+d}+\frac{b+d}{c+d}+\frac{b+d}{a+b} \\ &\geq \frac{4(a+c)}{a+b+c+d}+\frac{4(b+d)}{a+b+c+d}=4. \end{aligned}$$

Therefore, $M+N+2S \geq 8$, and $S \geq 2$. The equality holds if $a=b=c=d$ or $a=c, b=d=0$ or a=c=0, b=d.

Also solved by Arkady Alt, San Jose, CA; D.M. Bătinetu-Giurgiu, Bucharest, Neculai Stanciu Buzău and Titu Zvonaru Comănesti, all from Romania (two solutions); Bruno Salgueiro Fanego, Viveiro, Spain; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain, and the proposer.

- **5211:** *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

Let $n \geq 1$ be a natural number and let

$$f_n(x) = x^{x^{\cdot^{\cdot^{\cdot^{x}}}}},$$

where the number of x's in the definition of f_n is n. For example

$$f_1(x)=x, \quad f_2(x)=x^x, \quad f_3(x)=x^{x^x}, \ldots .$$

Calculate the limit

$$\lim_{x\to 1} \frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

We show the limit equals $(-1)^n$. Define $f_0(x)=1$. For $n \geq 2$ and $x>0$, we have $f_n(x)=e^{f_{n-1}(x)\ln x}$. Hence by the mean value theorem, we have

$$f_n(x)-f_{n-1}(x) = \ln x \left(f_{n-1}(x)-f_{n-2}(x)\right) e^{\xi},$$

where ξ lies between $f_{n-1}(x)\ln x$ and $f_{n-2}(x)\ln x$.

Since $\lim\limits_{x\to 1} f_{n-1}(x)\ln x = \lim\limits_{x\to 1} f_{n-2}(x)\ln x = 0$ and $\lim\limits_{x\to 1} \dfrac{\ln x}{1-x} = -1$, so

$$\lim_{x\to 1} \frac{f_n(x)-f_{n-1}(x)}{(1-x)^n} = -\lim_{x\to 1} \frac{f_{n-1}(x)-f_{n-2}(x)}{(1-x)^{n-1}}.$$

Clearly $\lim_{x\to 1} \frac{f_1(x)-f_0(x)}{1-x} = -1$. Hence by induction we have

$$\lim_{x\to 1} \frac{f_n(x)-f_{n-1}(x)}{(1-x)^n} = (-1)^n,$$

as claimed.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We will use induction to prove that $a_n = \lim_{x\to 1} \frac{f_n(x)-f_{n-1}(x)}{(1-x)^n} = (-1)^n$.
We have by applying L'Hôpital's rule twice,

$$a_2 = \lim_{x\to 1} \frac{f_2(x)-f_1(x)}{(1-x)^2} = \lim_{x\to 1} \frac{x^x - x}{(1-x)^2} = \lim_{x\to 1} \frac{x^x(1+\log x)-1}{-2(1-x)} = \lim_{x\to 1} \frac{x^x\left[(1+\log x)^2+\frac{1}{x}\right]}{2} = 1.$$

So the assertion holds for $n=2$.
We have $\frac{d}{dx}f_n(x) = \frac{d}{dx}e^{f_{n-1}(x)\log x} = f_n(x)\left(f'_{n-1}(x)\log x + \frac{f_{n-1}(x)}{x}\right)$. In particular,

$$f'_n(1) = f_n(1)\left(f'_{n-1}(1)\log(1) + \frac{f_{n-1}(1)}{1}\right) = 1.$$

So, by L'Hôpital's rule,

$$\begin{aligned}
a_n &= \lim_{x\to 1} \frac{f_n(x)-f_{n-1}(x)}{(1-x)^n} = \lim_{x\to 1} \frac{f'_n(x)-f'_{n-1}(x)}{-n(1-x)^{n-1}} \\
&= \lim_{x\to 1} \frac{f_n(x)\left(f'_{n-1}(x)\log x + \frac{f_{n-1}(x)}{x}\right) - f_{n-1}(x)\left(f'_{n-2}(x)\log x + \frac{f_{n-2}(x)}{x}\right)}{-n(1-x)^{n-1}} \\
&= \lim_{x\to 1} \frac{\left(f_n(x)-(f_{n-1}(x)\right)\left(f'_{n-1}(x)\log x + \frac{f_{n-1}(x)}{x}\right)}{-n(1-x)^{n-1}} \\
&+ \lim_{x\to 1} \frac{\left(f_{n-1}(x)\left(f'_{n-1}(x)\log x + \frac{f_{n-1}(x)}{x} - f'_{n-2}(x)\log x - \frac{f_{n-2}(x)}{x}\right)\right.}{-n(1-x)^{n-1}}.
\end{aligned}$$

So

$$\begin{aligned}
a_n &= \lim_{x\to 1} \frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}\left(1+\frac{\left(f'_{n-1}(x)\log x + \frac{f_{n-1}(x)}{x}\right)(1-x)}{n}\right) \\
&= \lim_{x\to 1} \frac{f_{n-1}(x)\left(f'_{n-1}(x)\log x + \frac{f_{n-1}(x)}{x} - f'_{n-2}(x)\log x - \frac{f_{n-2}(x)}{x}\right)}{-n(1-x)^{n-1}}
\end{aligned}$$

$$= \lim_{x\to 1} \frac{f'_{n-1}(x) - f'_{n-2}(x)}{-n(1-x)^{n-2}} \cdot \frac{\log x}{1-x} + \lim_{x\to 1} \frac{f_{n-1}(x) - f_{n-2}(x)}{-n(1-x)^{n-1}x}$$

$$= \frac{n-1}{n} \lim_{x\to 1} \frac{f_{n-1}(x) - f_{n-2}(x)}{(1-x)^{n-1}} \cdot (-1) + \frac{1}{(-n)} \lim_{x\to 1} \frac{f_{n-1}(x) - f_{n-2}(x)}{(1-x)^{n-1}}$$

$$= -a_{n-1} = -(-1)^{n-1} = (-1)^n.$$

Solution 3 by Adrian Naco, Polytechnic University, Tirana, Albania

At first we observe that the function is of the form

$$f_n(x) = x^{f_{n-1}(x)} = e^{f_{n-1}(x)\ln x}$$

and that

$$\begin{aligned}\frac{f_n(x) - f_{n-1}(x)}{(1-x)^n} &= \frac{e^{f_{n-1}(x)\ln x} - e^{f_{n-2}(x)\ln x}}{(1-x)^n} = e^{f_{n-2}(x)\ln x}\left(\frac{e^{[f_{n-1}(x)-f_{n-2}(x)]\ln x} - 1}{(1-x)^n}\right) \\ &= e^{f_{n-2}(x)\ln x} \cdot \frac{[f_{n-1}(x) - f_{n-2}(x)]\ln x}{(1-x)^n} \cdot \frac{e^{[f_{n-1}(x)-f_{n-2}(x)]\ln x} - 1}{[f_{n-1}(x) - f_{n-2}(x)]\ln x}. \qquad (2)\end{aligned}$$

The function $f_1(x) = x$ is continuous everywhere for $x > 0$ and

$$\lim_{x\to 1} f_1(x) = \lim_{x\to 1} x = 1.$$

One easily comes to the conclusion that the function $f_n(x) = e^{f_{n-1}(x)\ln x}$ is continuous everywhere for $x > 0$ as a composition of a product of two continuous functions $u(x) = f_{n-1}(x)\ln x$ and the exponential function $f_n(x) = e^{u(x)}$ and as a logical result implies that

$$\lim_{x\to 1} f_n(x) = e^{\lim_{x\to 1} [f_{n-1}(x)\ln x]} = e^{\left[\lim_{x\to 1} f_{n-1}(x)\right]\cdot\left[\lim_{x\to 1} \ln x\right]} = e^{1\cdot 0} = 1. \quad (3)$$

Using the known limit rule

$$\lim_{\alpha\to 0} \frac{e^\alpha - 1}{\alpha} = 1 \Rightarrow \lim_{x\to 1} \frac{e^{[f_{n-1}(x)-f_{n-2}(x)]\ln x} - 1}{[f_{n-1}(x) - f_{n-2}(x)]\ln x} = 1 \qquad (4)$$

since

$$\begin{aligned}\lim_{x\to 1} \alpha(x) &= \lim_{x\to 1} [f_{n-1}(x) - f_{n-2}(x)]\ln x \\ &= \left[\lim_{x\to 1} f_{n-1}(x) - \lim_{x\to 1} f_{n-2}(x)\right]\left(\lim_{x\to 1} \ln x\right) \\ &= (1-1)\cdot 0 = 0\end{aligned}$$

So from formula (2) and (4) we derive the inductive result for one step.

$$\begin{aligned}
&\lim_{x\to 1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}\\
= &\lim_{x\to 1} e^{f_{n-2}(x)\ln x}\cdot\frac{[f_{n-1}(x)-f_{n-2}(x)]\ln x}{(1-x)^n}\cdot\frac{e^{[f_{n-1}(x)-f_{n-2}(x)]\ln x}-1}{[f_{n-1}(x)-f_{n-2}(x)]\ln x}\\
= &\left(\lim_{x\to 1} e^{f_{n-2}(x)\ln x}\right)\cdot\left(\lim_{x\to 1}\frac{[f_{n-1}(x)-f_{n-2}(x)]\ln x}{(1-x)^n}\right)\cdot\left(\lim_{x\to 1}\frac{e^{[f_{n-1}(x)-f_{n-2}(x)]\ln x}-1}{[f_{n-1}(x)-f_{n-2}\,(x)]\ln x}\right)\\
= &\left(e^{\lim\limits_{x\to 1} f_{n-2}(x)\ln x}\right)\cdot\left(\lim_{x\to 1}\frac{[f_{n-1}(x)-f_{n-2}(x)]\ln x}{(1-x)^n}\right)\cdot\left(\lim_{x\to 1}\frac{e^{[f_{n-1}(x)-f_{n-2}(x)]\ln x}-1}{[f_{n-1}(x)-f_{n-2}\,(x)]\ln x}\right)\\
= &\ 1\cdot\left(\lim_{x\to 1}\frac{[f_{n-1}(x)-f_{n-2}(x)]\ln x}{(1-x)^n}\right)\cdot 1=\lim_{x\to 1}\frac{[f_{n-1}(x)-f_{n-2}(x)]\ln x}{(1-x)^n} \qquad (5)
\end{aligned}$$

Inductively we derive the general formula

$$\begin{aligned}
\lim_{x\to 1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n} &= \lim_{x\to 1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}\ln^0 x\\
&= \lim_{x\to 1}\frac{[f_{n-1}(x)-f_{n-2}(x)]\ln^1 x}{(1-x)^n}\\
&= \lim_{x\to 1}\frac{[f_{n-2}(x)-f_{n-3}(x)]\ln^2 x}{(1-x)^n}\\
&\quad \cdots\cdots\cdots\\
&= \lim_{x\to 1}\frac{[f_2(x)-f_1(x)]\ln^{n-2} x}{(1-x)^n}=\lim_{x\to 1}\frac{[x^x-x]\ln^{n-2} x}{(1-x)^n}\\
&= \lim_{x\to 1}\frac{\left[e^{x\ln x}-e^{\ln x}\right]\ln^{n-2} x}{(1-x)^n}=\lim_{x\to 1}\frac{e^{\ln x}\left[e^{(x-1)\ln x}-1\right]\ln^{n-2} x}{(1-x)^n}\\
&= \lim_{x\to 1} e^{\ln x}\frac{\left[e^{(x-1)\ln x}-1]\right]}{(x-1)\ln x}(x-1)\frac{\ln^{n-1} x}{(1-x)^n}\\
&= (-1)\left(\lim_{x\to 1} e^{\ln x}\right)\left(\lim_{x\to 1}\frac{\left[e^{(x-1)\ln x}-1]\right]}{(x-1)\ln x}\right)\left[\lim_{x\to 1}\frac{\ln x}{(1-x)}\right]^{n-1}\\
&= (-1)\cdot e^0\cdot 1\cdot\left[\lim_{x\to 1}\frac{\ln x}{(1-x)}\right]^{n-1}=-\left[\lim_{x\to 1}\frac{\ln x}{(1-x)}\right]^{n-1}.
\end{aligned}$$

Applying L'Hôpital's rule we have that

$$\begin{aligned}
\lim_{x\to 1} \frac{f_n(x) - f_{n-1}(x)}{(1-x)^n} &= -\left[\lim_{x\to 1} \frac{\ln x}{(1-x)}\right]^{n-1} = -\left[\lim_{x\to 1} \frac{(\ln x)'}{(1-x)'}\right]^{n-1} \\
&= -\left[\lim_{x\to 1} \frac{\frac{1}{x}}{(-1)}\right]^{n-1} = (-1)(-1)^{n-1} = (-1)^n.
\end{aligned}$$

Editor's comment: There was a mistake in the statement of the problem when it first appeared on the web. That version asked for the $\lim_{x\to 1} \dfrac{f_n(x) - f_{n-1}(x)}{(1-x)^{n+1}}$. *This mistake was corrected almost immediately but not before a few of the readers started working with the incorrect statement of the problem; although those readers noted the error and corrected it in their solutions, once again, mea culpa. Most all who submitted solutions to this problem approached it with induction.*

Also solved by Arkady Alt, San Jose, CA; Anastasios Kontronis, Athens, Greece; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy, and the proposer.