Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before January 15, 2010

- 5080: *Proposed by Kenneth Korbin, New York, NY*

  If p is a prime number congruent to 1 (mod4), then there are positive integers a, b, c, such that
  $$\arcsin\left(\frac{a}{p^3}\right)+\arcsin\left(\frac{b}{p^3}\right)+\arcsin\left(\frac{c}{p^3}\right)=90^o.$$
  Find a, b, and c if $p = 37$ and if $p = 41$, with $a < b < c$.

- 5081: *Proposed by Kenneth Korbin, New York, NY*

  Find the dimensions of equilateral triangle ABC if it has an interior point P such that $\overline{PA} = 5, \overline{PB} = 12$, and $\overline{PC} = 13$.

- 5082: *Proposed by David C. Wilson, Winston-Salem, NC*

  Generalize and prove:
  $$\begin{aligned}
  \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)} &= 1-\frac{1}{n+1}\\
  \frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots+\frac{1}{n(n+1)(n+2)} &= \frac{1}{4}-\frac{1}{2(n+1)(n+2)}\\
  \frac{1}{1\cdot 2\cdot 3\cdot 4}+\frac{1}{2\cdot 3\cdot 4\cdot 5}+\cdots+\frac{1}{n(n+1)(n+2)(n+3)} &= \frac{1}{18}-\frac{1}{3(n+1)(n+2)(n+3)}\\
  \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\cdots+\frac{1}{n(n+1)(n+2)(n+3)(n+4)} &= \frac{1}{96}-\frac{1}{4(n+1)(n+2)(n+3)(n+4)}
  \end{aligned}$$

- 5083: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let $\alpha > 0$ be a real number and let $f : [-\alpha, \alpha] \to \Re$ be a continuous function two times derivable in $(-\alpha, \alpha)$ such that $f(0) = 0$ and f'' is bounded in $(-\alpha, \alpha)$. Prove that the sequence $\{x_n\}_{n\geq 1}$ defined by
  $$x_n = \begin{cases} \displaystyle\sum_{k=1}^{n} f\left(\frac{k}{n^2}\right), & n > \dfrac{1}{\alpha};\\ 0, & n \leq \dfrac{1}{\alpha}\end{cases}$$
  is convergent and determine its limit.

- 5084: *Charles McCracken, Dayton, OH*

  A natural number is called a "repdigit" if all of its digits are alike.
  Prove that regardless of positive integral base b, no natural number with two or more digits when raised to a positive integral power will produce a repdigit.

- 5085: *Proposed by Valmir Krasniqi, (student, Mathematics Department,) University of Prishtinë, Kosova*

  Suppose that $a_k,\ (1 \le k \le n)$ are positive real numbers. Let $e_{j,k} = (n-1)$ if $j = k$ and $e_{j,k} = (n-2)$ otherwise. Let $d_{j,k} = 0$ if $j = k$ and $d_{j,k} = 1$ otherwise.

  Prove that
  $$\prod_{j=1}^{n}\sum_{k=1}^{n} e_{j,k}a_k^2 \ge \prod_{j=1}^{n}\left(\sum_{k=1}^{n} d_{j,k}a_k\right)^2.$$

Solutions

- 5062: *Proposed by Kenneth Korbin, New York, NY.*

  Find the sides and the angles of convex cyclic quadrilateral ABCD if
  $\overline{AB} = \overline{BC} = \overline{CD} = \overline{AD} - 2 = \overline{AC} - 2$.

  Solution 1 by David E. Manes, Oneonta, NY

  Let $x = \overline{AB} = \overline{BC} = \overline{CD}$ and let $y = \overline{BD}$. Then $\overline{AD} = \overline{AC} = x + 2$.

  Let $\alpha = \angle CAB,\ \beta = \angle ABD$, and $\gamma = \angle DBC$. Finally, in quadrilateral $ABCD$, we denote the angle at vertex A by $\angle A$ and similarly for the other three vertices. Then $\overline{AB} = \overline{BC}$ implies $\alpha = \angle BCA$. Since angles inscribed in the same arc are congruent, it follows that

  $$\begin{aligned}\alpha &= \angle CAB = \angle CDA,\\ \alpha &= \angle BCA = \angle BDA,\\ \beta &= \angle ABD = \angle ACD, \text{and}\\ \gamma &= \angle DBC = \angle DAC\end{aligned}$$

  Therefore,

  $$\angle A = \alpha + \gamma,\ \angle B = \beta + \gamma,\ \angle C = \alpha + \beta \text{ and } \angle D = 2\alpha = \beta \text{ since } \overline{AC} = \overline{AD}.$$

  From Ptolemy's Theorem, one obtains

  $$\begin{aligned}\overline{AC}\cdot\overline{BD} &= \overline{AB}\cdot\overline{CD} + \overline{AD}\cdot\overline{BC} \text{ or}\\ (x+2)y &= x^2 + x(x+2)\\ y &= \frac{2x(x+1)}{x+2}.\end{aligned}$$

  In triangles ACD and BCD, the law of cosines implies $\cos\gamma = \dfrac{2(x+2)^2 - x^2}{2(x+2)^2}$ and $\cos\gamma = \dfrac{y}{2x} = \dfrac{x+1}{x+2}$ respectively. Setting the two values equal yields the quadratic equation $x^2 - 2x - 4 = 0$ with positive solution $x = 1 + \sqrt{5}$. Hence,

  $$\overline{AB} = \overline{BC} = \overline{CD} = 1 + \sqrt{5} \text{ and } \overline{AD} = 3 + \sqrt{5}.$$

Moreover, note that

$$\begin{aligned} \cos\gamma &= \frac{x+1}{x+2} = \frac{2+\sqrt{5}}{3+\sqrt{5}} = \frac{1+\sqrt{5}}{4} \text{ implies that} \\ \gamma &= \arccos\left(\frac{1+\sqrt{5}}{4}\right) = 36^0 \end{aligned}$$

In $\triangle ACD, \gamma+\beta+2\alpha = 180^o$ or $\gamma + 2\beta = 180^0$ so that $\beta = \dfrac{180^0 - 36^0}{2} = 72^0$ and $\alpha = \beta/2 = 36^0$.

Therefore,

$$\angle A = \alpha + \gamma = 72^0 = 2\alpha = \angle D \text{ and}$$

$$\angle B = \beta + \gamma = 108^0 = \alpha + \beta = \angle C.$$

Solution 2 by Brian D. Beasley, Clinton, SC

We let $a = \overline{AB}$, $b = \overline{BC}$, $c = \overline{CD}$, $d = \overline{AD}$, $p = \overline{BD}$ and $q = \overline{AC}$. Then $a = b = c = d - 2 = q - 2$. According to the Wolfram MathWorld web site [1], for a cyclic quadrilateral, we have

$$pq = ac + bd \text{ (Ptolemy'sTheorem)} \quad \text{and} \quad q = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}.$$

Thus $a + 2 = \sqrt{2a^2 + 2a}$, so the only positive value of a is $a = 1 + \sqrt{5}$. Hence $a = b = c = 1 + \sqrt{5}$ and $d = p = q = 3 + \sqrt{5}$. Using the Law of Cosines, it is straightforward to verify that $\angle ABC = \angle BCD = 108°$ and $\angle CDA = \angle DAB = 72°$.

[1] Weisstein, Eric W. "Cyclic Quadrilateral." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/CyclicQuadrilateral.html

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

We show that the sides are $1+\sqrt{5}, 1+\sqrt{5}, 1+\sqrt{5}, 3+\sqrt{5}$ and the angles are $108^0,\ 72^0,\ 72^0,\ 108^0$.

Let $\alpha = \overline{AB} = \overline{BC} = \overline{CD} = \overline{AD} - 2 = \overline{AC} - 2, \beta = \angle CBA$ and R the circumradius of $ABCD$.

By solution 1 of SSM problem 4961,

$$R = \frac{1}{4}\sqrt{\frac{[aa + a(a+2)][a(a+2)+aa][aa+a(a+2)]}{(2a+1-a)(2a+1-a)(2a+1-a)[2a+1-(a+2)]}} = \frac{a}{2}\sqrt{\frac{2a}{a-1}}.$$

From this and the generalized sine theorem in $\triangle ABC$,

$$\frac{a}{2R} = \sin\left(\frac{180^0 - \beta}{2}\right) \Longrightarrow \cos\left(\frac{\beta}{2}\right) = \sqrt{\frac{a-1}{2a}}.$$

By the law of cosines in $\triangle ABC$,

$$\cos\beta = \frac{a^2 + a^2 - (a+2)^2}{2a^2} \Longrightarrow \cos\left(\frac{\beta}{2}\right) = \sqrt{\frac{1+\cos\beta}{2}} = \frac{\sqrt{3a^2 - 4a - 4}}{2a}.$$

Hence,

$$\sqrt{\frac{a-1}{2a}} = \frac{\sqrt{3a^2-4a-4}}{2a} \Longrightarrow a^2-2a-4=0 \Longrightarrow a = 1+\sqrt{5} = 2\phi,$$

so the sides are

$$\overline{AB} = \overline{BC} = \overline{CD} = 1+\sqrt{5} \text{ and } \overline{AD} = a+\mathit{2} = \mathit{3}+\sqrt{\mathit{5}}.$$

Then $\beta = 2\arccos\sqrt{\dfrac{\sqrt{5}}{2(1+\sqrt{5})}} = 108^0$, so the angles are $\angle CBA = 108^0,\ \angle DCB = \angle CBA = 108^0,\ \angle ADC = 180^0 - 108^0 = 72^0$ and $\angle BAD = 72^0$.

Also solved by Michael Brozinsky, Central Islip, NY; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Charles McCracken, Dayton, OH; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5063: *Proposed by Richard L. Francis, Cape Girardeau, MO.*

Euclid's inscribed polygon is a constructible polygon inscribed in a circle whose consecutive central angle degree measures form a positive integral arithmetic sequence with a non-zero difference.

a) Does Euclid's inscribed n-gon exist for any prime n greater than 5?

b) Does Euclid's n-gon exist for all composite numbers n greater than 2?

Solution by Joseph Lupton, Jacob Erb, David Ebert, and Daniel Kasper, students at Taylor University, Upland, IN

a) For an inscribed polygon to fit this description, there has to be an arithmetic sequence of positive integers where the number of terms in the sequence is equal to the number of sides of the polygon and the terms sum to 360. So if the first term is f and the constant difference between the terms is d, the sum of the terms is

$$f\cdot n + \frac{n(n-1)}{2}d = 360.$$

Thus, $f\cdot n + \dfrac{n(n-1)}{2}d = 360 \Longrightarrow n\Big|360$. That is, n is a prime number greater than five and $n\Big|2^3\cdot 3^3\cdot 5$. But there is no prime number greater than five that divides 360. So there is no Euclidean polygon that can be inscribed in a circle whose consecutive central angle degree measures form a positive integral arithmetic sequence with a non-zero difference.

b) Euclid's inscribed n-gon does not exist for all composite numbers greater than two. Obviously, if n gets too large, then the terms $\frac{n(n-1)}{2}d$ will be greater than 360 even if $d=1$ which is the minimal d allowed. There is no Eculidean inscribed n-gon for $n=21$. If there were, the the sum of central angles would be $f\cdot n + n\cdot d\cdot\dfrac{n-1}{2}$ implies that 21 divides 360. Similarly, there is no 14-gon for if there were, it would imply that 7 divides 360.

- **Comments and elaborations by David Stone and John Hawkins, Statesboro GA**

We note that this problem previously appeared as part of Problem 4708 in this journal in March, 1999; however the solution was not published. Also, a Google search on the internet turned up a paper by the proposer in the Bulletin of the Malaysian Mathematical Sciences Society in which the answer to both questions is presented as being "no". {See "The Euclidean Inscribed Polygon" (Bull. Malaysian Math Sc. Soc (Second series) 27 (2004), 45-52).}

David and John solved the problem and then elaborated on it by considering the possibility that the inscribed polygon many not enclose the center of the circle. And it is here that things start to get interesting.

(In the case where the inscribed polygon does not include the center of the circle, and letting a be the first term in the arithmetic sequence and d the common difference, they noted that the largest central angle must be the sum of the previous $n-1$ central angles, and they proceeded as follows:)

$$\begin{aligned} a+(n-1)d = S_{n-1} &= \frac{n-1}{2}\Big(2a+(n-2)d\Big) \text{ or} \\ 2a+2(n-1)d &= 2a(n-1)+(n-1)(n-2)d \text{ or} \\ 2a(n-2) &= -(n-1)(n-4)d. \end{aligned}$$

For $n=3$, this happens exactly when $a=d$; although $n=3$ is of no concern for the stated problem, we shall return to this case later.

For $n \geq 4$, this condition is never satisfied because the left-hand side is positive and the right-hand side ≤ 0.

David and John then determined all Euclidean inscribed n-gons as follows:

The cited paper by the poser points out that 3^0 is the smallest constructible angle of positive integral degree. In fact, it is well known that an angle is constructible if, and only if, its degree measure is an integral multiple of 3^0. This implies that a and d must both be multiples of 3. We wish to find all solutions of the Diophantine equation

(1) $\quad n(2a+(n-1)d) = 2^4 \cdot 3^2 \cdot 5$, where a and d are multiples of 3.

Letting $a=3A$ and $d=3D$, the above equation becomes

(2) $\quad n\Big(2A+(n-1)D\Big) = 2^4 \cdot 3 \cdot 5 = 240$, so n must be a divisor of 240.

Moreover, the cofactor $2A+(n-1)D$ is bounded below. That is

$$\begin{aligned} 2A+(n-1)D &\geq 2+(n-1) = n+1. \text{ So} \\ \frac{240}{n} &= 2A+(n-1)D \geq 1, \text{ and} \\ n(n+1) &\leq 240. \end{aligned}$$

These conditions allow only $n = 3,\ 4,\ 5,\ 6,\ 8,\ 10,\ 12,$ and 15.

First we show that $n=12$ fails. For in this case (2) becomes

$$\begin{aligned} 12(2A+11D) = 240, &\text{ or} \\ 2A+11D = 20&, \end{aligned}$$

and this linear Diophantine equation has no positive solutions.

All other possible values of n do produce corresponding Euclidean $n-$gons.

The case $n = 3$ is perhaps the most interesting. There are twenty triangles inscribed in semi-circle: $(3A,\ 6A,\ 9A)$ for $A = 1, 2, \ldots 20$, each having $a = d$, and nineteen more triangles which properly enclose the center of the circle: $(3t, 120, 240 - 3t)$, for $t = 21,\ 22, \ldots,\ 39$, each with $d = 120 - a$.

We consider in detail the case $n = 4$, in which case Equation (2) becomes $4(2A + 3D) = 2^4 \cdot 3 \cdot 5$, or $2A + 3D = 60$. The solution of this Diophantine equation is given by

$$\begin{cases} A = 3t \\ D = 20 - 2t \end{cases}$$

where the integer parameter t satisfies $0 < t < 10$.

We exhibit the results in tabular form, with all angles in degrees:

t	A	$a = 3A$	D	$d = 3D$	*Central angles of inscribed quarilateral*
1	3	9	18	54	9, 63, 117, 171
2	6	18	16	48	18, 66, 114, 162
3	9	27	14	42	27, 69, 111 153
4	12	36	12	36	36, 72, 108, 144
5	15	45	10	30	45, 75, 105, 135
6	18	54	8	24	54, 78, 102, 126
7	21	63	6	18	63, 81, 99, 117
8	24	72	4	12	72, 84, 96, 108
9	27	81	2	6	81, 87, 93, 99

That is, the central angles are $(9t,\ 60 + 3t,\ 120 - 3t,\ 180 - 9t)$ for $t = 1, 2, \ldots, 9$. Thus we have nine Euclidean inscribed quadrilaterals.

Similarly for $n = 5$, we have eleven Euclidean inscribed pentagons, with central angles $(6t,\ 36 + 3t,\ 72,\ 108 - 3t,\ 144 - 6t)$ for $t = 1,\ 2, \ldots,\ 11$.

Similarly for $n = 6$, we have three Euclidean inscribed hexagons, with central angles $(45,\ 51,\ 57,\ 63,\ 75), (30,\ 42,\ 54,\ 66,\ 78,\ 90)$ and $(15,\ 33,\ 52,\ 69,\ 105)$.

For $n = 8$, we have two Euclidean inscribed octagons with central angles $(24,\ 30,\ 36,\ 42,\ 48,\ 54,\ 60,\ 66)$ and $(3,\ 15,\ 27,\ 39,\ 51,\ 63,\ 75,\ 87)$.

For $n = 10$, we have one Euclidean inscribed decagon, with central angles $(9,\ 15,\ 21,\ 27,\ 33,\ 39,\ 45,\ 51,\ 57,\ 63)$.

For $n = 15$, we have one Euclidean inscribed 15-gon with central angles $(3,\ 6,\ 9,\ 12,\ 15,\ 18,\ 21,\ 24\ 27,\ 30,\ 33,\ 36,\ 39,\ 42,\ 45)$.

There is a grand total of 66 Euclidean inscribed n-gons!

A final note: If $n(n + 1)$ divides 240, then $a = d = 3\dfrac{240}{n(n+1)} = \dfrac{720}{n(n+1)}$ produces a Euclidean inscribed $n-$gon.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Boris Rays, Brooklyn, NY, and the proposer.

- 5064: *Proposed by Michael Brozinsky, Central Islip, NY.*

The Lemoine point of a triangle is that point inside the triangle whose distances to the three sides are proportional to those sides. Find the maximum value that the constant of proportionality, say λ, can attain.

Solution 1 by David E. Manes, Oneonta, NY

The maximum value of λ is $\sqrt{3}/6$ and is attained when the triangle is equilateral.

Given the triangle ABC let $[ABC]$ denote its area. The distance from the Lemoine point to the three sides are in the ratio $\lambda a,\ \lambda b,\ \lambda c$ where $\lambda = \dfrac{2[ABC]}{a^2+b^2+c^2}$ and a, b, c denote the length of the sides $BC,\ CA$ and AB respectively. Let $\alpha = \angle BAC, \beta = \angle CBA$, and $\gamma = \angle ACB$. Then

$$[ABC] = \frac{1}{2}bc \cdot \sin\alpha = \frac{1}{2}ac \cdot \sin\beta = \frac{1}{2}ab \cdot \sin\gamma.$$

Therefore,

$$a^2+b^2+c^2 \geq ab+bc+ca = [ABC]\left(\frac{1}{\sin\alpha}+\frac{1}{\sin\beta}+\frac{1}{\sin\gamma}\right).$$

The function $f(x) = \dfrac{1}{\sin x}$ is convex on the interval $(0, \pi)$. Jensen's inequality then implies

$$f(\alpha)+f(\beta)+f(\gamma) \geq 3f\left(\frac{\alpha+\beta+\gamma}{3}\right) = 3f\left(\frac{\pi}{3}\right) = \frac{3}{\sin\left(\frac{\pi}{3}\right)} = 2\sqrt{3}$$

with equality if and only if $\alpha = \beta = \gamma = \pi/3$. Therefore, $a^2+b^2+c^2 \geq 4\sqrt{3}\cdot[ABC]$ so that

$$\lambda = \frac{2[ABC]}{a^2+b^2+c^2} \leq \frac{2[ABC]}{4\sqrt{3}\cdot[ABC]} = \frac{\sqrt{3}}{6}$$

with equality if and only if the triangle ABC is equilateral.

Solution 2 by John Nord, Spokane, WA

Without loss of generality we can denote the coordinates of $\triangle ABC$ as $A(0,0), B(1,0), C(b,c)$, the coordinates of the Lemoine point L as (x_1, y_1), the constant of proportionality from L to the sides as λ, the coordinates on AB of the foot of the perpendicular from L to AB as $D(x_1, 0)$, the coordinates on BC of the foot of the perpendicular from L to BC as $E(x_2, y_2)$ and the coordinates on AC of the foot of the perpendicular from L to AC as $F(x_3, y_3)$.

The distance from L to AB equals $LD = \lambda \cdot 1$.
The distance from L to BC equals $LE = \lambda \cdot \sqrt{(1-b)^2+c^2}$ and
The distance from L to AC equals $LF = \lambda \cdot \sqrt{b^2+c^2}$.

The coordinates of E can be found by finding the intersection of LE and BC. That is, by solving:

$$\begin{cases} y = \dfrac{c}{b-1}x + \dfrac{c}{1-b}, \text{ and} \\[2ex] y = \dfrac{1-b}{c}x + y_1 + \dfrac{b-1}{c}x_1. \end{cases}$$

And the coordinates of F can be found by finding the intersection of LF and AC. That is, by solving,

$$\begin{cases} y = \dfrac{c}{b}x \text{ and} \\ y = \dfrac{-b}{c}x + y_1 + \dfrac{b}{c}x_1. \end{cases}$$

Once we have computed (x_2, y_2) and (x_3, y_3) in terms of b, c, x_1 and λ, we apply the distance relationships above. This results in:

$$x_1 = \frac{b + b^2 + c^2}{2(1 - b + b^2 + c^2)} \qquad y_1 = \lambda = \frac{c}{2(1 - b + b^2 + c^2)}.$$

The maximum value of λ is obtained by solving the system of partial derivatives

$$\begin{cases} \dfrac{\partial \lambda}{\partial b} = 0 \\ \dfrac{\partial \lambda}{\partial c} = 0. \end{cases}$$

This yields: $c = \dfrac{\sqrt{3}}{2}$ and $b = \dfrac{1}{2}$. Substituting these values into y_1 above gives $\lambda = \dfrac{\sqrt{3}}{6}$ as the maximum value of the constant of proportionality.

Solution 3 by Charles Mc Cracken, Dayton, OH

The Lemoine point is also the intersection of the symmedians.

The medians of a triangle divide the triangle in two equal areas.

The medians intersect at the centroid, G.

Any point other than G is closer than G to one side of the triangle.

In $\triangle ABC$ let a denote the side (and its length) opposite $\angle A$, b the side opposite $\angle B$, and c the side opposite $\angle C$. Let L denote the Lemoine point.

If the distance from L to side a is λa, then λa less the distance from G to a we call γa.

Similarly for sides b and c.

For $\lambda = \gamma$, L must coincide with G.

This will happen when the medians and symmedians coincide.

This occurs when the triangle is equiangular $(60^0 - 60^0 - 60^0)$ and hence equilateral $(a = b = c)$.

In that case, $\lambda = \dfrac{\sqrt{3}}{6} \equiv 0.289$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain, John Hawkins and David Stone (jointly), Statesboro, GA; Kee-Wai Lau, Hong Kong, China; Tom Leong, Scranton, PA, and the proposer.

- 5065: *Mihály Bencze, Brasov, Romania.*

Let n be a positive integer and let $x_1 \le x_2 \le \cdots \le x_n$ be real numbers. Prove that

$$1) \qquad \sum_{i,j=1}^{n} |(i-j)(x_i - x_j)| = \frac{n}{2} \sum_{i,j=1}^{n} |x_i - x_j|.$$

$$2) \qquad \sum_{i,j=1}^{n} (i-j)^2 = \frac{n^2(n^2-1)}{6}.$$

Solution 1 by Paul M. Harms, North Newton, KS

1) Both summations in part 1) have the same terms for $i > j$ that they have for $i < j$ and have 0 for $i = j$. Equality will be shown for $i > j$.

Each row below is the left summation of part 1) of the problem for $i > j$ and for a fixed j starting with $j = 1$.

$$\begin{array}{l} 1(x_2 - x_1) + 2(x_3 - x_1) + \ldots + (n-1)(x_n - x_1) \\ 1(x_3 - x_2) + 2(x_4 - x_2) + \ldots + (n-2)(x_n - x_2) \\ \qquad\qquad\qquad \vdots \\ 1(x_{n-1} - x_{n-2}) + 2(x_n - x_{_{n-2)}} \\ 1(x_n - x_{n-1}) \end{array}$$

The coefficient of x_1 is $(-1)[1+2+\ldots+(n-1)] = \dfrac{-(n-1)n}{2}$. Note that the coefficient of x_n (looking at the diagonal from lower left to upper right is
$1+2+\ldots+(n-1) = \dfrac{(n-1)n}{2}$.

The coefficient of x_2 is $(-1)[1+2+\ldots+(n-2)]+1 = \dfrac{-(n-2)(n-1)}{2}+1$, where the one is the coefficient of x_2 in row 1.

The coefficient of x_{n-1} is the negative of the coefficient of x_2.

The coefficient of x_r where r is a positive integer less than $\dfrac{n+1}{2}$ is

$$\begin{aligned} (-1)[1+2+\ldots(n-r)]+0+1+\ldots(r-1) &= \frac{(-1)(n-r)(n-r+1)}{2} + \frac{(r-1)r}{2} \\ &= \frac{(-1)n(n-2r+1)}{2} \\ &= (-1)\frac{n}{2}[(n-r)+(1-r)]. \end{aligned}$$

The coefficients of x_r andx_{n+1-r} are the negatives of each other.

If we write out the right summation of part 1) for $i > j$, we can obtain a triangular form like that above except that each coefficient of the difference of the $x's$ is 1. Using the form just explained, the coefficient of x_1 is $(-1)(n-1)$ and the coefficient of x_n along the diagonal is $(n-1)$.

The coefficient of x_2 is $(-1)(n-2)+1$ where the $(+1)$ is the coefficient of x_2 in row 1.

For x_r, where r is a positive integer less than $\dfrac{n+1}{2}$, the coefficient is
$(-1)(n-r)+(r-1)$ where $(r-1)$ comes from the x_r having coefficients of one in each of the first $(r-1)$ rows. The coefficient of x_r on the right side of the inequaity of part 1) is then $\dfrac{n}{2}(-1)[(n-r)+(1-r)]$ which is the same as the left side of the inequality.

Also, the coefficients of x_r and x_{n+1-r} are negative of each other.

2) To show part 2), first consider the summation of each of the three terms $i^2, j^2, -2ij$. For each j, the summation of i^2 from $i=1$ to n is $1^2+2^2+\ldots n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Then the summation of i^2 where both i and j go from 1 to n is $\dfrac{n(n+1)(2n+1)}{6}$. The summation of j^2 is the same value.

The summation of ij is

$$\begin{aligned} 1(1+2+\ldots+n)+2(1+2+\ldots+n)+\ldots+n(1+2+\ldots+n) &= (1+2+\ldots+n)^2 \\ &= \frac{n^2(n+1)^2}{2^2} \end{aligned}$$

The total summation of the left side of part 2) is

$$\begin{aligned} \frac{2n^2(n+1)(2n+1)}{6} - \frac{2n^2(n+1)^2}{2^2} &= n^2(n+1)\left[\frac{2n+1}{3} - \frac{n+1}{2}\right] \\ &= \frac{n^2(n+1)(n-1)}{6}. \end{aligned}$$

Solution 2 by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy

We begin with 1). The result is achieved by a double induction. For $n=1$ there is nothing to say. Let's suppose that 1) holds for any $1 \le n \le m$. For $n=m+1$ the equality reads as

$$\sum_{i,j=1}^{m+1} |i-j|\,|x_i - x_j| =$$
$$\sum_{i,j=1}^{m} |i-j|\,|x_i - x_j| + \sum_{i=1}^{m+1} |i-m-1|\,|x_i - x_{m+1}| + \sum_{j=1}^{m+1} |m+1-j|\,|x_{m+1} - x_j| =$$
$$\frac{m}{2}\sum_{i,j=1}^{m} |x_i - x_j| + 2\sum_{i=1}^{m+1} |i-m-1|\,(x_{m+1} - x_i).$$

(in the second passage the induction hypotheses has been used) and we need it equal to

$$\frac{m+1}{2}\sum_{i,j=1}^{m+1} |x_i - x_j| = \frac{m}{2}\sum_{i,j=1}^{m} |x_i - x_j| + \frac{1}{2}\sum_{i,j=1}^{m} |x_i - x_j| + (m+1)\sum_{i=1}^{m} |x_i - x_{m+1}|.$$

Comparing the two quantities we have to prove

$$2\sum_{i=1}^{m+1}(m+1-i)(x_{m+1} - x_i) = \frac{1}{2}\sum_{i,j=1}^{m} |x_i - x_j| + (m+1)\sum_{i=1}^{m} |x_i - x_{m+1}|$$

or

$$\sum_{i=1}^{m}(x_{m+1} - x_i)(m+1-2i) = \frac{1}{2}\sum_{i,j=1}^{m} |x_i - x_j|$$

or

$$-\sum_{i=1}^{m} x_i(m+1-2i) = \frac{1}{2}\sum_{i,j=1}^{m} |x_i - x_j| \qquad \text{since} \qquad \sum_{i=1}^{m}(m+1-2i) = 0.$$

Here starts the second induction. For $m = 1$ there is nothing to do as well. Let's suppose that the equality holds true for any $1 \leq m \leq r$. For $m = r + 1$ we have to prove that

$$-\sum_{i=1}^{r+1} x_i(r+2-2i) = \frac{1}{2}\sum_{i,j=1}^{r} |x_i - x_j| + \frac{1}{2}\sum_{i=1}^{r+1}(x_{r+1} - x_i) + \frac{1}{2}\sum_{i=1}^{r+1}(x_{r+1} - x_i).$$

which, by using the induction hypotheses is

$$-\sum_{i=1}^{r} x_i(r+1-2i) - \sum_{i=1}^{r} x_i + rx_{r+1} = -\sum_{i=1}^{r} x_i(r+1-2i) + \sum_{i=1}^{r+1}(x_{r+1} - x_i).$$

or

$$-\sum_{i=1}^{r} x_i + rx_{r+1} = (r+1)x_{r+1} - x_{r+1} - \sum_{i=1}^{r} x_i.$$

namely the expected result.

To prove 2) we employ 1) by calculating $\frac{n}{2}\sum_{i,j=1}^{n} |i-j|$. The symmetry of the absolute value yields

$$\frac{n}{2}\sum_{i,j=1}^{n} |i-j| = n\sum_{1\leq i<j\leq n}(j-i) = n\sum_{i=1}^{n}\sum_{j=i+1}^{n}(j-i) = n\sum_{i=1}^{n}\sum_{k=1}^{n-i} k = \frac{n}{2}\sum_{i=1}^{n}(n-i)(n-i+1).$$

The last sum is equal to $\frac{n}{2}\sum_{k=1}^{n-1} k(k+1)$.

In the last step we show that $\sum_{k=1}^{n-1} k(k+1) = \frac{n^3-n}{3}$.

For $n = 1$ both sides are 0. Let's suppose it is true for $1 \leq n \leq m-1$.
For $n = m$ we have

$$\sum_{k=1}^{m-1} k(k+1) + m(m+1) = \frac{m^3-m}{3} + m(m+1) = m(m+1)\frac{m+2}{3} = \frac{(m+1)^3 - (m+1)}{3}.$$

Finally,

$$\frac{n}{2}\frac{n^3-n}{3} = n^2\frac{n^2-1}{6}$$

The proof is complete.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Michael C. Faleski, University Center, MI; Kee-Wai Lau, Hong Kong, China; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5066: *Proposed by Panagiote Ligouras, Alberobello, Italy.*

  Let a, b, and c be the sides of an acute-angled triangle ABC. Let $abc = 1$. Let H be the orthocenter, and let d_a, d_b, and d_c be the distances from H to the sides BC, CA, and AB respectively. Prove or disprove that

$$3(a+b)(b+c)(c+a) \geq 32(d_a + d_b + d_c)^2.$$

Solution by Kee-Wai Lau, Hong Kong, China

We prove the inequality. First we have $(a+b)(b+c)(c+a) \geq (2\sqrt{ab})(2\sqrt{bc})(2\sqrt{ca}) = 8$. Hence it suffices to prove that $d_a + d_b + d_c \leq \dfrac{\sqrt{3}}{2}$. Let s, r, R be respectively the semi-perimeter, in-radius and circumradius of triangle ABC. Let the foot of the perpendicular from A to BC be D and the foot of the perpendicular from B to AC be E so that $\triangle BCE \sim \triangle BHD$. Hence,

$$\begin{aligned} d_a &= \overline{DH} = \frac{(\overline{BD})(\overline{CE})}{\overline{BE}} \\ &= \frac{(c\cos B)(a\cos C)}{c\sin A} = 2R\cos B\cos C, \text{ and similarly,} \\ d_b &= 2R\cos C\cos A \text{ and } d_c = 2R\cos A\cos B. \end{aligned}$$

Therefore, by the well known equality

$$\begin{aligned} \cos A\cos B + \cos B\cos C + \cos C\cos A &= \frac{r^2 + s^2 - 4R^2}{4R^2}, \text{ we have} \\ d_a + d_b + d_c &= \frac{r^2 + s^2 - rR^2}{2R}. \end{aligned}$$

And by a result of J. C. Gerretsen: Ongelijkheden in de Driehoek Nieyw Tijdschr.Wisk. 41(1953), 1-7, we have $s^2 \leq 4R^2 + 4Rr + 3r^2$. Thus

$$d_a + d_b + d_c = \frac{2r(R+r)}{R} \leq 3r,$$

which follows from L. Euler's result that $R \geq 2r$.

It remains to show that $r \leq \dfrac{1}{2\sqrt{3}}$. But this follows from the well known result that $s \geq 3\sqrt{3}r$ and the fact that $1 = abc = 4rsR \geq 4r(3\sqrt{3})r(2r) = 24\sqrt{3}r^3$.

This completes the solution.

Also solved by the proposer.

- 5067: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be complex numbers such that $a + b + c = 0$. Prove that

$$\max\{|a|, |b|, |c|\} \leq \frac{\sqrt{3}}{2}\sqrt{|a|^2 + |b|^2 + |c|^2}.$$

Solution by Tom Leong, Scranton, PA

Since $a+b+c=0$, $|a|$, $|b|$, and $|c|$ form the sides of a (possibly degenerate) triangle. It follows from the triangle inequality that the longest side, $\max\{|a|,|b|,|c|\}$, cannot exceed half of the perimeter, $\frac{1}{2}(|a|+|b|+|c|)$, of the triangle. Using this fact along with the Cauchy-Schwarz inequality gives the desired result:

$$\begin{aligned}
\max\{|a|,|b|,|c|\} &\leq \frac{1}{2}(|a|+|b|+|c|) \\
&= \frac{1}{2}(1\cdot|a|+1\cdot|b|+1\cdot|c|) \\
&\leq \frac{1}{2}\sqrt{1^2+1^2+1^2}\sqrt{|a|^2+|b|^2+|c|^2} \\
&= \frac{\sqrt{3}}{2}\sqrt{|a|^2+|b|^2+|c|^2}.
\end{aligned}$$

Also solved by Brian D. Beasley, Clinton, SC; Michael Brozinsky, Centeral Islip, NY; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Manh Dung Nguyen (student, Special High School for Gifted Students), HUS, Vietnam; Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy; Boris Rays, Brooklyn, NY; Dmitri V. Skjorshammer (student, Harvey Mudd College), Claremont, CA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.