Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before January 15, 2014

- **5271:** *Proposed by Kenneth Korbin, New York, NY*

  Given convex cyclic quadrilateral $ABCD$ with $\overline{AB} = x, \overline{BC} = y$, and $\overline{BD} = 2\overline{AD} = 2\overline{CD}$.

  Express the radius of the circum-circle in terms of x and y.

- **5272:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  The Jacobsthal numbers begin $0, 1, 1, 3, 5, 11, 21, \cdots$ with general term $J_n = \dfrac{2^n - (-1)^n}{3}$, $\forall n \geq 0$. Prove that there are infinitely many Pythagorean triples like $(3, 4, 5)$ and $(13, 84, 85)$ that have "hypotenuse" a Jacobsthal number.

- **5273:** *Proposed by Titu Zvonaru, Comănesti, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania*

  Solve in the positive integers the equation $abcd + abc = (a+1)(b+1)(c+1)$.

- **5274:** *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia*

  Let x, y, z, α be real positive numbers. Show that if

  $$\sum_{cyclic} \frac{(n+1)\, x^3 + nx}{x^2+1} = \alpha$$

  then

  $$\sum_{cyclic} \frac{1}{x} > \frac{9n}{\alpha} - \frac{\alpha}{n} + \frac{9n\alpha}{9n^2+\alpha^2}$$

  where n is a natural number.

- **5275:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Find all real solutions to the following system of equations

$$\left.\begin{array}{r}
\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+x_1}}}}_{n}+\underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2+x_1}}}}_{n}=x_2\sqrt{2},\\
\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+x_2}}}}_{n}+\underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2+x_2}}}}_{n}=x_3\sqrt{2},\\
\ldots\ldots\ldots\ldots\ldots\ldots\\
\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+x_{n-1}}}}}_{n}+\underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2+x_{n-1}}}}}_{n}=x_n\sqrt{2},\\
\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+x_n}}}}_{n}+\underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2+x_n}}}}_{n}=x_1\sqrt{2},
\end{array}\right\}$$

where $n \geq 2$.

- **5276:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

(a) Let $a \in (0,1]$ be a real number. Calculate

$$\int_0^1 a^{\lfloor \frac{1}{x} \rfloor} dx,$$

where $\lfloor x \rfloor$ denotes the integer part of x.

(b) Calculate

$$\int_0^1 2^{-\lfloor \frac{1}{x} \rfloor} dx.$$

Solutions

- **5254:** *Proposed by Kenneth Korbin, New York, NY*

Five different triangles, with integer length sides and with integer area, each have a side with length 169. The size of the angle opposite 169 is the same in all five triangles. Find the sides of the triangles.

Solution 1 by Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Iran

Let a, b and c be the lengths of three sides of the triangles, A is the measure of the angle opposite the side of length 169, and S is the area of triangle. Note that, given the conditions in the hypothesis, $\cos A$ must be a rational number based on the Law of Cosines. We found eleven such triangles $(S, \cos A, a, b, c)$, where $S = \sqrt{(p(p-a)(p-b)(p-c)}$ and $p = \dfrac{a+b+c}{2}$. They are as follows:

1. (2184, 84/85, 105, 169, 272)

2. (8580, 84/85, 169, 264, 425)

3. (18720, 84/85, 169, 425, 576)

4. (26364, 84/85, 169, 520, 663)

5. (30030, 84/85, 169, 561, 700)

6. (62244, 84/85, 169, 855, 952)

7. (65910, 84/85, 169, 884, 975)

8. (73554, 84/85, 169, 943, 1020)

9. (83694, 84/85, 169, 1020, 1073)

10. (90090, 84/85, 169, 1071, 1100)

11. (92274, 84/85, 169, 1092, 1105)

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

We will assume that $a = 169$ and b and c are the other two sides, with $b \le c$. Since b and c are to be integers, the Law of Cosines dictates that $\cos A$ is to be rational. Also, the requirement that each triangle is to have integral area insures that $\sin A$ must be rational (using the formula $Area = \frac{1}{2}bc\sin A$). One way to achieve both and still satisfy $\sin^2 A + \cos^2 A = 1$ is to make

$$\cos A = \frac{x}{z} \quad \text{and} \quad \sin A = \frac{y}{z}$$

for some Pythagorean triple (x, y, z). After experimenting with several triples, we had the best results by choosing

$$\cos A = \frac{84}{85} \quad \text{and} \quad \sin A = \frac{13}{85}.$$

Then, the Law of Cosines yields

$$\begin{aligned} 169^2 &= b^2 + c^2 - 2bc\left(\frac{84}{85}\right) \\ &= c^2 - \frac{168}{85}bc + \left(\frac{84}{85}b\right)^2 + b^2 - \left(\frac{84}{85}b\right)^2 \end{aligned}$$

$$= \left(c - \frac{84}{85}b\right)^2 + \left(\frac{13}{85}b\right)^2,$$

which reduces to

$$(85c - 84b)^2 + (13b)^2 = [(169)(85)]^2.$$

(Note that the assumption $b \le c$ makes $85c - 84b > 0$.)

We now know that $(13b, 85c - 84b, (169)(85))$ must be a Pythagorean triple and hence, there are positive integers k, m, n such that

$$m > n,\ \gcd(m,n) = 1,\ m \not\equiv n \pmod 2,\ \text{and}\ (169)(85) = k\left(m^2 + n^2\right).$$

Then, either $13b = 2kmn$ and $85c - 84b = k(m^2 - n^2)$ or $13b = k(m^2 - n^2)$ and $85c - 84b = 2kmn$. E. g., when $k = (13)(85) = 1,105$, we get $m = 3$ and $n = 2$. When we set $13b = 2(1,105)(3)(2)$ and $85c - 84b = 1,105(3^2 - 2^2)$, we obtain $b = 1,020$ and $c = 1,073$ while the assignment $13b = 1,105(3^2 - 2^2)$ and $85c - 84b = 2(1,105)(3)(2)$ yields $b = 425$ and $c = 576$. Proceeding in this way, we found 11 feasible values for the sides b and c. Each presented an integral area for the triangle and each resulted in $\cos A = \frac{84}{85}$ (by the Law of Cosines). Since $\cos x$ is injective on $[0, \pi]$, each of our solutions produced the same value for $\angle A$. Our results are summarized in the following table.

k	m	n	a	b	c	*Area*
13	33	4	169	264	425	8,580
13	32	9	169	943	1,020	73,554
$5 \cdot 13$	14	5	169	855	952	62,244
$5 \cdot 13$	11	10	169	105	272	2,184
13^2	7	6	169	1,092	1,105	92,274
$13 \cdot 17$	8	1	169	1,071	1,100	90,090
$13 \cdot 17$	7	4	169	561	700	30,030
$5 \cdot 13^2$	4	1	169	520	663	26,364
$5 \cdot 13 \cdot 17$	3	2	169	1,020	1,073	83,694
$5 \cdot 13 \cdot 17$	3	2	169	425	576	18,720
$13^2 \cdot 17$	2	1	169	884	975	65,910

Comment by editor: **David Stone and John Hawkins of Georgian Southern University in Statesboro GA** exhibited two families of triangles satisfying the conditions of the problem. The first family contained 11 triangles with the angle opposite the side of length 169 having a common value of $\cos^{-1}\left(\frac{84}{85}\right) = 8.7974°$. The triangles that they obtained for this family are exhibited in the above solutions. But in their second family they listed 5 additional triangles for which the angle opposite the side of length 169 have a common value of $\cos^{-1}\left(\frac{1517}{1525}\right) = 5.8713°$

They obtained their triangles by denoting the sides of the triangles as $(a, b, 169)$ with $a \ge b$ and the angle θ opposite 169, and then they used the following tools:

1) Law of cosines, $\cos\theta = \dfrac{a^2 + b^2 - 169^2}{2ab}$.

2) Triangle Inequality: $-169 \le b - a \le 169$; thus, for any given value of a it must be that $a - 169 \le a$.

3) Heron's formula: with $s = \dfrac{a+b+169}{2}$, and where $s(s-a)(s-b)(s-169)$ is a perfect square. That is, where $\left[(a+b)^2 - 169^2\right]\left[169^2 - (a+b)^2\right]$ is a perfect square.

4) MATLAB and Excel. They coded nested loops to find values of a and b which satisfy (2) and (3) and then computed $\cos\theta$ by (1). Then they put the results into an Excel file and sorted by $\cos\theta$. From there they said: it was easy to see the families sharing a common angle.

They wrote: For $a, b \le 40,000$ we found 262 triangles with integer sides and integer area and having 169 as a side. In our table, we have only listed the two families containing five or more elements with a common angle opposite 169. For each triangle we also show its area. They then listed the above table and made observations on it. They wrote: the last triangle in the family $(a = 1105, b = 1092, c = 169)$ is a (13,84,85) right triangle magnified by 13. They also noted that two triangles have sides 169 and 425, while two others have two sides of 169 and 1020 (an appearance of the SSA or Ambiguous case from Trigonometry!).

They then listed their second table and made the following comments on it.

a	b	c	$Area$
350	183	169	3276
1037	900	169	47736
1525	1452	169	113256
1582	1525	169	123396
1625	1586	169	131820

Empirically, the common angle (opposite 169) equals

$$\cos^{-1}\left[\frac{350^2 + 183^2 - 169^2}{2\,(350)\,(183)}\right] = \cos^{-1}\left(\frac{1517}{1525}\right) \approx 5.8713°.$$

Comment 1: We did not have complete confidence in trusting floating point arithmetic to give us triangles with an identical angle. For instance, to see that $(272, 105, 169)$ and $(425, 264, 169)$ have the same angle opposite the side of length 169, we must have

$$\frac{272^2 + 105^2 - 169^2}{2(272)(105)} = \frac{425^2 + 264^2 - 169^2}{2(425)(264)}.$$

Cross-multiplying, we can check this with **integer** arithmetic:

$$(425)\,(264)\left[272^2 + 105^2 - 169^2\right] = 6333465600 - (272)\,(105)\left[425^2 + 262^2 - 169^2\right].$$

In each of our families, we checked the first entry again each other triangle to verify true equality of angles.

Comment 2: Our MATLAB file ran a and b up to 40,000, but found no solutions near this peak value. We do not believe that there are any more such triangles (other than the 262 we found.)

Comment 3: There is a nice geometric way to visualize each family of triangles. We explain by focusing on the first group of 11 triangles. Let two rays OA and OB emanate from a vertex O, separated by our angle $\approx 8.7974°$. Starting at O, mark off the "a values" along OA and the "b values" along OB. For instance, designate A_1 as the point 272 units along OA and B_1 the points 105 units along OB. We have drawn our first triangle –the distance A_1 to B_1 across the "wedge" is 169. Similarly we have A_2= 425 and B_2=264, and the distance A_2 to B_2 across the "wedge" is 169.

Eventually, we will draw all eleven of our triangles in the wedge in nested fashion. Because the distance across the wedge will eventually surpass 169, no more triangles are possible. So we have a nice geometric argument that any such family of triangles must be finite. (In fact, by trigonometry, the maximum value for a (and b) to form an isosceles triangle with this angle and bridge 169 is approximately 1101.75. Note that the largest triangle in this family is near this limiting size.)

Finally, note that each of the quadrilaterals $A_iA_jB_jB_i,\ 1 \leq i < j \leq 11$ has integer sides and integer area and a pair of opposing sides equal to 169. For instance, the quadrilateral $A_1A_2B_2B_1$ has sides
$\left(\overline{A_1A_2}, \overline{A_2B_2}, \overline{B_1B_2}, \overline{B_1A_1}\right) = (A_2 - A_1, 169, B_2 - B_1, 169) = (153, 169, 159, 169)$ and area $Area(\triangle A_2OB_2) - (\triangle A_1OB_1) = 8580 - 2184 = 6396$. An almost unimaginable family of 55 such quadrilaterals.

Also solved by Brian E. Beasley, Presbyterian College, Clinton, SC; Bruno Salgueiro Fanego, Viveiro, Spain, and the proposer.

- **5255:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

Let n be a natural number. Let $\phi(n), \sigma(n)$ and $\tau(n)$ be the Euler phi-function, the sum of the different divisors of n and the number of different divisors of n, respectively. Prove:
(a) $\forall n \geq 2,\ \exists$ natural numbers a and b such that $\phi(a) + \tau(b) = n$.
(b) $\forall k \geq 1,\ \exists$ natural numbers a and b such that $\phi(a) + \sigma(b) = 2^k$.
(c) $\forall n \geq 2,\ \exists$ natural numbers a and b such that $\tau(a) + \tau(b) = n$.
(d) $\forall k \geq 1,\ \exists$ natural numbers a and b such that $\sigma(a) + \sigma(b) = 2^k$.
(e) $\forall n \geq 3,\ \exists$ natural numbers a, b and c such that $\phi(a) + \sigma(b) + \tau(c) = n$
(f) $\exists$ infinitely many natural numbers n such that $\phi(\tau(n)) = \tau(\phi(n))$.

Solution 1 by Brian D. Beasley, Presbyterian Colleg, Clinton, SC

(a) Given $n \geq 2$, let $a = 1$ and $b = 2^{n-2}$. Then $\phi(a) = 1$ and $\tau(b) = n - 1$. (Note that we may take $b = p^{n-2}$ for any prime p.)

(b) Given $k \geq 1$, let $a = 1$ and $b = 2^{k-1}$. Then $\phi(a) = 1$ and
$$\sigma(b) = 1 + 2 + 2^2 + \cdots + 2^{k-1} = 2^k - 1.$$

(c) We may use the same a and b as in part (a), since $\tau(1) = \phi(1) = 1$.

(d) We may use the same a and b as in part (b), since $\sigma(1) = \phi(1) = 1$.

(e) Given $n \geq 3$, let $a = b = 1$ and $c = 2^{n-3}$. Then $\phi(a) = \sigma(b) = 1$ and $\tau(c) = n - 2$. (Note that we may take $c = p^{n-3}$ for any prime p.)

(f) Let p be a prime and take $n = 2^{p-1}$. Then
$$\phi(\tau(n)) = \phi(p) = p - 1 \quad \text{and} \quad \tau(\phi(n)) = \tau(2^{p-2}) = p - 1.$$

Since there are infinitely many primes, the result follows.

Solution 2 by Kee-Wai Lau, Hong Kong, China

(a) $\phi(n) + \tau\left(2^{n-\phi(n)-1}\right) = n.$

(b) $\phi(2) + \sigma\left(2^{k-1}\right) = 2^k.$

(c) $\tau(1) + \tau(1) = 2$ and $\tau(2) + \tau\left(2^{n-3}\right) = n$ for $n \geq 3$.

(d) $\sigma(1) + \sigma\left(2^{k-1}\right) = 2^k.$

(e) $\phi(n-1) + \sigma(1) + \tau(2^{n-2-\phi(n-1)}) = n.$

(f) $\phi(\tau(2^{p-1})) = \tau(\phi(2^{p-1}) = p - 1$ for any odd prime p.

Solution 3 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

We will make use of some well-known rules, where p denotes a prime.

$$\phi(p^m) = p^{m-1}(p-1),\ \sigma(2^m) = 2^{m+1} - 1,\ \text{and}\ \tau(p^m) = m + 1.$$

(a) For any prime p, $\phi(n) + \tau\left(p^{n-\phi(n)-1}\right) = \phi(n) + [n - \phi(n)] = n.$

(b) $\phi(1) + \sigma(2^{k-1}) = 1 + \left[2^k - 1\right] = 2^k.$

(c) For any pirme p, and any m with $2 \leq m \leq n$, we have

$$\tau(p^{n-m}) + \tau(p^{m-2}) = (n - m + 1) + (m - 2 + 1) = n.$$

(d) $\sigma(2^{k-1}) + \sigma(1) = \left(2^k - 1\right) + 1 = 2^k.$

(e) For any prime p, $\phi(1) + \sigma(1) + \tau(p^{n-3}) = 1 + 1 + (n - 2) = n.$

(f) For any prime p, $\phi(\tau(2^{p-1})) = \phi(p) = p - 1$ and
$\tau(\phi(2^{p-1})) = \tau(2^{p-2}) = (p-2) + 1 = p - 1$. So $\phi(\tau(2^{p-1})) = \tau(\phi(2^{p-1}))$.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo Sate University, San Angelo, TX; Ed Gray, Highland Beach, FL; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Iran; David E. Manes, SUNY College at Oneonta, Oneonta, NY, and the proposer.

- **5256:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

Let a be a positive integer. Compute:

$$\lim_{n\to\infty} n\left(a - e^{\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{na}}\right).$$

Solution 1 by Ángel Plaza and Kisin Sadarangani, University de Las Palmas, de Gran Canaria, Spain

Let H_n be the nth harmonic number, that is $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$. Note first that $e^{\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{na}} \to a$ when n tends to infinity, because

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{na} &= \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n}} + \frac{\frac{1}{n}}{1+\frac{2}{n}} + \ldots + \frac{\frac{1}{n}}{1+\frac{(a-1)n}{n}} \\
&= \int_0^{a-1} \frac{1}{1+x}\, dx = \ln a.
\end{aligned}$$

The proposed limit may be obtained as follows:

$$\begin{aligned}
\lim_{n\to\infty} n\left(a - e^{\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{na}}\right) &= \lim_{n\to\infty} -an\left(e^{H_{an}-H_n-\ln a} - 1\right) \\
&= \lim_{n\to\infty} -an\left(H_{an} - H_n - \ln a\right) \\
&= \lim_{n\to\infty} -an \cdot \frac{1-a}{2an} = \frac{a-1}{2}.
\end{aligned}$$

Where we have used that $H_n = \ln n + \gamma + \dfrac{1}{2n} - \dfrac{1}{12n^2} + \cdots$, being γ is the Euler-Mascheroni constant. Hence $H_{an} - H_n \sim \ln a + \frac{1}{2an} - \frac{1}{2n} + o\left(\frac{1}{n^2}\right)$.

Solution 2 by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania

The limit equals -2 if $a = 1$ and $\frac{a-1}{2}$ if $a > 1$. First we consider the case when $a = 1$. We have,

$$\lim_{n\to\infty} n\left(1 - e^{\frac{1}{n+1}+\frac{1}{n}}\right) = \lim_{n\to\infty}\left(\frac{1-\exp\left(\frac{2n+1}{n(n+1)}\right)}{\frac{2n+1}{n(n+1)}} \cdot \frac{2n+1}{n+1}\right) = -2.$$

Now we consider the case when $a > 1$. We will be using, in our analysis, the following asymptotic expansion for the nth harmonic number (see 1, [Entry 23 p. 59])

$$1 + \frac{1}{n} + \cdots + \frac{1}{n} = \gamma + \ln n + \frac{1}{2n} - \frac{1}{8n^2} + \frac{15}{2n^4} - \cdots \quad (n \to \infty).$$

Equivalently,

$$1 + \frac{1}{n} + \cdots + \frac{1}{n} = \gamma + \ln n + \frac{1}{2n} + O\left(\frac{1}{n^2}\right) \quad (n \to \infty).$$

It follows that

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{na} = H_{na} - H_n = \ln a + \frac{1-a}{2na} + O\left(\frac{1}{n^2}\right) \quad (n \to \infty).$$

Thus

$$\begin{aligned}
n\left(a - e^{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{na}}\right) &= n\left(a - a \cdot e^{\frac{1-a}{2na}+O\left(\frac{1}{n^2}\right)}\right) \\
&= a \cdot \frac{1-\exp\left(\frac{1-a}{2na} + O\left(\frac{1}{n^2}\right)\right)}{\frac{1-a}{2na} + O\left(\frac{1}{n^2}\right)} \cdot \left(\frac{1-a}{2a} + O\left(\frac{1}{n}\right)\right),
\end{aligned}$$

which in turn implies that

$$\lim_{n\to\infty} n\left(a - e^{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{na}}\right) = \frac{a-1}{2}.$$

The problem is solved.

[1] H. M. Srivastava and J. Choi, Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, 2001.

Solution 3 by Ed Gray, Highland Beach, FL

1) Let $S = e^{\dfrac{1}{n+1}+\dfrac{1}{n+2}+\cdots+\dfrac{1}{na}}$

2) $\ln(S) = \dfrac{1}{n+1}+\dfrac{1}{n+2}+\cdots+\dfrac{1}{an}$

3) $\ln(S) = \displaystyle\sum_{k=1}^{na}\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{k}.$

We use the Euler's approximation for the partial sum of the harmonic series. That is

4) $T_m = \displaystyle\sum_{k=1}^{m}\frac{1}{k} = \ln(m)+\gamma+\frac{1}{2m}-\frac{1}{12m^2}+\frac{1}{120m^4}-\cdots$, where γ is the Euler-Mascheroni constant $0.577\cdots$

In our approximation, we will only keep the term $\dfrac{1}{2m}$ to avoid unnecessary computations. Then from (3) and (4),

5) $\ln(S) = \ln(na)+\gamma+\dfrac{1}{2na}-\left(\ln(n)+\gamma+\dfrac{1}{2n}\right)$ or

6) $\ln(S) = \ln(na)-\ln(n)+\dfrac{1}{2na}-\dfrac{1}{2n}$

7) $\ln(S) = \ln\left(\dfrac{na}{n}\right)+\dfrac{1}{2na}-\dfrac{1}{2n}$

8) $\ln(S) = \ln a+\dfrac{1}{2na}-\dfrac{1}{2n}$

9) $S = e^{\ln a}\cdot e^{\frac{1}{2na}}\cdot e^{-\frac{1}{2n}}$, or

10) $S = a\cdot e^{\frac{1}{2na}}\cdot e^{-\frac{1}{2n}}$

For large n the exponents are small, and we keep only the first two terms in the expansion for e^y

11) $e^{\frac{1}{2na}} = 1+\dfrac{1}{2na}$

12) $e^{-\frac{1}{2na}} = 1 - \dfrac{1}{2n}$

13) The product is: $1 - \dfrac{1}{2n} + \dfrac{1}{2na} - \dfrac{1}{4an^2}$, and step 10 becomes

14) $S = a - \dfrac{a}{2n} + \dfrac{1}{2n} - \dfrac{1}{4n^2}$. Then

15) $a - S = \dfrac{a}{2n} - \dfrac{1}{2n} + \dfrac{1}{4n^2}$

16) $n(a-S) = \dfrac{a}{2} - \dfrac{1}{2} + \dfrac{1}{4n}$

So the limit as n approaches infinity is $\dfrac{a-1}{2}$.

Solution 4 by Paul M. Harms, North Newton, KS

When m is a positive integer $1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{m} = \ln m + \gamma + R(m)$ where γ is the Euler-Mascheroni constant and $R(m)$ is an error term that approaches $\dfrac{1}{2}m$ as m gets large. Let a be a positive integer greater than one. We have

$$\begin{aligned}
\frac{1}{n+1} + \frac{1}{n+1} + \cdots \frac{1}{na} &= 1 + \frac{1}{2} + \cdots \frac{1}{na} - \left(1 + \frac{1}{2} + \cdots \frac{1}{n}\right) \\
&= \ln na + \gamma + R(na) - (\ln n + \gamma + (n)) \\
&= \ln a + R(na) - R(n).
\end{aligned}$$

Then the limit in the problem involves

$$n\left(a - e^{\ln a} e^{R(na)-R(n)}\right) = na\left(1 - e^{R(na)-R(n)}\right).$$

For large n this can be approximated by

$$\frac{a\left(1 - 2^{\frac{1}{2}na - \frac{1}{2}n}\right)}{\frac{1}{n}}.$$

Thinking of n as a continuous variable and using L'Hôpital's Rule, the limit of the last expression is the limit of $\left(ae^{\frac{1}{2}na-\frac{1}{2}n}\left(\frac{1}{2}a - \frac{1}{2}\right)\right)$ as $n \to \infty$. The result is $\dfrac{a-1}{2}$.

Solution 5 by G. C. Greubel, Newport News, VA

We are asked to evaluate the limit

$$\lim_{n\to\infty} n\left(a - e^{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{na}}\right).$$

The primary difficulty is reducing the exponential to some aspect easier to work with. With this in mind consider the series of the exponential. This is given by

$$\phi_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{na}.$$

This can be quickly be seen as

$$\begin{aligned}\phi_n &= \sum_{k=1}^{an}\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{k} \\ &= H_{an} - H_n\end{aligned}$$

where H_n is the Harmonic number. With this there is a basis to expand upon. In order to proceed further the expansion of a Harmonic number in terms of factors of $1/n$ is required. The required expansion is obtained from Wolfram Mathworld Harmonic numbers site[1] and is given by

$$H_n \sim \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6} + \mathcal{O}\left(\frac{1}{n^8}\right).$$

where γ is Euler's constant. When use of this is made the result becomes

$$\begin{aligned}\phi_n &= \ln(an) - \ln(n) + \frac{1}{2n}\left(\frac{1}{a} - 1\right) - \frac{1}{12n^2}\left(\frac{1}{a^2} - 1\right) \\ &+ \frac{1}{120n^4}\left(\frac{1}{a^4} - 1\right) - \mathcal{O}\left(\frac{1}{n^6}\right) \\ &= \ln a + \frac{(1-a)}{2an} - \frac{(1-a^2)}{12a^2n^2} + \frac{(1-a^4)}{120a^4n^4} - \mathcal{O}\left(\frac{1}{n^6}\right).\end{aligned}$$

Now that a valid approximation for large values of n has been obtained it can be used to reduce the exponential portion of the limit. With this in mind the result becomes

$$\begin{aligned}e^{\phi_n} &= 1 + \phi_n + \frac{1}{2}\phi_n^2 + \cdots \\ &\approx 1 + \left[\ln a + \frac{1-a}{2an} - \frac{1-a^2}{12a^2n^2} + \mathcal{O}\left(\frac{1}{n^4}\right)\right] + \frac{1}{2}\left[\ln^2 a + \frac{2(1-a)\ln a}{4a^2 n}\right. \\ &\left.\mathcal{O}\left(\frac{1}{n^2}\right)\right] + \frac{1}{3!}\left[\ln^3 a + \frac{3(1-a)}{2an}\ln^2 a + \mathcal{O}\left(\frac{1}{n^2}\right)\right] + \cdots \\ &\approx e^{\ln a} + \frac{1-a}{2an}\left(1 + \frac{\ln a}{1!} + \frac{\ln^2 a}{2!} + \cdots\right) + \mathcal{O}\left(\frac{1}{n^2}\right)\end{aligned}$$

[1] The Wolfram Mathworld site for Harmonic numbers is found at http://mathworld.wolfram.com/HarmonicNumber.html and is stated as equation (13).

$$\approx \quad e^{\ln a} + \frac{1-a}{2an} e^{\ln a} + \mathcal{O}\left(\frac{1}{n^2}\right)$$

$$e^{\phi_n} \quad \approx \quad a + \frac{1-a}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

With this result it can now be seen that

$$a - e^{\phi_n} \approx \frac{a-1}{2n} - \mathcal{O}\left(\frac{1}{n^2}\right) \text{ and}$$
$$n\left(a - e^{\phi_n}\right) \approx \frac{a-1}{2} - \mathcal{O}\left(\frac{1}{n}\right).$$

Now the limit is easy to compute and is given by

$$\lim_{n\to\infty} n\left(a - e^{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{an}}\right) = \frac{a-1}{2}.$$

Also solved by Arkady Alt, San Jose, CA; Bruno Sagueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Vergata, Rome, Italy; David Stone and John Hawkins, Southern Georgia University, Statesboro, GA, and the proposer.

5257: *Proposed by Pedro H.O. Pantoja, UFRN, Brazil*

Prove that:

$$1 + \frac{1}{2}\cdot\sqrt{1+\frac{1}{2}} + \frac{1}{3}\cdot\sqrt[3]{1+\frac{1}{2}+\frac{1}{3}} + \cdots + \frac{1}{n}\cdot\sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}} \sim \ln(n),$$

where $f(x) \sim g(x)$ means $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = 1.$

Solution 1 by Arkady Alt, San Jose, CA

Let $S_n = 1 + \frac{1}{2}\cdot\sqrt{h_2} + \frac{1}{3}\cdot\sqrt[3]{h_3} + ... + \frac{1}{n}\cdot\sqrt[n]{h_n}$,where $h_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$.

Since $\dfrac{1}{k+1} < \ln(k+1) - \ln k < \dfrac{1}{k}$($\iff \left(1+\dfrac{1}{k}\right)^k < e < \left(1+\dfrac{1}{k}\right)^{k+1}$) then

$\sum\limits_{k=1}^{n} (\ln(k+1) - \ln k) < h_n \iff \ln(n+1) < h_n$ and $h_k - 1 < \sum\limits_{k=2}^{n} (\ln k - \ln(k-1)) \iff h_n < 1 + \ln n$

and, therefore, $\dfrac{S_n - S_{n-1}}{\ln n - \ln(n-1)} = \dfrac{\dfrac{\sqrt[n]{h_n}}{n}}{\ln\left(1+\dfrac{1}{n-1}\right)} =$

$$\frac{\sqrt[n]{h_n}}{\ln\left(1+\frac{1}{n-1}\right)^n} \in \left(\frac{\sqrt[n]{\ln(n+1)}}{\ln\left(1+\frac{1}{n-1}\right)^n}, \frac{\sqrt[n]{\ln n+1}}{\ln\left(1+\frac{1}{n-1}\right)^n}\right).$$

Since $\lim\limits_{n\to\infty} \sqrt[n]{\ln(n+1)} = 1$, $\lim\limits_{n\to\infty} \sqrt[n]{1+\ln n} = 1$, $\lim\limits_{n\to\infty} \ln\left(1+\dfrac{1}{n-1}\right)^n = 1$ then

$\lim\limits_{n\to\infty} \dfrac{S_n - S_{n-1}}{\ln n - \ln(n-1)} = 1$ and by Stolz Theorem we obtain

$$\lim_{n\to\infty} \frac{S_n}{\ln n} = \lim_{n\to\infty} \frac{S_n - S_{n-1}}{\ln n - \ln(n-1)} = 1.$$

Solution 2 by Ángel Plaza, Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Spain

Let L be the $\lim\limits_{n\to\infty} \dfrac{1+\frac{1}{2}\cdot\sqrt{1+\frac{1}{2}}+\frac{1}{3}\cdot\sqrt[3]{1+\frac{1}{2}+\frac{1}{3}}+\cdots+\frac{1}{n}\cdot\sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}}}{\ln(n)}$.

Since $\lim\limits_{n\to\infty} \ln(n) = \infty$, by the Stolz-Cesàro theorem,

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{\frac{1}{n}\cdot\sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}}}{\ln(n)-\ln(n-1)} \\ &= \lim_{n\to\infty} \frac{\sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}}}{\ln\left(\frac{n}{n-1}\right)^n}. \end{aligned}$$

Note that $\lim\limits_{n\to\infty} \sqrt[n]{1+\dfrac{1}{2}+\cdots+\dfrac{1}{n}} = \lim\limits_{n\to\infty} \dfrac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{1+\frac{1}{2}+\cdots+\frac{1}{n-1}} = 1$, by the Stolz-Cesàro theorem, and also that $\lim\limits_{n\to\infty} \ln\left(\dfrac{n}{n-1}\right)^n = 1$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

It is well known that $1+\frac{1}{2}+\cdots+\frac{1}{n} = \ln n + O(1)$ as $n\to\infty$ and $\ln(1+x) = x + O\left(x^2\right)$, $e^x = 1 = x + O\left(x^2\right)$ as $x\to 0$. Hence

$$\frac{\ln\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)}{n} = \frac{\ln\ln n}{n} + O\left(\frac{1}{n\ln n}\right)$$

and

$$\frac{1}{n}\cdot\sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}} = \frac{1}{n}e^{\ln\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)/n} = \frac{1}{n}\left(1+\frac{\ln\ln n}{n}+O\left(\frac{1}{n\ln n}\right)\right).$$

Since $\displaystyle\sum_{n=3}^{\infty} \frac{\ln \ln n}{n^2}$ and $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2 \ln n}$ converge, so

$$1 + \frac{1}{2} \cdot \sqrt{1 + \frac{1}{2}} + \frac{1}{3} \cdot \sqrt[3]{1 + \frac{1}{2} + \frac{1}{3}} + \cdots + \frac{1}{n} \cdot \sqrt[n]{1 + \frac{1}{2} + \cdots + \frac{1}{n}} = \ln(n) + O(1),$$

and we are done.

Editor's comment: **D. M. Bătinetu-Giurgiu, of the "Matei Basarab" National College in Bucharest, Romania and Neculai Stanciu, of George Emil Palade School in Buzău, Romania,** submitted two solutions to the problem. Their first solution was similar in approach to the second solution presented above, but in their second solution they generalized the problem as follows:

If $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ are sequences of positive real numbers such that:

- $\{y_n\}_{n\geq 1}$ is increasing and unbounded,
- $\exists t \in \Re_+$ such that $\displaystyle\lim_{n\to\infty} n^t \{y_{n+1} - y_n\} = a \in \Re_+$,
- $\displaystyle\lim_{n\to\infty} n^t x_n = a$ exists $\in \Re_+$, and $z_n = \displaystyle\sum_{k=1}^{n} x_k$, then

  $\{y_n\}_{n\geq 1} \sim \{z_n\}_{n\geq 1}$. I.e., $\displaystyle\lim_{n\to\infty} \frac{z_n}{y_n}$.

Proof. By the Cesaro-Stolz theorem we have:

$$\lim_{n\to\infty} \frac{z_n}{y_n} = \lim_{n\to\infty} \frac{z_{n+1} - z_n}{y_{n+1} - y_n} = \lim_{n\to\infty} \frac{x_{n+1}}{y_{n+1} - y_n} = \lim_{n\to\infty} \frac{(n+1)^t x_{n+1}}{\left(\frac{n+1}{n}\right)^t n^t (y_{n+1} - y_n)} = \frac{a}{1 \cdot a} = 1.$$

Remark: If we take $y_n = \ln n$, $h_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k}$, $x_n = \dfrac{1}{n} \sqrt[n]{h_n}$, and $z_n = \displaystyle\sum_{k=1}^{n} x_k$, then by the above we obtain $\{y_n\}_{n\geq 1} \sim \{z_n\}_{n\geq 1}$ which is problem 5257.

Also solved by Bruno Sagueiro Fanego, Viveiro, Spain; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Vergata, Rome, Italy, and the proposer.

5258: *Proposed by José Luis Díaz-Barrero and José Gibergans-Báguena, Polytechnical University of Catalonia, Barcelona, Spain*

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be real numbers such that $1 + \displaystyle\sum_{k=1}^{n} \cos^2 \alpha_k = n$. Prove that:

$$\sum_{1\leq i<j\leq n} \tan \alpha_i \tan \alpha_j \leq \frac{n}{2}.$$

Solution 1 by Arkady Alt, San Jose, CA

Let $x_i = \tan^2 \alpha_i, i = 1, 2, ..., n$ then $x_i \geq 0, i = 1, 2, ..., n, 1 + \sum\limits_{k=1}^{n} \cos^2 \alpha_k = n \iff$

$\sum\limits_{k=1}^{n} \dfrac{1}{1 + x_i} = n - 1$ and, since $\sum\limits_{1 \leq i < j \leq n}^{n} \tan \alpha_i \tan \alpha_j \leq \sum\limits_{1 \leq i < j \leq n}^{n} |\tan \alpha_i| \, |\tan \alpha_j| = \sum\limits_{1 \leq i < j \leq n}^{n} \sqrt{x_i x_j}$,

then it is sufficient to prove $\sum\limits_{1 \leq i < j \leq n}^{n} \sqrt{x_i x_j} \leq \dfrac{n}{2}$.

Let $a_i = \dfrac{x_i}{1 + x_i}, 1, 2, .., n$ then $\sum\limits_{i=1}^{n} a_i = \sum\limits_{i=1}^{n} \left(1 - \dfrac{1}{1 + x_i}\right) = n - \sum\limits_{i=1}^{n} \dfrac{1}{1 + x_i} = 1$ and,

since $x_i = \dfrac{a_i}{1 - a_i}, 1, 2, ..., n$ our problem is:

Prove inequality $\sum\limits_{1 \leq i < j \leq n}^{n} \sqrt{\dfrac{a_i a_j}{(1 - a_i)(1 - a_j)}} \leq \dfrac{n}{2}$, for $a_i \geq 0, i = 1, 2, ..., n$ such

that $\sum\limits_{k=1}^{n} a_i = 1$.

We have $\sum\limits_{1 \leq i < j \leq n}^{n} \sqrt{\dfrac{a_i a_j}{(1 - a_i)(1 - a_j)}} \leq \sum\limits_{1 \leq i < j \leq n}^{n} \dfrac{1}{2} \left(\dfrac{a_j}{1 - a_i} + \dfrac{a_i}{1 - a_j}\right) =$

$$\frac{1}{2} \left(\sum_{1 \leq i < j \leq n}^{n} \frac{a_j}{1 - a_i} + \sum_{1 \leq i < j \leq n}^{n} \frac{a_i}{1 - a_j}\right) = \frac{1}{2} \left(\sum_{i=1}^{n-1} {}_{j=i+1}^{n} \frac{a_j}{1 - a_i} + \sum_{j=2}^{n} {}_{i=1}^{j} \frac{a_i}{1 - a_j}\right) =$$

$$\frac{1}{2} \left(\sum_{i=1}^{n-1} \frac{1}{1 - a_i} \sum_{j=i+1}^{n} a_j + \sum_{j=2}^{n} \frac{1}{1 - a_j} \sum_{i=1}^{j-1} a_i\right) = \frac{1}{2} \cdot \frac{1}{1 - a_1} \sum_{j=2}^{n} a_j + \frac{1}{2} \sum_{i=2}^{n-1} \frac{1}{1 - a_i} \sum_{j=i+1}^{n} a_j +$$

$$\sum_{j=2}^{n-1} \tfrac{1}{1 - a_j} \sum_{i=1}^{j-1} a_i + \frac{1}{1 - a_n} \sum_{i=1}^{n-1} a_i = 1 + \frac{1}{2} \left(\sum_{i=2}^{n-1} \frac{1}{1 - a_i} \sum_{j=i+1}^{n} a_j + {}_{j=2}^{n-1} \frac{1}{1 - a_j} \sum_{i=1}^{j-1} a_i\right) =$$

$$1 + \frac{1}{2} \left(\sum_{i=2}^{n-1} \frac{1}{1 - a_i} \sum_{j=i+1}^{n} a_j + \sum_{i=2}^{n-1} \frac{1}{1 - ai} \sum_{j=1}^{i-1} a_i\right) = 1 + \frac{1}{2} \left(\sum_{i=2}^{n-1} \frac{1}{1 - a_i} \left(\sum_{j=i+1}^{n} a_j + \sum_{j=1}^{i-1} a_i\right)\right) =$$

$$1 + \frac{1}{2} \sum_{i=2}^{n-1} \frac{1 - a_i}{1 - a_i} = 1 + \frac{n - 2}{2} = \frac{n}{2}.$$

Solution 2 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Vergata, Rome, Italy

Proof: We first note that if

$\alpha_1 = \alpha_2 = \ldots = \alpha_{n-1} = 0$, $\alpha_n = \pi/2$, the constraints of the problem are satisfied, but

$$\sum_{1 \leq i < j \leq n} \tan \alpha_i \tan \alpha_j$$

is undefined; so we add the assumption $\alpha_i \neq \pi/2 + 2k\pi$, $k \in Z$,

$i = 1, \ldots, n$. Both $\cos^2 x$ and $\tan x$ are π–periodic so we can assume

$\alpha_i \in (-\pi/2, \pi/2)$ and set $\alpha_i = \arctan a_i$. This yields

$$1 + \sum_{k=1}^{n} \frac{1}{1+a_k^2} = n \implies \sum_{1\le i<j\le n} a_i a_j \le n.$$

By defining $a_k = \sqrt{b_k}$ the inequality becomes

$$2 \sum_{1\le i<j\le n} \sqrt{b_i b_j} \le n \qquad \text{whenever} \qquad \sum_{i=1}^{n} \frac{1}{b_i+1} = n-1.$$

By convexity of $1/(1+x)$ for $x>0$ we have

$$n-1 = \sum_{i=1}^{n} \frac{1}{b_i+1} \le \frac{n}{1+\dfrac{b_1+\ldots+b_n}{n}},$$

that is, $b_1 + \ldots + b_n \le n/(n-1)$. Now

$$2 \sum_{1\le i<j\le n} \sqrt{b_i b_j} \le \sum_{1\le i<j\le n} (b_i+b_j) = (n-1)(b_1+\ldots+b_n) \le (n-1)n/(n-1) = n,$$

and we are done.

Solution 3 Adrian Naco, Polytechnic University, Tirana, Albania.

Let $x_i = \tan\alpha_i, \forall i \in \{1,2,.....,n\}$. Applying the trigonometric formula,
$\cos^2\alpha_i = \dfrac{1}{1+\tan^2\alpha_i}$, the condition and the initial inequality give respectively,

$$1 + \sum_{1}^{n} \frac{1}{1+x_i^2} = n \qquad \text{and} \qquad \sum_{1\le i<j\le n} x_i x_j \le \frac{n}{2}.$$

Let us assume

$$a_i = \frac{1}{(n-1)(x_i^2+1)} \quad \Rightarrow \quad x_i^2 = \tfrac{1-(n-1)a_i}{(n-1)a_i} \quad \text{and} \quad \sum_{i=1}^{n} a_i = 1$$

$$y_{i,j} = \frac{1-(n-1)a_i}{(n-1)a_j} \quad \Rightarrow \quad y_{i,j}y_{j,i} = x_i^2 x_j^2 \quad \text{and} \quad y_{i,j} \ge 0, \forall i,j$$

Thus we have that

$$2 \sum_{1\le i<j\le n} x_i x_j \;\le\; 2 \sum_{1\le i<j\le n} |x_i||x_j| = 2 \sum_{1\le i<j\le n} \sqrt{x_i^2 x_j^2}$$

$$
\begin{aligned}
&\leq 2\sum_{1\leq i<j\leq n}\frac{1}{2}\left(x_i^2+x_j^2\right)=\sum_{1\leq i<j\leq n}\left(x_i^2+x_j^2\right)\\
&= \sum_{1\leq i<j\leq n}\left[\frac{1-(n-1)a_i}{(n-1)a_j}+\frac{1-(n-1)a_j}{(n-1)a_i}\right]\\
&= \frac{1}{n-1}\sum_{1\leq i<j\leq n}\left[\frac{1}{a_j}+\frac{1}{a_i}\right]-\sum_{1\leq i<j\leq n}\left[\frac{a_j}{a_i}+\frac{a_i}{a_j}\right]\\
&= \frac{1}{n-1}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[\frac{1}{a_j}+\frac{1}{a_i}\right]-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[\frac{a_j}{a_i}+\frac{a_i}{a_j}\right]\\
&= \frac{1}{n-1}\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{1}{a_j}+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{1}{a_i}\right]-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_j}{a_i}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_i}{a_j}\\
&= \frac{1}{n-1}\left[\sum_{i=1}^{n-1}(n-i)\frac{1}{a_i}+\sum_{i=2}^{n}(i-1)\frac{1}{a_i}\right]-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_j}{a_i}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_i}{a_j}\\
&= \frac{1}{n-1}\sum_{i=1}^{n}(n-1)\frac{1}{a_i}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_j}{a_i}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_i}{a_j}\\
&= \sum_{i=1}^{n}\frac{1}{a_i}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_j}{a_i}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_i}{a_j}\\
&= \sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{n}a_j\right)}{a_i}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_j}{a_i}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_i}{a_j}\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{a_j}{a_i}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_j}{a_i}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_i}{a_j}\\
&= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_j}{a_i}+\sum_{i=1}^{n}\frac{a_i}{a_i}+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_i}{a_j}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_j}{a_i}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_i}{a_j}\\
&= \sum_{i=1}^{n}\frac{a_i}{a_i}=\sum_{i=1}^{n}1=n.
\end{aligned}
$$

Finally we have that

$$\sum_{1\leq i<j\leq n}\tan\alpha_i\tan\alpha_j=\sum_{1\leq i<j\leq n}x_ix_j\leq\frac{n}{2}.$$

The equality holds for $x_i = \tan\alpha_i = \tan\alpha_j = x_j, 1 \le i < j \le n$ or equivalently for $\alpha_i = k\pi + \alpha_j, 1 \le i < j \le n, k \in Z$.

Solution 4 by Bruno Salgueiro Fanego, Viveiro, Spain

Note that $n = 1 + \sum_{k=1}^{n} \cos^2\alpha_k = 1 + \sum_{k=1}^{n}(1 - \sin^2\alpha_k) = 1 + \sum_{k=1}^{n} 1 - \sum_{k=1}^{n} \sin^2\alpha_k = 1 + n - \sum_{k=1}^{n} \sin^2\alpha_k \iff \sum_{k=1}^{n} \sin^2\alpha_k = 1 \iff \sum_{k=1}^{n} \frac{\tan^2\alpha_k}{1+\tan^2\alpha_k} = 1$, and that the inequality to prove is equivalent to $\left(\sum_{k=1}^{n} \tan\alpha_k\right)^2 - \sum_{k=1}^{n} \tan^2\alpha_k = 2 \sum_{1\le i<j\le n} \tan\alpha_i \tan\alpha_j \le n = \sum_{k=1}^{n} 1 \iff$

$\left(\sum_{k=1}^{n} \tan\alpha_k\right)^2 \le \sum_{k=1}^{n} 1 + \sum_{k=1}^{n} \tan^2\alpha_k = \sum_{k=1}^{n} \left(1 + \tan^2\alpha_k\right) \iff \frac{\left(\sum_{k=1}^{n} \tan\alpha_k\right)^2}{\sum_{k=1}^{n}\left(1+\tan^2\alpha_k\right)} \le 1 =$

$\sum_{k=1}^{n} \frac{\tan^2\alpha_k}{+\tan^2\alpha_k}$ which is just Bergström′s inequality $\frac{\left(\sum_{k=1}^{n} a_k\right)^2}{\sum_{k=1}^{n} b_k} \le \sum_{k=1}^{n} \frac{a_k^2}{b_k}$ applied to $a_k = \tan\alpha_k \in \Re$ and $b_k = 1 + \tan^2\alpha_k \in \Re;\ 1 \le k \le n$.

Equality occurs if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$, that is if and only if $\frac{1}{2}\sin(2\alpha_1) = \frac{1}{2}\sin(2\alpha_2) = \cdots = \frac{1}{2}\sin(2\alpha_n)$, and $\sum_{k=1}^{n} \sin^2\alpha_k = 1$.

Also solved by the proposers.