Problems Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
December 15, 2014

- **5313:** *Proposed by Kenneth Korbin, New York, NY*

  Find the sides of two different isosceles triangles if they both have perimeter 256 and area 1008.

- **5314:** *Proposed by Roger Izard, Dallas TX*

  A biker and a hiker like to workout together by going back and forth on a road which is ten miles long. One day, at 8 AM, at the starting end of the road, they went out together. The biker soon got far past the hiker, reached the end of the road, reversed his direction, and soon passed by the hiker at 9:06 AM. Then, the biker got down to the beginning part of the road, reversed his direction, and got back to the hiker at 9:24 AM. The biker and the hiker were, then, going in the same direction. Calculate in miles per hour the speeds of the hiker and the biker.

- **5315:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  The hexagonal numbers have the form $H_n = 2n^2 - n,\ n = 1, 2, 3, \ldots$. Prove that infinitely many hexagonal numbers are the sum of two hexagonal numbers.

- **5316:** *Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

  Let $\{u_n\}_{n\geq 0}$ be a sequence defined recursively by

  $$u_{n+1} = \sqrt{\frac{u_n^2 + u_{n-1}^2}{2}}.$$

  Determine $\lim_{n\to\infty} u_n$ in terms of u_0, u_1.

- **5317:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $a_k, b_k > 0,\ 1 \leq k \leq n$, be real numbers such that $a_1 + a_2 + \ldots + a_n = 1$. Prove that

  $$\frac{1}{n^3}\left(\sum_{k=1}^{n} b_k\right)^5 \leq \sum_{k=1}^{n} \frac{b_k^5}{a_k}.$$

- **5318:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Prove that $(1+x)^x \le 1+x^2$ for $0 \le x \le 1$.

Solutions

- **5295:** *Proposed by Kenneth Korbin, New York, NY*

  A convex cyclic hexagon has sides
  $$\left(5,\ 7\sqrt{17},\ 23\sqrt{13},\ 25\sqrt{13},\ 25\sqrt{17},\ 45\right).$$
  Find the diameter of the circumcircle and the area of the hexagon.

  Solution by Kee-Wai Lau, Hong Kong, China

  We show that diameter of the circumcircle is 125 and the area of the hexagon is $8(86\sqrt{34}+81\sqrt{39})$.

  Let O be the center and d be the diameter of the circumcircle, which we denote by C. It is easy to see that the anlgle subtended at O by a side of the hexagon with length s equals $2\sin^{-1}\left(\dfrac{s}{d}\right)$. We first suppose that O lies inside the hexagon, so that
  $$f(d) = \pi, \qquad (1)$$
  where
  $$f(d) = \sin^{-1}\left(\frac{5}{d}\right)+\sin^{-1}\left(\frac{7\sqrt{17}}{d}\right)+\sin^{-1}\left(\frac{23\sqrt{13}}{d}\right)+2\sin^{-1}\left(\frac{25\sqrt{13}}{d}\right)+\sin^{-1}\left(\frac{25\sqrt{17}}{d}\right)+\sin^{-1}\left(\frac{4.}{d}\right).$$

  $$\begin{aligned} a &= \sin^{-1}\left(\frac{23\sqrt{13}}{125}\right)+\sin^{-1}\left(\frac{\sqrt{13}}{5}\right)+\sin^{-1}\left(\frac{\sqrt{1}}{25}\right) \text{ and} \\ b &= \sin^{-1}\left(\frac{7\sqrt{17}}{125}\right)+\sin^{-1}\left(\frac{\sqrt{17}}{5}\right)+\sin^{-1}\left(\frac{\sqrt{9}}{25}\right) \end{aligned}$$

  Then $f(125)=a+b$. Since $a=\sin^{-1}\left(\dfrac{4\sqrt{39}}{25}\right)+\sin^{-1}\left(\dfrac{1}{25}\right)=\sin^{-1}\ 1=\dfrac{\pi}{2}$ and

  $b=\sin^{-1}\left(\dfrac{4\sqrt{34}}{25}\right)+\sin^{-1}\left(\dfrac{9}{25}\right)=\sin^{-1}\ 1=\dfrac{\pi}{2}$ so (1) holds if and only if $d=125$.

  Now the distances from O to the sides $\left(5,\ 7\sqrt{17},\ 23\ \sqrt{13},\ 25\sqrt{13},\ 25\sqrt{17},\ 45\right)$ are $\left(10\sqrt{39},\ 43\sqrt{2},\ 27\sqrt{3},\ 25\sqrt{3},\ 25\sqrt{2},\ 10\sqrt{34}\right)$. So the area of the hexagon equals
  $$\frac{1}{2}\left(50\sqrt{39}+301\sqrt{34}+621\sqrt{39}+625\sqrt{39}+10\sqrt{39}+625\sqrt{34}+450\sqrt{34}\right)$$

$$= \quad 8\left(86\sqrt{34}+81\sqrt{39}\right).$$

We next suppose that O lies on or is outside the hexagon. Since the longest side of the hexagon is $25\sqrt{17}$, so $d \geq 25\sqrt{17}$. Moreover,

$$\sin^{-1}\left(\frac{5}{d}\right)+\sin^{-1}\left(\frac{7\sqrt{17}}{d}\right)+\sin^{-1}\left(\frac{23\sqrt{13}}{d}\right)+\sin^{-1}\left(\frac{25\sqrt{13}}{d}\right)+\sin^{-1}\left(\frac{25\sqrt{17}}{d}\right)+\sin^{-1}\left(\frac{45}{d}\right)$$

$$= \quad \sin^{-1}\left(\frac{25\sqrt{17}}{d}\right),$$

and hence,

$$\sin^{-1}\left(\frac{23\sqrt{13}}{d}\right)+\sin^{-1}\left(\frac{25\sqrt{13}}{d}\right) < \sin^{-1}\left(\frac{25\sqrt{17}}{d}\right). \qquad (2)$$

If $d < \sqrt{15002} = \sqrt{(2)(13)(577)}$, then by (2)

$$\sin^{-1}\left(\frac{25\sqrt{17}}{d}\right) > \sin^{-1}\left(\frac{23}{\sqrt{1154}}\right)+\sin^{-1}\left(\frac{25}{\sqrt{1154}}\right) = \sin^{-1}1 = \frac{\pi}{2} \text{ which is false.}$$

If $d \geq \sqrt{15002}$, then the left hand side of (2) equals

$$\begin{aligned}
\sin^{-1}\left(\frac{25\sqrt{13}}{d}\sqrt{1-\frac{6877}{d^2}}+\frac{23\sqrt{13}}{d}\sqrt{1-\frac{8125}{d^2}}\right) &\geq \sin^{-1}\left(\frac{25\sqrt{13}}{d}\sqrt{1-\frac{6877}{15002}}+\frac{23\sqrt{13}}{d}\sqrt{1-\frac{8125}{15002}}\right)\\
&= \sin^{-1}\left(\frac{\sqrt{15002}}{d}\right)\\
&> \sin^{-1}\left(\frac{25\sqrt{17}}{d}\right),
\end{aligned}$$

which is also false. Thus we conclude that O must lie inside the hexagon, and this completes the solution.

Also solved by Ed Gray, Highland Beach, FL, and the proposer.

- **5296:** *Proposed by Roger Izard, Dallas, TX*

Consider the "Star of David," a six pointed star made by overlapping the triangles ABC and FDE. Let

$$\overline{AB}\cap\overline{DF}=G,\text{ and }\overline{AB}\cap\overline{DE}=H,$$
$$\overline{AC}\cap\overline{DF}=L,\text{ and }\overline{AC}\cap\overline{FE}=K,$$
$$\overline{BC}\cap\overline{DE}=I,\text{ and }\overline{BC}\cap\overline{FE}=J,$$

in such a way that:

$$\frac{CK}{AC}=\frac{EI}{DE}=\frac{BI}{BC}=\frac{GD}{DF}=\frac{AG}{AB}=\frac{FK}{EF}\text{ and}$$

$$\frac{AL}{AC}=\frac{DH}{DE}=\frac{BH}{AB}=\frac{EJ}{EF}=\frac{FL}{DF}=\frac{CJ}{CB}.$$

Let $r=\dfrac{CK}{AC}$ and let $p=\dfrac{AL}{AC}$. Prove that $r+p=\dfrac{3pr+1}{2}$.

Solution by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

We construct a drawing of the figure and determine lengths of some of the sides in terms of r, p and the sides of the given triangles.
The following shows that the side lengths of the smaller triangle based on

$$r=\frac{CK}{AC}=\frac{EI}{DE}=\frac{BI}{BC}=\frac{GD}{DF}=\frac{AG}{AB}=\frac{FK}{EF}\text{ and}$$

$$p=\frac{AL}{AC}=\frac{DH}{DE}=\frac{BH}{AB}=\frac{EJ}{EF}=\frac{FL}{DF}=\frac{CJ}{CB}.$$

We see that $AC+AL+LK+KC=pAC+LK+rAC$, so $LK=(1-r-p)AC$.
Similarly,

$$\begin{aligned}HI&=(1-r-p)DE\\KJ&=(1-r-p)EF\\GH&=(1-r-p)AB\\IJ&=(1-r-p)BC\\GL&=(1-r-p)DF.\end{aligned}$$

We apply the Law of Cosines to the two triangles having A as principal vertex.
In $\triangle ABC,\ AC^2+AB^2-2AC\cdot AB\cos A=BC^2$, and in
$\triangle AGL,\ (pAC)^2+(rAB)^2-2prAC\cdot AB\cos A=GL^2=(1-r-p)^2DF^2$.
Solving each equation for $2AC\cdot AB\cos A$ and equating the results, we have

$$2AC\cdot AB\cos A=AC^2+AB^2-BC^2=\frac{p^2AC^2+r^2AB^2-(1-r-p)^2DF^2}{pr}.$$

Clearing fractions yields

$$prAC^2+prAB^2-prBC^2=p^2AC^2+r^2AB^2-(1-r-p)^2DF^2$$

so

$$(pr-pr)AC^2+(pr-r^2)AB^2-prBC^2+(1-r-p)^2DF^2=0.$$

By considering the other vertices B, C, D, E, F in turn, we obtain analogous equations:

$$\begin{aligned}
(pr-p^2)AB^2+(pr-r^2)BC^2-prAC^2+(1-r-p)^2DE^2 &= 0\\
(pr-p^2)BC^2+(pr-r^2)AC^2-prAB^2+(1-r-p)^2FE^2 &= 0\\
(pr-p^2)DE^2+(pr-r^2)DF^2-prFE^2+(1-r-p)^2AB^2 &= 0\\
(pr-p^2)EF^2+(pr-r^2)DE^2-prDF^2+(1-r-p)^2BC^2 &= 0\\
(pr-p^2)DF^2+(pr-r^2)EF^2-prDE^2+(1-r-p)^2AC^2 &= 0.
\end{aligned}$$

Summing these six equations and letting $S = AB^2 + AC^2 + BC^2 + DE^2 + DF^2 + EF^2$ yields a very nice result:

$$\begin{aligned}
(pr-p^2)S+(pr-r^2)S-prS+(1-r-p)^2S &= 0, \text{ or}\\
\left\{(pr-p^2)+(pr-r^2)-pr+(1-r-p)^2\right\}S &= 0.
\end{aligned}$$

Because S is not zero, this gives

$$(pr-p^2)+(pr-r^2)-pr+(1-r-p)^2=0.$$

Expanding the trinomial and collecting like terms gives us

$$\begin{aligned}
3pr-2r-2p+1 &= 0. \text{ So,}\\
2(r+p) &= 1+3pr. \text{ Thus,}\\
r+p &= \frac{3pr+1}{2}.
\end{aligned}$$

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China, and the proposer.

- **5297:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

Let $s_n = n^2,\ t_n = \dfrac{n(n+1)}{2},\ p_n = \dfrac{n(3n-1)}{2}$, for positive integers n, be the square, triangular and pentagonal numbers respectively. Prove, independently of each other, that

$$
\begin{aligned}
i) \quad t_a + p_b &= t_c \\
ii) \quad t_a + s_b &= p_c \\
iii) \quad p_a + s_b &= s_c,
\end{aligned}
$$

for infinitely many positive integers, a, b, and c.

Solution by Carl Libis, Lane College, Jackson, TN

$$
\begin{aligned}
i) \quad t_n + p_{n+1} &= \frac{n(n+1)}{2} + \frac{(n+1)(3n+2)}{2} = \frac{n^2+n+3n^2+5n+2}{2} \\
&= \frac{4n^2+6n+2}{2} = \frac{(2n+1)(2n+2)}{2} = t_{2n+1}
\end{aligned}
$$

$$
ii) \quad s_n + t_{n-1} = n^2 + \frac{(n-1)n}{2} = \frac{2n^2}{2} + \frac{n^2-n}{2} = \frac{3n^2-n}{2} = \frac{n(3n-1)}{2} = p_n
$$

$$
\begin{aligned}
iii) \quad p_{4n+1} + s_n &= \frac{(4n+1)(12n+2)}{2} + n^2 = \frac{48n^2+20n+2}{2} + \frac{2n^2}{2} \\
&= \frac{50n^2+20n+2}{2} = 25n^2+10n+1 = (5n+1)^2 = s_{5n+1}
\end{aligned}
$$

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Michael Brozinsky, Central Islip, NY; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), Angelo State University, San Angelo, TX; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Becca Rousseau, Ellie Erehart, and Davis Weerheim (jointly), students at Taylor University, Upland, IN; David Stone and John Hawkins (jointly), Georgia Southern University, Statesboro, GA, and the proposer.

- **5298:** *Proposed by D. M. Bătinetu-Giurgiu,"Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

Let $(a_n)_{n\geq 1}$ be an arithmetic progression and m a positive integer. Calculate:

$$
\lim_{n\to\infty}\left(\left(\sum_{k=1}^{m}\left(1+\frac{1}{n}\right)^{n+a_k} - me\right)n\right).
$$

Solution by Anastasios Kotronis, Athens, Greece

Let $a_n = a_1 + (n-1)d$ where a_1 is the initial term and d is the common difference of successive terms. Then

$$\begin{aligned}
\sum_{k=1}^{m}\left(1+\frac{1}{n}\right)^{n+a_k} &= \sum_{k=1}^{m}\left(1+\frac{1}{n}\right)^{n+a_1+(k-1)d} = \left(1+\frac{1}{n}\right)^{n+a_1-d}\sum_{k=1}^{m}\left(1+\frac{1}{n}\right)^{kd} \\
&= \exp\left((n+a_1-d)\ln\left(1+\frac{1}{n}\right)\right)\sum_{k=1}^{m}\exp\left(kd\ln\left(1+\frac{1}{n}\right)\right) \\
&= \exp\left((n+a_1-d)\left(\frac{1}{n}-\frac{1}{2n^2}+\mathcal{O}(n^{-3})\right)\right)\sum_{k=1}^{m}\exp\left(kd\left(\frac{1}{n}+\mathcal{O}(n^{-2})\right)\right) \\
&= \left(e+\frac{e(a_1-d-1/2)}{n}+\mathcal{O}(n^{-2})\right)\sum_{k=1}^{m}\left(1+\frac{kd}{n}+\mathcal{O}(n^{-2})\right) \\
&= \left(e+\frac{e(a_1-d-1/2)}{n}+\mathcal{O}(n^{-2})\right)\left(m+\frac{dm(m+1)}{2n}+\mathcal{O}(n^{-2})\right) \\
&= em+\frac{em\left(d(m-1)+2a_1-1\right)}{2n}+\mathcal{O}(n^{-2}) = em+\frac{em\left(a_m+a_1-1\right)}{2n}+\mathcal{O}(n^{-2})
\end{aligned}$$

so the desired limit is $\dfrac{em\left(a_m+a_1-1\right)}{2}$.

Also solved by Ed Gray, Highland Beach, FL; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Kee-Wai Lau, Hong Kong, China, and the proposers.

- **5299:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Without the aid of a computer, show that

$$\ln^2 2\int_0^1 \frac{x^{3/2}2^x\sin x}{(1+x\ln 2)^2}\,dx \;\geq\; \frac{1-\ln 2}{1+\ln 2}\int_0^1 \sqrt{x}\,\sin x\,dx.$$

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University Rome, Italy

The two functions $\sqrt{x}\sin x$ and $\dfrac{x2^x}{(1+x\ln 2)^2}$ are both increasing in $[0,1]$. Indeed, $\dfrac{1}{2\sqrt{x}}\sin x+\sqrt{x}\cos x$ and $\dfrac{2^x(1+x\ln 2+x^2\ln^2 2)}{(1+x\ln 2)^2}$ are the derivatives respectively of the first and the second function.

Chebyshev's inequality yields

$$\ln^2 2 \int_0^1 \frac{x^{3/2} 2^x \sin x}{(1+x\ln 2)^2} dx \geq \ln^2 2 \int_0^1 \frac{x 2^x}{(1+x\ln 2)^2} dx \int_0^1 \sqrt{x} \sin x dx.$$

Moreover,

$$\ln^2 2 \int_0^1 \frac{x 2^x}{(1+x\ln 2)^2} dx = \frac{2^x}{1+x\ln 2}\Big|_0^1 = \frac{1-\ln 2}{1+\ln 2},$$

hence, the result.

Solution 2 by Ed Gray, Highland Beach, FL

The method will be to increase the integral on the right to get a function that is integrable, and decrease the integral on the left to get a function which is integral in such a way that the inequality is maintained. We will also evaluate $\frac{1}{(\ln(2))^2} \cdot \frac{1-\ln 2}{1+\ln 2}$, and use its value as a coefficient on the right hand side of the inequality.

For $0 \leq x \leq 1$,

$$\sin(x) \leq x, \sqrt{x}\sin(x) \leq x^{3/2}.$$

So,

$$\int_0^1 \sqrt{(x)} \sin(x) dx < \int_0^1 x^{3/2} dx = \frac{2}{5} x^{5/2}\Big|_0^1 = 0.4.$$

Also,

$$\frac{1}{(\ln(2))^2} = 2.08136898, \ \frac{1-\ln(2)}{1+\ln(2)} = 0.181232218, \text{ and } \frac{1}{(\ln(2))^2} \cdot \left(\frac{1-\ln(2)}{1+\ln(2)}\right) = 0.3772111.$$

- (1) $\int_0^1 \left(\frac{x^{3/2}(2^x)\sin(x)}{(1+x\ln(2))^2}\right) dx \geq (0.4)(0.3772111) = 0.150884$. We need to reduce the value of the integral to get an approximation that still satisfies the inequality.
- (2) Consider $1 + x > 1 + x\ln(2)$. Squaring,
- (3) $1 + 2x + x^2 > \left(1 + x\ln(2)\right)^2$, and
- (4) $1 + 2x > \left(1 + x\ln(2)\right)^2$. This inequality holds since both functions are monotonically increasing, and the relationship holds for $x = 1$.

Then:

- (5) $\frac{1}{1+2x} < \frac{1}{(1+x\ln(2))^2}$. So,
- (6) $\frac{x^{3/2}\left(2^x\right)\sin(x)}{1+2x} < \left(\frac{x^{3/2}\left(2^x\right)\sin(x)}{(1+x\ln 2)^2}\right)$. For $0 \leq x \leq 1$,
- (7) $\sin(x) > x - \frac{x^3}{6}$, so,
- (8) $\frac{x^{5/2}\left(2^x\right)\left(1-\frac{x^3}{6}\right)}{1+2x} < \frac{x^{3/2} 2^x \sin(x)}{\left(1+x\ln(2)\right)^2}$, or

- (9) $\dfrac{x^{5/2}\left(2^x\right)\left(1-\dfrac{x^3}{6}\right)}{1+2x} < \dfrac{x^{3/2}\left(2^x \sin(x)\right.}{(1+2x)} < \dfrac{x^{3/2}\left(2^x \sin(x)\right)}{(1+x\ln(2))^2}$

We now express $\dfrac{2^x}{1+2x}$ in a Taylor series expansion about 0.5.

- (10) $f(x) = f(.5) + f'(.5)(x-.5) + \dfrac{f''(.5)}{2!}(x-.5)^2 + \dfrac{f'''(.5)}{3!}(x-.5)^3 + \ldots.$

As one can imagine, the derivatives get quite messy, so let's bring in Bing to compute them for us, (which does not violate the spirit of not using a computer because it is not evaluating the integral, just saving time. In any case, the series out to $(x-.5)^5$ is

$f(x) = \dfrac{2^x}{1+2x} \approx 0.7071 - .2169(x-.5) + .3868(x-.5)^2 - .3475(x-.5)^3 + .3543(x-.5)^4 - .3534(x-.5)^5$

The following table gives a "feel" for the goodness of fit for the approximation over the range of $0 \leq x \leq 1$.

x	$2^x/(1+2x)$	*Approximate value*
0	1.0	.988951
0.1	.893144	.890731
0.2	.820499	.820131
0.3	.769465	.769412
0.4	.733060	.733060
0.5	.707107	.7072107
0.6	.688962	.688960
0.7	.675877	.676858
0.8	.669654	.669457
0.9	.666452	.665419
1.0	.666667	.662985

Not only is this a good fit, but if we define the expansion by $f(x)$, we see that $f(x) < \dfrac{2^x}{1+2x}$ and the equation in step (9) becomes

- (11) $x^{5/2)}\left(1-\dfrac{x^2}{6}\right)f(x) < x^{5/2}\left(1-\dfrac{x^2}{6}\right)\left(\dfrac{2^x}{1+2x}\right) < x^{3/2}\dfrac{2^x \sin x}{1+2x} < \dfrac{x^{3/2}2^x \sin(x)}{(1+x\ln(2))^2}.$

We now need to write the series expansion of $f(x)$, to obtain a polynomial in x. Then by multiplying by $(x^{5/2})\left(1-\dfrac{x^2}{6}\right)$ we will obtain a polynomial in x for which we can easily perform the integration from 0 to 1. We save the reader the details. The integrand is:

- (12) $.058909x^{19/2} - .20633866x^{17/2} - 0301165x^{15/2} + .924424x^{13/2} - 1.7479965x^{11/2} + 1.7168252x^{9/2} - 1.1521715x^{7/2} + .988952x^{5/2}$. Integrating with respect to x gives us

- (13) $.00561x^{21/2} - .0217198x^{19/2} - .003543x^{17/2} + .123256x^{15/2} - .268923x^{13/2} + .31215x^{11/2} - .256038x^{9/2} + .282558x^{7/2}$.

So, returning to the equation in (1), we see that $.1733502 > .150884$, and this proves the inequality.

Solution 3 by Kee-Wai Lau, Hong Kong, China

For $0 \le x \le 1$, let $f(x) = \dfrac{2^x}{(1+x\ln 2)^2}$ so that $\dfrac{df(x)}{dx} = \dfrac{(\ln 2)2^x(x\ln 2 - 1)}{(1+x\ln 2)^2} < 0$,

and $f(x) \ge f(1) = \dfrac{2}{(1+\ln 2)^2}$. Hence, $\displaystyle\int_0^1 \frac{x^{3/2}2^x \sin x}{(1+x\ln 2)^2}dx \ge \frac{2}{(1+\ln 2)^2}\int 0^1 x^{3/2}\sin x dx$.

By the substitution $x = y^{3/5}$, we obtain

$$\int_0^1 x^{3/2}\sin x dx = \frac{3}{5}\int_0^1 \sqrt{y}\sin(y^{3/5})dy \ge \frac{3}{5}\int_0^1 \sqrt{y}\sin y dy.$$

Hence to prove the inequality of the problem, we need only show that

$\dfrac{6\ln^2 2}{5(1+\ln 2)} \ge 1 - \ln 2$, or equivalently $\ln^2 2 \ge \dfrac{5}{11}$. Since $\left(\dfrac{17}{25}\right)^2 = \dfrac{289}{625} > \dfrac{5}{11}$,

so it suffices to show that $\ln 2 > \dfrac{17}{25}$, or $e^{-17/25} > \dfrac{1}{2}$. But this follows from the fact that $e^{-17/25} > 1 - \displaystyle\sum_{n=1}^{5}\frac{(-1)^{n-1}}{n!}\left(\frac{17}{25}\right)^n = \frac{148386317}{292986750} > \frac{1}{2}$.

Remark: If we use the rapidly convergent series $\ln 2 = \dfrac{2}{3}\displaystyle\sum_{k=0}^{\infty}\frac{1}{(2k+1)9^k}$, as listed in *"Natural logarithm of 2–Wikipedia"* in the internet, we obtain easily $\ln 2 > \dfrac{2}{3}\left(1 + \dfrac{1}{27}\right) = \dfrac{56}{81} > \dfrac{17}{25}$.

Also solved by the proposer.

- **5300:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $n \ge 1$ be an integer. Prove that

$$\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^{2n} x} = \sum_{k=0}^{n-1}\binom{n-1}{k}\cdot\frac{1}{2n-2k-1}.$$

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

If $n = 1$,

$$\begin{aligned}\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^2 x} &= \int_{\pi/4}^{\pi/2} \csc^2 x\, dx \\ &= -\cot x]_{\pi/4}^{\pi/2} \\ &= 1 \\ &= \sum_{k=0}^{0} \binom{0}{k} \frac{1}{2-2k-1}.\end{aligned}$$

Hence, the statement is true when $n = 1$.

If $n \geq 2$, then we use the standard calculus approach for evaluating

$$\int \csc^{2m} x\, dx.$$

To begin,

$$\begin{aligned}\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^{2n} x} &= \int_{\pi/4}^{\pi/2} \csc^{2n} x\, dx \\ &= \int_{\pi/4}^{\pi/2} \left(1 + \cot^2 x\right)^{n-1} \left(\csc^2 x\, dx\right).\end{aligned}$$

If we substitute $u = \cot x$ and simplify, we get

$$\begin{aligned}\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^{2n} x} &= -\int_{1}^{0} \left(1 + u^2\right)^{n-1} du \\ &= \int_{0}^{1} \left(1 + u^2\right)^{n-1} du.\end{aligned}$$

Finally, by the Binomial Theorem,

$$\begin{aligned}\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^{2n} x} &= \int_{0}^{1} \sum_{k=0}^{n-1} \binom{n-1}{k} u^{2(n-1-k)}\, du \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} \int_{0}^{1} u^{2n-2k-2}\, du \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} \left.\frac{u^{2n-2k-1}}{2n-2k-1}\right]_{0}^{1} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot \frac{1}{2n-2k-1}.\end{aligned}$$

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

$$
\begin{aligned}
\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^2 x} &= \int_{\pi/4}^{\pi/2} \left(\frac{1}{\sin^2 x}\right)^n dx \\
&= \int_{\pi/4}^{\pi/2} \left(1 + \frac{1}{\tan^2 x}\right)^n dx \\
&\underset{(t=1/\tan x)}{=} \int_1^0 \left(1+t^2\right)^n \frac{-dt}{1+t^2} \\
&= \int_0^1 \left(1+t^2\right)^{n-1} dt \\
&= \int_0^1 \left(\sum_{k=0}^{n-1} \binom{n-1}{k} \cdot 1^k \cdot \left(t^2\right)^{n-1-k}\right) dt \\
&= \sum_{k=0}^{n-1} \int_0^1 \binom{n-1}{k} \cdot t^{2n-2k-2} dt \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot \frac{t^{2n-2k-1}}{2n-2k-1}\Bigg|_{t=0}^{t=1} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot \frac{1}{2n-2k-1}.
\end{aligned}
$$

Solution 3 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

Letting $t = \sin x$ yields $\displaystyle\int_{\{1/\sqrt{2}\}}^1 \frac{1}{t^{2n}} \cdot \frac{1}{\sqrt{1-t^2}} dt.$

Moreover, $y = \sqrt{\dfrac{1}{t^2} - 1}$ yields

$$
\int_1^0 (y^2+1)^n \frac{\sqrt{1+y^2}}{y} \frac{-y}{(1+y^2)^{3/2}} dy = \int_0^1 (1+y^2)^{n-1} dy.
$$

Therefore,

$$
\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{2n-2k-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} \int_0^1 t^{2n-2k-2} dt
$$

$$
\begin{aligned}
&= \int_0^1 t^{2n-2} \sum_{k=0}^{n-1} \binom{n-1}{k} t^{-2k} dt \\
&= \int_0^1 t^{2n-2} \left(1 + \frac{1}{t^2}\right)^{n-1} \\
&= \int_0^1 (1+t^2)^{n-1} dt.
\end{aligned}
$$

and this concludes the proof.

Also solved by Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Anastasios Kotronis, Athens, Greece; Kee-Wai Lau, Hong Kong, China, and the proposer.