Problems   Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before December 15, 2015

- **5361:** *Proposed by Kenneth Korbin, New York, NY*

  Convex quadrilateral $ABCD$ has perimeter $P = 75 + 61\sqrt{15}$ and has $\angle B = \angle D = 90°$. The lengths of the diagonals are 112 and 128. Find the lengths of the sides.

- **5362:** *Proposed by Michael Brozinsky, Central Islip, NY*

  Two thousand forty seven death row prisoners were arranged from left to right with the numbers 1 through 2047 on their backs in this left to right order. Prisoner 1 was given a gun and shoots prisoner number 2 dead, and then gives the gun to prisoner number 3 who shoots prisoner number 4 and then gives the gun to number 5 and so on, so that every second originally numbered prisoner is shot dead.

  This process is then repeated from right to left, starting with the person (in this case number 2047) who last received the gun and then continues to proceed from right to left, and then the direction switches again, and then again until only one prisoner remains standing. What is the number of the prisoner who survives the left to right, right to left shootout? Note that if there had been 2048 prisoners, number 2047 would have no one to whom to hand the gun in the left to right direction after shooting number 2048, and so he would then start the gun in its opposite direction shooting the living prisoner to his immediate left i.e., number 2045. In this case, number 2047 gets to shoot two prisoners before he hands the gun off to another prisoner.

- **5363:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzaău, Romania*

  Let $x \in \Re$ and $A(x) = \begin{pmatrix} x+1 & 1 & 1 & 1 \\ 1 & x+1 & 1 & 1 \\ 1 & 1 & x+1 & 1 \\ 1 & 1 & 1 & x+1 \end{pmatrix}$.

  Compute $A(0) \cdot A(x) \cdot A(y) \cdot A(z), \forall x, y, z \in \Re$.

- **5364:** *Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

  Prove that $\sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2k}{k} 4^{-n} = 1.$

- **5365:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $n \geq 3$ be a positive integer. Find all real solutions of the system

$$\left.\begin{array}{c} a_2^3(a_2^2 + a_3^2 + \ldots + a_{j+1}^2) = a_1^2 \\ a_3^3(a_3^2 + a_4^2 + \ldots + a_{j+2}^2) = a_2^2 \\ \cdots\cdots\cdots \\ a_n^3(a_n^2 + a_1^2 + \ldots + a_{j-1}^2) = a_{n-1}^2 \end{array}\right\}$$

  for $1 < j < n$.

- **5366:** *Proposed by Ovidiu Furdui and Alina Sintămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Find all non-constant, differentiable functions $f : R \to R$ which verify the functional equation $f(x+y) - f(x-y) = 2f'(x)f(y)$, for all $x, y \in R$.

Solutions

- **5343:** *Proposed by Kenneth Korbin, New York, NY*

  Four different Pythagorean Triangles each have hypotenuse equal to $4p^4 + 1$ where p is prime.

  Express the sides of these triangles in terms of p.

Solution 1 by Brian D. Beasley, Presbyterian College, Clinton, SC

We designate the lengths of the legs of these triangles by a and b, so that $a^2 + b^2 = (4p^4+1)^2$. We then make use of the well-known identity

$$(w^2 + x^2)(y^2 + z^2) = (wy + xz)^2 + (wz - xy)^2.$$

Since $4p^4 + 1 = (2p^2)^2 + (1)^2 = (2p^2-1)^2 + (2p)^2$, we make the appropriate substitutions into the above identity to obtain the following four expressions of $(4p^4+1)^2$ as the sum of two squares:

$$\begin{aligned} (4p^4+1)^2 &= (4p^4-1)^2 + (4p^2)^2 \\ &= (4p^4 - 8p^2 + 1)^2 + (8p^3 - 4p)^2 \\ &= (4p^4 - 2p^2 + 2p)^2 + (4p^3 - 2p^2 + 1)^2 \\ &= (4p^4 - 2p^2 - 2p)^2 + (4p^3 + 2p^2 - 1)^2. \end{aligned}$$

Hence the four triangles have the following lengths for their legs:

$a = 4p^4 - 1,\ b = 4p^2;$

$a = 4p^4 - 8p^2 + 1,\ b = 8p^3 - 4p;$

$a = 4p^4 - 2p^2 + 2p,\ b = 4p^3 - 2p^2 + 1;$

$a = 4p^4 - 2p^2 - 2p,\ b = 4p^3 + 2p^2 - 1.$

Addendum. We note that for $p \geq 2$, these eight values of a and b are positive and distinct. We also observe that the condition that p be prime does not seem to be necessary.

Solution 2 by Trey Smith, Angelo State University, San Angelo, TX

It is well known that if $m > n$ are both positive integers then

$$(m^2 - n^2,\ 2mn,\ m^2 + n^2)$$

is a Pythagorean triple.

1. Letting $m_1 = 2p^2$ and $n_1 = 1$ yields the Pythagorean triple
$$(\ 4p^4 - 1,\ 4p^2,\ 4p^4 + 1\ ).$$
2. Letting $m_2 = 2p^2 - 1$ and $n_2 = 2p$ yields the Pythagorean triple
$$(\ 4p^4 - 8p^2 + 1,\ 8p^3 - 4p,\ 4p^4 + 1\ ).$$
3. $4p^4 + 1 = (2p^2 + 2p + 1)(2p^2 - 2p + 1)$, and
$2p^2 + 2p + 1 = p^2 + 2p + 1 + p^2 = (p+1)^2 + p^2$. Letting $m_3 = p + 1$ and $n_3 = p$ yields the Pythagorean triple $(\ 2p + 1,\ 2p(p+1),\ 2p^2 + 2p + 1\ )$. Multiplying each side of the associated Pythagorean triangle by $2p^2 - 2p + 1$ yields the triple
$$(\ (2p + 1)(2p^2 - 2p + 1),\ 2p(p + 1)(2p^2 - 2p + 1),\ 4p^4 + 1\ ).$$
4. Using a similar argument to 3 above, and letting $m_4 = p$ and $n_4 = p - 1$ then multiplying each side of the associated Pythagorean triangle by $2p^2 + 2p + 1$ yields the triple
$$(\ (2p - 1)(2p^2 + 2p + 1),\ 2p(p - 1)(2p^2 + 2p + 1),\ 4p^4 + 1\ ).$$

It is worth noting that the above computations produce the demonstrated four Pythagorean triangles for any given prime p. There are, however, cases where a particular choice of p yields more than four Pythagorean triangles. For example, when $p = 3$ we have the triples
$(36,\ 323,\ 325)$,
$(80,\ 315,\ 325)$,
$(91,\ 312,\ 325)$,
$(125,\ 300,\ 325)$,
$(165,\ 280,\ 325)$,
$(195,\ 260,\ 325)$,
$(204,\ 253,\ 325)$.

Solution 3 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

We remove the restriction that p be prime, requiring only that p be an integer ≥ 2. It is very well known that very Pythagorean triangle (a, b, c) has the from

$$a \ = \ k(2mn)$$

$$\begin{aligned} b &= k\left(m^2-n^2\right) \\ c &= k\left(m^2+n^2\right), \end{aligned}$$

where $k \geq 1$, and m and n are relatively prime integers of opposite parity with $m > n$. Thus we need to write $4p^4+1$ in the form of $k\left(m^2+n^2\right)$ in four different ways.

We have the obvious choices

$$4p^4+1 = 1\cdot\left[\left(2p^2\right)^2+1^2\right] \quad \textit{and } 4p^4+1 = 4p^4-4p^2+1+4p^2 = 1\cdot\left[\left(2p^2-1\right)^2+(2p)^2\right].$$

A different factorization produces two more triangles:

$$\begin{aligned} 4p^4+1 &= \left(2p^2-2p+1\right)\cdot\left(2p^2+2p+1\right) \\ &= \left[p^2+(p-1)^2\right]\left[(p+1)^2+p^2\right] \\ &= \left(2p^2-2p+1\right)\left[(p+1)^2+p^2\right] \text{ and} \\ &= \left(2p^2+2p+1\right)\left[p^2+(p-1)^2\right]. \end{aligned}$$

We summarize the results in Table 1:

k	m	n	$a=k(2mn)$	$b=k(m^2-n^2)$	$c=k(m^2+n^2)$
1	$2p^2$	1	$4p^2$	$4p^4-1$	$4p^4+1$
1	$2p^2-1$	$2p$	$8p^3-4p$	$4p^4-8p^2+1$	$4p^4+1$
$2p^2-2p+1$	$p+1$	p	$4p^4-2p^2+2p$	$4p^3-2p^2+1$	$4p^4+1$
$2p^2+2p+1$	p	$p-1$	$4p^4-2p^2-2p$	$4p^3-2p^2+1$	$4p^4+1$

It appears that we have four triangles with the required hypotenuse, but we need to check they are really distinct. Since all of the "a legs" are even and the "b legs" odd, we only need to compare the values for a and show they are all distinct. This requires 6 comparisons.

For instance, if it were the case that the first two triangles were the same for some value of p, we would have $4p^2 = 8p^3-4p$.

then $0 = 8p^3-4p^2-4p = 4p(p-1)(2p+1)$, which is impossible.

The other comparisons also prove to be impossible.

Therefore, we do have four distinct Pythagorean triangles with hypotenuse $4p^2+1$.

An example with $p=2$.

k	m	n	$a=k(2mn)$	$b=k(m^2-n^2)$	$c=k(m^2+n^2)$
1	8	1	16	63	65
1	7	4	56	33	65
5	3	2	60	250	65
13	2	1	52	39	65

Note that these four triples are all possible with triples with hypotenuse 65, so the result proved is, in general, the best possible.

The four triples produced for $p = 3$, so that $4 \cdot 3^4 + 1 = 325$:

k	m	n	a	b	c
1	18	1	36	323	325
1	17	6	204	253	325
13	4	3	312	91	325
25	3	2	300	125	325

A Deeper Look: There are many more such triangles having hypotenuse $4p^4 + 1$. Consider the following construction suggested the last row of our table.

The generating pair $m = 2, n = 1$ produces a Pythagorean triangle with hypotenuse 5. If we can find a value of p such that 5 divides $4p^4 + 1$, then we can let $k = \dfrac{4p^4+1}{5}$, $m = 2$ and $n = 1$ and produce the triangle.

$$a = k(2mn) = 4k;\ b = k\left(m^2 - n^2\right) = 3k;\ c = \left(m^2 + n^2\right) = \frac{4p^4+1}{5} \cdot 5 = 4p^4 + 1.$$

Are there any such p? Well,

$$5\big|(4p^4+1) \iff 4p^4 + 1 \equiv 0(\text{mod } 5) \iff -p^4 \equiv -1(\text{mod } 5) \iff p^4 \equiv (1\text{mod } 5).$$

By Fermat's Little Theorem, this last condition is true for all p relatively prime to 5. That is, for any p not divisible by 5, we have a Pythagorean triangle with hypotenuse $4p^4 + 1$.

For instance, with $p = 2,\ k = \dfrac{4 \cdot 2^4 + 1}{5} = \dfrac{65}{3} = 13$, and this construction re-creates the last row of our table.

Let's designate the triple found via this construction at $PT(2;\ 13, 2, 1)$.

In general, we designate by $PT(p; k, m, n)$ the triangle having hypotenuse $4 \cdot p^4 + 1$, generated by $k = \dfrac{4 \cdot p^4 + 1}{m^2 + n^2}$, m and n, where m and n are relatively prime integers of opposite parity with $m > n$.

With $p = 3,\ k = \dfrac{4 \cdot 3^4 + 1}{5} = \dfrac{325}{5} = 65$, and this construction yields a *new* triangle with hypotenuse 325; $(260, 195, 325)$ that is $PT(3;\ 65, 2, 1)$. Note that the four solutions given in Table 1 are $PT(p;\ 1, 2p^2, 1)$, $PT(p;\ 1, 2p^2 - 1, 2p)$, $PT(p;\ 2p^2 - 2p + 1, p + 1, p)$ and $PT(p;\ 2p^2 + 2p + 1, p, p - 1)$.

Continuing in this vein, the generating pair $m = 3, n = 2$ produces a Pythagorean triangle with hypotenuse 13. If we can find a value of p such that 13 divides $4p^4 + 1$, then we can let

$$k = \frac{4p^4+1}{13}, m = 3 \text{ and } n = 2 \text{ and produce the triangle}$$

$$a = k(2mn) = 12k,\ b = k\left(m^2 - n^2\right) = 5k,\ c = k\left(m^2 + n^2\right) = \frac{4p^4+1}{13} \cdot 13 = 4p^4 + 1.$$

Are there any such p? Well,

$$13 \big| 4p^4+1 \iff 4p^4+1 \equiv 0 (\text{mod } 13) \iff 4p^4 \equiv -1 (\text{mod } 13) \iff 4p^4 \equiv 12 (\text{mod } 13)$$

$$\iff p^4 \equiv 3 (\text{mod } 13).$$

It is easy to check that this last congruence is satisfied if and only if $p = 2, 3, 10$ or $11 (\text{mod } 13)$. Using any such p will produce a triangle generated by

$$k = \frac{4p^4+1}{13}, m = 3 \text{ and } n = 2 \text{ and of the form}$$

$$a = 12k,\ b = 5k,\ c = \frac{4p^4+1}{13} \cdot 13 = 4p^4+1.$$

This process can be used for any fundamental generating pair m and n.

Theorem: This construction produces all Pythagorean triples having the desired hypotenuse, $4p^4+1$.

First, some evidence. For instance, we re-examined the table for $p = 2$.

k	m	n	a	b	c	PT
1	8	1	16	63	65	PT(2;1,8,1)
1	7	4	56	33	65	PT(2;1,7,4)
5	3	2	60	25	65	PT(2;5,3,2)
13	2	1	52	39	65	PT(2;13,2,1)

For $p = 3$, we also look at all Pythagorean triples with hypotenuse $4 \cdot 3^4 + 1 = 325$, where the first four triples are those shown above, produced by our procedure shown in Table 1.

k	m	n	a	b	c	PT
1	18	1	36	323	325	PT(3;1,18,1)
1	17	6	204	253	325	PT(3;1,17,6)
13	4	3	312	91	325	PT(3;13,4,3)
25	3	2	300	125	325	PT(3;25,3,2)
			80	315	325	PT(3;5,8,1)
			280	165	325	PT(3;5,7,4)
			260	195	325	PT(3;13,2,1)

Proof of the theorem. Suppose we are given a Pythagorean triple (a, b, c) which has hypotenuse of the form $4p^2+1$. We can immediately computer p from
$c = 4p^4+1;\quad p = \sqrt[4]{\dfrac{c-1}{4}}.$

We can also computer $k = \gcd(a, b)$.

This gives us a primitive Pythagorean triple $\left(\frac{a}{k}, \frac{b}{k}, \frac{c}{k}\right)$, in which we may choose $\frac{a}{k}$ to be the even leg.

That is, we must find appropriate m and n so that

$$\frac{a}{k} = 2mn,\quad \frac{b}{k} = m^2 - n^2,\quad \frac{c}{k} = \frac{4p^2+1}{k} = m^2 + n^2.$$

By solving the last two equations, we find that $m = \sqrt{\dfrac{b+c}{2k}}$ and $n = \sqrt{\dfrac{c-b}{2k}}$.

These must be coprime integers of opposite parity, because $\left(\dfrac{a}{k}, \dfrac{b}{k}, \dfrac{c}{k}\right)$is a primitive Pythagorean triple.

Therefore, (a, b, c) is $PT(p; k, m, n)$.

Caveat: Producing triples by using this construction is rather random. Given an appropriate generating pair (m, n) we must find p (and thus k) by solving the congruence $4p^4 + 1 \equiv 0 \bmod \left(m^2 + n^2\right)$.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University San Angelo, TX; Jerry Chu (Student, Saint George's School), Spokane, WA; Bruno Salgueiro Fanego (two solutions), Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY, and the proposer.

- **5344:** *Proposed by Y. N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan*

Let $\triangle ABC$ be isosceles with $AB = AC$. Let D be a point on side BC. A line through point D intersects rays AB and AC at points E and F respectively. Prove that $ED \cdot DF \geq BD \cdot DC$.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

Let Γ be the circle which passes through B, C and E and let J be the other point of intersection of the line DE with Γ. Since E and F are on the rays with origin A and with orientations $\overline{AB}$ and $\overline{AC}$ respectively, we have that $DF = DJ = JF \geq DJ$ with equality if, and only if $J = C = F$, that is, if, and only if the line EF=BC, so

$$ED \cdot DF \geq ED \cdot DJ \qquad (1)$$

with equality if, and only if the line through point D given in the statement of the problem is the line BC, and, by the intersecting chords theorem, the absolute value of the power of D with respect to Γ is ED$\cdot DJ$ and also $BD \cdot DC$ that is

$$ED \cdot DJ = BD \cdot DC. \qquad (2)$$

From (1) and (2) we deduce the inequality to be shown and that equality occurs if, and only if, the line through point D is the line BC.

Solution 2 by Titu Zvonaru, Comănesti, Romania

We denote by M the midpoint of $BC, a = MB, = MC, h = AM$ and $\tan(\angle FDC) = m$. Suppose that F lies between A and C. A parallel line to EF through M intersects AB and AC at points E' and F' respectively. By Similitude, we obtain:

$$\begin{aligned} \frac{DF}{MF'} &= \frac{DC}{MC} \iff DF = \frac{MF' \cdot DC}{MC}, \\ \frac{DE}{ME'} &= \frac{DB}{MB} \iff DF = \frac{ME' \cdot DB}{MB}. \end{aligned}$$

(1)

Since

$$ED{\cdot}DF \geq BD{\cdot}DC \iff \frac{ME' \cdot DB}{MB}\cdot\frac{MF' \cdot DC}{BC} \geq BD{\cdot}DC \iff ME{\cdot}MF \geq MB{\cdot}MC,$$

we deduce that it suffices to prove the statement of the problem if D is the midpoint of BC. In the following we will assume that D is the midpoint of BC.

Lt T be the projection of F to BC. It results that

$$\frac{TC}{DC} = \frac{FT}{AD} \iff \frac{MC - DT}{DC} = \frac{DT \cdot m}{AD} \Rightarrow DT = \frac{ah}{h + am}.$$

By the Pythagorean Theorem, we obtain

$$DF = \sqrt{DT^2 + FT^2} = \sqrt{\frac{a^2h^2}{(h+am)^2} + \frac{a^2h^2}{(h+am)^2}m^2} = \frac{ah}{h+am}\sqrt{1+m^2},$$

and similarly, $DE = \dfrac{ah}{h - am}\sqrt{1+m^2}$.

It results that:

$$ED{\cdot}DF \geq BD{\cdot}DC \iff \frac{a^2h^2}{h^2 - a^2m^2}\left(1+m^2\right) \iff h^2{+}h^2m^2 \geq h^2{-}a^2m^2 \iff \left(a^2+h^2\right)m^2 \geq 0,$$

which is true. The equality holds if and only if $m = 0$, that is, the line through D is BC.

Solution 3 by Ed Gray, Highland Beach, FL

To be specific in the case you wish to draw a diagram, let the point D be on the left of middle of side BD so that point E is on side AB in the triangle closer to B than to A. The point F on the extension of AC and is external to the triangle ABC. We shall be interested in triangles EBE and DCF.

In $\triangle BED$, let $\alpha = \angle EBD$ and let $\beta = \angle EDB$. So $\angle DEB = 180 - \alpha - \beta$. Also $\angle BCA = \alpha$ because $\triangle ABC$ is isosceles.

In $\triangle$CDF, $\angle FDC = \beta; FCD = 180 - \alpha$, and although $\triangle EBD$ and $\triangle FCD$ are not similar to one another, the law of sines holds in each triangle.

In $\triangle BED$; $\dfrac{ED}{\sin\alpha} = \dfrac{BD}{\sin(180-\alpha-\beta)} = \dfrac{BD}{\sin(\alpha+\beta)}$. So, $ED = \dfrac{BD\sin a}{\sin(\alpha+\beta)}$.

In $\triangle DCF$; $\dfrac{DC}{\sin(\alpha-\beta)} = \dfrac{DF}{\sin(180-\alpha)} = \dfrac{DF}{\sin\alpha}$. So, $DF = \dfrac{DC\sin\alpha}{\sin(\alpha-\beta)}$.

To show $ED \cdot DF \geq BD \cdot DC$ we must show that

$$\frac{(BD\sin a)\cdot(DC\sin\alpha)}{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)} \geq BD \cdot DC, \text{ or}$$

$$\frac{\sin^2\alpha}{\sin(\alpha+\beta)\sin(\alpha-\beta)} \geq 1, \quad \text{or}$$

$$\sin^2\alpha \quad \geq \quad \sin(\alpha+\beta)\sin(\alpha-\beta)$$

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ \sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta,\ \text{or}\\ \sin^2\alpha &\geq (\sin\alpha\cos\beta+\cos\alpha\sin\beta)(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\\ &= \sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta.\end{aligned}$$

Adding $\cos^2\alpha$ to both sides of the above inequality we obtain

$$\begin{aligned}1 &\geq \cos^2\alpha-\cos^2\alpha\sin^2\beta+\sin^2\alpha\cos^2\beta=\cos^2\alpha(1-\sin^2\beta)+\sin^2\alpha\cos^2\beta\\ 1 &\geq \cos^2\alpha\cos^2\beta+\sin^2\alpha\cos^2\beta=(\cos^2\beta)(\cos^2\alpha+\sin^2\alpha)\\ 1 &\geq \cos^2\beta,\ \text{and this proves the conjecture.}\end{aligned}$$

Also solved by Michael Brozinsky, Central Islip, NY; Jerry Chu (student, Saint George's School), Spokane, WA; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins, Georgia Southern University, Statesboro GA, and the proposer.

- **5345:** *Proposed by Arkady Alt, San Jose, CA*

Let $a,b>0$. Prove that for any x,y the following inequality holds

$$|a\cos x+b\cos y|\leq\sqrt{a^2+b^2+2ab\cos(x+y)},$$

and find when equality occurs.

Solution 1 by Michael Brozinsky, Central Islip, NY

Since $\sqrt{u^2}=|u|$, the left hand side of the given inequality can be written as

$$a^2\cos^2 x+2ab\cos x\cos y+b^2\cos^2 y,$$

and so using the identities $\sin^2 u=1-\cos^2 u$ and $\cos(x+y)=\cos x\cos y-\sin x\sin y$, it must be shown that

$$a^2\sin^2 x+b^2\sin^2 y\geq 2ab\sin x\sin y.$$

This is true from the AM-GM inequality, with equality if, and only if, $a\sin x=b\sin y$.

Solution 2 by Paul M. Harms, North Newton, KS

Since each side of the inequality is a nonnegative number, the inequality holds if the square of the left side is less than or equal to the square of the right side. We need to show that

$$(a\cos x+b\sin y)^2=a^2\cos^2 x+2ab\cos x\cos y+b^2\cos^2 y\leq a^2+b^2+2ab\cos(x+y).$$

The last inequality is equivalent to

$$\begin{aligned}
0 &\leq a^2\left(1-\cos^2 x\right)+b^2\left(1-\cos^2 y\right)+2ab\left(\cos(x+y)-\cos x\cos y\right)\\
&= a^2\sin^2 x+b^2\sin^2 y+2ab((\cos x\cos y-\sin x\sin y)-\cos x\cos y)\\
&= (a\sin x-b\sin y)^2.
\end{aligned}$$

Clearly, $0\leq(a\sin x-b\sin y)^2$ so the problem inequality holds. Equality will hold when $a\sin x=b\sin y$ or $\dfrac{a}{b}=\dfrac{\sin x}{\sin y}$.

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Dionne Bailey, Elsie Campbell, Charles Diminnie, and Karl Havlak, Angelo State University, San Angelo, TX; Brian D. Beasley, Presbyterian College, Clinton, SC; Jerry Chu (student, Saint George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Ethan Gegner (student, Taylor University), Upland, IN; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vergata, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro GA; Vu Tran (student, Purdue University), West Lafayette, IN; Nicusor Zlota, "Traian Vula" Technical College, Focsani, Romania, and the proposer.

- **5346:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

Show that in any triangle ABC, with the usual notations, the following hold,

$$\frac{h_b+h_c}{h_a}r_a^2+\frac{h_c+h_a}{h_b}r_b^2+\frac{h_a+h_b}{h_c}r_c^2 \;\geq\; 2s^2,$$

where r_a is the excircle tangent to side a of the triangle and s is the triangle's semiperimeter.

Solution 1 by Moti Levy, Rehovot, Israel

From geometry of the triangle:

$$h_a=\frac{2}{\dfrac{1}{r_b}+\dfrac{1}{r_c}},\; h_b=\frac{2}{\dfrac{1}{r_a}+\dfrac{1}{r_c}}\;,\; h_c=\frac{2}{\dfrac{1}{r_b}+\dfrac{1}{r_a}}. \qquad (1)$$

Solving (1) for r_a, r_b and r_c, we get

$$r_a \;=\; \frac{h_ah_bh_c}{h_ah_b+h_ah_c-h_bh_c}$$

$$
\begin{aligned}
r_b &= \frac{h_c h_a h_b}{h_a h_b - h_a h_c + h_b h_c} \\
r_c &= \frac{h_a h_b h_c}{-h_a h_b + h_a h_c + h_b h_c} \qquad (2)
\end{aligned}
$$

Suppose $h_a \geq h_b \geq h_c$. It follows from (2) that $r_a \leq r_b \leq r_c$. It is also easy to see that $h_a \geq h_b \geq h_c$ implies $\dfrac{h_b + h_c}{h_a} \leq \dfrac{h_c + h_a}{h_b} \leq \dfrac{h_a + h_b}{h_c}$.

So now we can apply Chebyshev's sum inequality,

$$\frac{h_b + h_c}{h_a} r_a^2 + \frac{h_c + h_a}{h_b} r_b^2 + \frac{h_a + h_b}{h_c} r_c^2 \geq \frac{1}{3}\left(\frac{h_b + h_c}{h_a} + \frac{h_c + h_a}{h_b} + \frac{h_a + h_b}{h_c}\right)\left(r_a^2 + r_b^2 + r_c^2\right).$$

Since $x + \frac{1}{x} \geq 2$, for $x \geq 0$,

$$\frac{h_b + h_c}{h_a} + \frac{h_c + h_a}{h_b} + \frac{h_a + h_b}{h_c} = \frac{h_b}{h_a} + \frac{h_c}{h_a} + \frac{h_c}{h_b} + \frac{h_a}{h_b} + \frac{h_a}{h_c} + \frac{h_b}{h_c} \geq 6.$$

$$\frac{h_b + h_c}{h_a} r_a^2 + \frac{h_c + h_a}{h_b} r_b^2 + \frac{h_a + h_b}{h_c} r_c^2 \geq 2\left(r_a^2 + r_b^2 + r_c^2\right).$$

To complete the solution, we use the well known inequality

$$r_a^2 + r_b^2 + r_c^2 \geq s^2,$$

(which can be shown by proving that $\tan^2 \dfrac{\alpha}{2} + \tan^2 \dfrac{\beta}{2} + \tan^2 \dfrac{\gamma}{2} \geq 1$, and that $r_a = s \tan \dfrac{\alpha}{2}, \;\; r_b = s \tan \dfrac{\beta}{2}, \;\; r_c = s \tan \dfrac{\gamma}{2}$).

Reference: Bottemi O., et al. Geometric inequalities (Noordhoff, 1969), 2.35 p. 27, 5.34 p. 57.

Solution 2 by Nikos Kalapodis, Patras, Greece

Applying the Cauchy-Schwartz inequality,

$$\left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right) \geq \left(a_1 b_1 + a_2 b_2 + a_3 b_3\right)^2$$

for $a_1 = \dfrac{r_a}{\sqrt{h_a}}, \; a_2 = \dfrac{r_b}{\sqrt{h_b}}, \; a_3 = \dfrac{r_c}{\sqrt{h_c}}$ and $b_1 = \sqrt{h_a}, \; b_2 = \sqrt{h_b}, \; b_3 = \sqrt{h_c}$ we have

$$\left(\frac{r_a^2}{h_a} + \frac{r_b^2}{h_b} + \frac{r_c^2}{h_c}\right)(h_a + h_b + h_c) \geq (r_a + r_b + r_c)^2,$$

i.e.,

$$\frac{h_b + h_c}{h_a} r_a^2 + \frac{h_c + h_a}{h_b} r_b^2 + \frac{h_a + h_b}{h_v} r_c^2 \geq 2\left(r_a r_b + r_b r_c + r_c r_a\right). \qquad (1)$$

Taking into account the well-known formulas $S^2 = s(s-a)(s-b)(s-c)$ and $S = r_a(s-a) = r_b(s-b) - r_c(s-c)$ for the area S of triangle ABC, we have

$$r_a r_b + r_b r_c + r_c r_a = \frac{S^2}{(s-a)(s-b)} + \frac{S^2}{(s-b)(s-c)} + \frac{S^2}{(s-c)(s-a)}$$

$$= \quad s(s-c)+s(s-a)+s(s-b)$$

$$= \quad s\left(3s-(a+b+c)\right)$$

$$= \quad s(3s-2s)=s^2 \qquad (2)$$

Unsing (1)and (2) we obtain the required inequality.

Solution 3 by Titu Zvonaru, Comănesti, Romania

We suppose that $a \geq b \geq c$. Denoting by F the area of triangle ABC w have

$$a \geq b \geq c \iff \frac{1}{a} \leq \frac{1}{b} \leq \frac{1}{c} \iff \frac{F}{a} \leq \frac{F}{b} \leq \frac{F}{c} \iff h_a \leq h_b \leq h_c$$

$$\iff \frac{h_a+h_b+h_c}{h_a} \geq \frac{h_a+h_b+h_c}{h_b a} \geq \frac{h_a+h_b+h_c}{h_c}$$

$$\iff \frac{h_b+h_c}{h_a} \geq \frac{h_c+h_a}{h_b} \geq \frac{h_a+h_b}{h_c}.$$

and

$$a \geq b \geq c \iff s-a \leq s-b \leq s-c \iff \frac{F}{s-a} \geq \frac{F}{s-b} \geq \frac{F}{s-c} \iff r_a \geq r_b \geq r_c.$$

Applying the Chebyshev inequality and the well known inequality

$$x^2+y^2+z^2 \geq xy+yz+zx,$$

we obtain

$$\frac{h_b+h_c}{h_a}r_a^2+\frac{h_c+h_a}{h_b}r_b^2+\frac{h_a+h_b}{h_c}r_c^2$$

$$\geq \quad \frac{1}{3}\left(\frac{h_b+h_c}{h_a}+\frac{h_c+h_a}{h_b}+\frac{h_a+h_b}{h_c}\right)\left(r_a^2+r_b^2+r_c^2\right)$$

$$\geq \quad \frac{1}{3}\left(\frac{h_a}{h_b}+\frac{h_b}{h_a}+\frac{h_b}{h_c}+\frac{h_c}{h_b}+\frac{h_c}{h_a}+\frac{h_a}{h_c}\right)(r_ar_b+r_br_c+r_cr_a)$$

$$\geq \quad \frac{1}{3}(2+2+2)\left(\frac{F^2}{(s-a)(s-b)}+\frac{F^2}{(s-b)(s-c)}+\frac{F^2}{(s-c)(s-a)}\right)$$

$$= \quad 2\cdot\frac{F^2(s-c+s-a+s-b)}{(s-c)(s-b)(s-c)}$$

$$= \quad 2 \cdot \frac{s(s-a)(s-b)(s-c)s}{(s-a)(s-b)(s-c)} = 2s^2.$$

The equality holds if and only if $a = b = c$, that is, when triangle ABC is equilateral.

Solution 4 by Kee-Wai Lau, Hong Kong, China

Since $h_a = b \sin C, h_b = c \sin A, h_c = a \sin B$, so by the sine formula we have

$$\begin{aligned}
\frac{h_b + h_c}{h_a} &= \frac{c \sin A + a \sin B}{b \sin C} \\
&= \frac{\sin A(\sin B + \sin C)}{\sin B \sin C} \\
&= \frac{\sin A}{\sin B + \sin C}\left(4 + \frac{(\sin B - \sin C)^2}{\sin B \sin C}\right) \\
&\geq \frac{4 \sin A}{\sin B + \sin C} \\
&= \frac{4 \sin\left(\dfrac{A}{2}\right)}{\cos\left(\dfrac{B-C}{2}\right)} \\
&\geq 4 \sin\left(\frac{A}{2}\right).
\end{aligned}$$

Similarly, $\dfrac{h_c + h_a}{h_b} \geq 4 \sin\left(\dfrac{B}{2}\right)$ and $\dfrac{h_a + h_b}{h_c} \geq 4 \sin\left(\dfrac{C}{2}\right)$. Hence using the well-known relations $r_a = s \tan\left(\dfrac{A}{2}\right)$, $r_b = s \tan\left(\dfrac{B}{2}\right)$, $r_c = s \tan\left(\dfrac{C}{2}\right)$, we see that

$$\frac{1}{s^2}\left(\frac{h_b + h_c}{h_a} r_a^2 + \frac{h_c + h_a}{h_b} r_b^2 + \frac{h_a + h_b}{h_c} r_c^2\right) \geq 4\left(f(A/2) + f(B/2) + f(C/2)\right),$$

where $f(x) = \sin x \tan^2 x$, for $0 < x < \dfrac{\pi}{2}$. Since

$$\frac{d^2 f(x)}{dx^2} = \sin x + \tan x \sec x + 4 \tan x \sec^3 x + 2 \tan^3 x \sec x > 0,$$

so, $f(A/2) + f(B/2) + f(C/2) \geq 3f\left(\dfrac{A+B+C}{6}\right) = \dfrac{1}{2}$, and therefore the inequality of the problem holds.

Also solved by Jerry Chu (student, Saint George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Albert Stadler, Herrliberg, Switzerland; Nicusor Zlota, "Traian Vula" Technical College, Focsani, Romania, and the proposers.

- **5347:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $0 < a < b$ be real numbers and let $f, g : [a, b] \to R_+^*$ be continuous functions. Prove that there exists $c \in (a, b)$ such that

$$\left(\frac{1}{f(c)} + \frac{1}{\displaystyle\int_c^b g(t)\,dt}\right)\left(g(c) + \int_a^c f(t)\,dt\right) \geq 4$$

(R_+^* represents the set of non-negative real numbers.)

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

In order to avoid non-sense expressions, as zero denominators, we may assume that f, g are not identically null. The proposed inequality may be written as

$$\frac{g(c) + \displaystyle\int_a^c f(t)dt}{2} \geq \frac{2}{\dfrac{1}{f(c)} + \dfrac{1}{\displaystyle\int_c^b g(t)dt}}.$$

The right-hand side tends to zero for $c \to b$, because $\displaystyle\int_c^b g(t)dt \to 0$. On the other hand, g, and f are not identically null so the limit of the left-hand side is positive for $c \to b$, since at least $\displaystyle\int_a^b f(t)dt > 0$ and the conclusion follows.

Solution 2 by Henry Ricardo, New York Math Circle, NY

Define $F(x) = \displaystyle\int_a^x f(t)dt \cdot \int_x^b g(t)dt$. Since $F(a) = F(b) = 0$, Rolle's theorem tells us that there exists $c \in (a, b)$ such that $0 = F'(c) = f(c)\displaystyle\int_c^b g(t)dt - g(c)\int_a^c f(t)dt$, or

$$f(c)\int_c^b g(t)dt \;=\; g(c)\int_a^c f(t)dt. \qquad (1)$$

Since f and g are non-negative, the AM-GM inequality yields

$$\left(\frac{1}{f(c)} + \frac{1}{\int_c^b g(t)dt}\right)\left(g(c) + \int_a^c f(t)dt\right) \geq \frac{2}{\sqrt{f(c)\int_c^b g(t)dt}} \cdot 2\sqrt{g(c)\int_a^c f(t)dt} = 4$$

by statement (1).

Comment by solver: We are tacitly assuming that $f(c) \neq 0$. It is better to alter the problem's hypothesis so that at least f is strictly positive on $[a, b]$.

Solution 3 by Michael Brozinsky, Central Islip, NY

Assume the contrary that no such c exists so that

$$\left(\frac{1}{f(c)}+\frac{1}{\int_c^b g(t)\,dt}\right)\left(g(c)+\int_a^c f(t)\,dt\right)\geq 4(*) \text{ for all } x \text{ on}(a,b).$$

Now $\displaystyle\int_x^b g(t)dt$ and $\displaystyle\int_a^x f(t)dt$ are continuous and positive functions of x for $a\leq x\leq b$ since $f(t)$ and $g(t)$ are nonnegative and continuous. Hence from $(*)$ we have $\dfrac{g(x)}{f(x)}<4$ for all x on (a,b) $(**)$ and also $\displaystyle\int_a^x f(t)dt<4\cdot\int_x^b g(t)dt$ $(***)$. From $(***)$, $(**)$ then implies that $\displaystyle\int_a^x f(t)dt<4\cdot\int_x^b 4f(t)dt$ and so letting $x\longrightarrow b$ we have a contradiction that $\displaystyle\int_a^b f(t)dt\leq 0$. Hence there exists a c on (a,b) such that $F(c)>4$, in fact, there exists a c on (a,b) such that $F(c)>M$ where M is an arbitrary positive number as the above proof shows replacing the $4'$s by M throughout.

Solution 4 by Paolo Perfetti, Department of Mathematics, Tor Vergata, Rome, Italy

We argue by contradiction assuming that

$$\left(\frac{1}{f(c)}+\frac{1}{\int_c^b g(t)dt}\right)\left(g(c)+\int_a^c f(t)dt\right)<4$$

for any $c\in(a,b)$.

Cauchy Schwarz yields

$$4>\left(\frac{1}{f(c)}+\frac{1}{\int_c^b g(t)dt}\right)\left(g(c)+\int_a^c f(t)dt\right)\geq\left(\sqrt{\frac{\int_a^c f(t)dt}{f(c)}}+\sqrt{\frac{g(c)}{\int_c^b g(t)dt}}\right)^2$$

Now we prove the Lemma

Lemma *There exists $d\in(a,b)$ such that* $\dfrac{\int_a^d f(t)dt}{f(d)}\geq\dfrac{\int_d^b g(t)dt}{g(d)}$.

Proof

$\dfrac{\int_a^d f(t)dt}{f(d)}\geq\dfrac{\int_d^b g(t)dt}{g(d)}$ if and only if

$$g(d)\int_a^d f(x)dx\geq f(d)\int_d^b g(x)dx \tag{1}$$

Now let $g(b)=g_0>0$. A value d can be chosen so close to b such that $|g(x)-g_0|\leq g_0/2$ for any $x\in(d,b]$. For the same reasons $|f(x)-f_0|\leq f_0/2$ for any $x\in(d,b]$ where $f_0=f(b)$. Moreover we can suppose $\displaystyle\int_a^d f(x)dx\geq\frac{1}{2}\int_a^b f(x)dx=I/2>0$. We can write

$$g(d)\int_a^d f(x)dx \geq \frac{1}{2}g_0\int_a^d f(x)dx \geq \frac{1}{2}g_0\frac{I}{2}$$

and

$$\frac{3}{2}f(b)\frac{3}{2}g_0(b-d) \geq f(d)\int_d^b g(x)dx.$$

To prove (1) it suffices

$$\frac{1}{2}g_0\frac{I}{2} \geq \frac{3}{2}f(b)\frac{3}{2}g_0(b-d) \iff I \geq 9f(b)(b-d)$$

and this clearly holds provided that d is very close to b. .

Thanks to the lemma, we can write

$$4 > \left(\sqrt{\frac{\int_a^c f(t)dt}{f(c)}} + \sqrt{\frac{g(c)}{\int_c^b g(t)dt}}\right)^2 \geq \left(\sqrt{\frac{\int_d^b g(t)dt}{g(d)}} + \sqrt{\frac{g(d)}{\int_d^b g(t)dt}}\right)^2 \geq 4$$

since $x + 1/x \geq 2$ for any $x > 0$, contradiction.

Also solved by Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel, and the the proposer.

- **5348:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $k \geq 1$ be an integer. Prove that

$$\int_0^1 \ln^k(1-x)\ln x\,\mathrm{d}x = (-1)^{k+1}k!(k+1-\zeta(2)-\zeta(3)-\cdots-\zeta(k+1)),$$

where ζ denotes the Riemann zeta function.

Solution 1 by Moubinool Omarjee of Lycée Henri IV, Paris, France

We change the variable letting $u = -\ln(1-x)$.

$$\begin{aligned}
\int_0^1 \ln^k(1-x)\ln x dx &= (-1)^k\int_0^{+\infty} u^k\ln(1-e^{-u})e^{-u}du \\
&= (-1)^{k+1}\int_0^{+\infty}\sum_{n=1}^{\infty}\frac{1}{n}u^k e^{-u(n+1)}du \\
&= (-1)^{k+1}\sum_{n=2}^{\infty}\frac{1}{n-1}\int_0^{+\infty} u^k e^{-un}du \\
&= (-1)^{k+1}\sum_{n=2}^{\infty}\frac{1}{n-1}\frac{1}{n^{k+1}}\Gamma(k+1)
\end{aligned}$$

$$= (-1)^{k+1}\sum_{n=2}^{\infty}\frac{1}{n-1}\frac{1}{n^{k+1}}k!$$

$$= (-1)^{k+1}k!\sum_{n=2}^{\infty}\frac{1}{n-1}\frac{1}{n^{k+1}}$$

$$= (-1)^{k+1}k!\sum_{n=2}^{\infty}\left(\frac{1}{n(n-1)}-\frac{1}{n^2}-\frac{1}{n^3}-\ldots-\frac{1}{n^{k+1}}\right)$$

$$= (-1)^{k+1}k!\sum_{n=2}^{\infty}\left(\frac{1}{n(n-1)}-\sum_{n=2}^{\infty}\frac{1}{n^2}-\sum_{n=2}^{\infty}\frac{1}{n^3}-\ldots-\sum_{n=2}^{\infty}\frac{1}{n^{k+1}}\right)$$

$$= (-1)^{k+1}k!\left(k+1-\zeta(2)-\zeta(3)-\ldots-\zeta(k+1)\right)$$

Solution 2 by Anastasios Kotronis, Athens, Greece

It is straightforward to see that $\sum_{n\geq 1}\frac{x^n}{n}\ln^k x$ converges uniformly on $[0,1]$ and, integrating by parts, that for n,k non negative integers:

$$\int x^n \ln^k x\,dx = x^{n+1}\left(\frac{\ln^k x}{n+1}-\frac{k\ln^{k-1}x}{(n+1)^2}+\frac{k(k-1)\ln^{k-2}x}{(n+1)^3}-\ldots+\frac{(-1)^k k!}{(n+1)^{k+1}}\right)+c.$$

so we have

$$\int_0^1 \ln^k(1-x)\ln x\,dx \overset{1-x=y}{=\!=\!=} \int_0^1 \ln(1-y)\ln^k y\,dy = -\int_0^1\sum_{n\geq 1}\frac{y^n}{n}\ln^k y\,dy = -\sum_{n\geq 1}\frac{1}{n}\int_0^1 y^n\ln^k y\,dy$$

$$=(-1)^{k+1}k!\sum_{n\geq 1}\frac{1}{n(n+1)^{k+1}} = (-1)^{k+1}k!\sum_{n\geq 2}\frac{1}{(n-1)n^{k+1}}$$

$$= (-1)^{k+1}k!\sum_{n\geq 2}\frac{1-1+\frac{1}{n^{k+2}}}{1-\frac{1}{n}} = (-1)^{k+1}k!\sum_{n\geq 2}\left(\frac{n}{n-1}-\sum_{m=0}^{k+1}\frac{1}{n^m}\right)$$

$$= (-1)^{k+1}k!\sum_{n\geq 2}\left(\frac{n}{n-1}-1-\frac{1}{n}-\sum_{m=2}^{k+1}\frac{1}{n^m}\right)$$

$$= (-1)^{k+1}k!\left(\sum_{n\geq 2}\left(\frac{1}{n-1}-\frac{1}{n}\right)-\sum_{n\geq 2}\sum_{m=2}^{k+1}\frac{1}{n^m}\right)$$

$$= (-1)^{k+1}k!\left(1-\sum_{m=2}^{k+1}\sum_{n\geq 2}\frac{1}{n^m}\right) = (-1)^{k+1}k!\left(1-\sum_{m=2}^{k+1}(\zeta(m)-1)\right)$$

$$= (-1)^{k+1}k!\left(k+1-\zeta(2)-\zeta(3)-\cdots-\zeta(k+1)\right).$$

Solution 3 by Moti Levy, Rehovot, Israel

Clearly,

$$\int_0^1 \ln^k(1-x)\ln x dx = \int_0^1 \ln(1-x)\ln^k x dx.$$

The Taylor series of $\ln(1-x)$ is $\ln(1-x) = -\sum_{m=1}^{\infty}\frac{x^m}{m}, \qquad |x|<1.$

$$\int_0^1 \ln(1-x)\ln^k x dx = -\int_0^1 \left(\sum_{m=1}^{\infty}\frac{x^m}{m}\right)\ln^k x dx.$$

The order of summation and integration can be interchanged (since $\int_0^1\left(\sum_{m=1}^{\infty}\frac{x^m}{m}\right)\left|\ln^k x\right| dx < \int_0^1 |\ln(1-x)\ln x|\, dx = 2-\frac{1}{6}\pi^2 < \infty$).
Hence,

$$\int_0^1 \ln(1-x)\ln^k x dx = -\sum_{m=1}^{\infty}\frac{1}{m}\int_0^1 x^m \ln^k x dx.$$

After integration by parts of $\int_0^1 x^m\ln^k x dx$, we get the recurrence,

$$\int_0^1 x^m\ln^k x dx = -\frac{k}{m+1}\int_0^1 x^m \ln^{k-1} x dx.$$

It follows from the recurrence relation that,

$$\int_0^1 x^m \ln^k x dx = (-1)^k\frac{k!}{(m+1)^{k+1}}.$$

$$\begin{aligned}\int_0^1 \ln(1-x)\ln^k x dx &= -\sum_{m=1}^{\infty}\frac{1}{m}(-1)^k\frac{k!}{(m+1)^{k+1}}\\ &= (-1)^{k+1}k!\sum_{m=1}^{\infty}\frac{1}{m(m+1)^{k+1}}\\ &= (-1)^{k+1}k!\sum_{m=1}^{\infty}\left(\frac{1}{m}-\frac{1}{m+1}-\frac{1}{(m+1)^2}-\frac{1}{(m+1)^3}-\cdots-\frac{1}{(m+1)^{k+1}}\right).\end{aligned}$$

$$\sum_{m=1}^{\infty}\left(\frac{1}{m}-\frac{1}{m+1}\right)=1, \quad \sum_{m=1}^{\infty}\frac{1}{(m+1)^l} = -1+\sum_{m=1}^{\infty}\frac{1}{m^l} = -1+\zeta(l).$$

$$\int_0^1 \ln(1-x)\ln^k x dx = (-1)^{k+1}k!\left(k+1-\sum_{l=2}^{k+1}\zeta(l)\right).$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News, VA; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vergata, Rome, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Albert Stadler, Herrliberg, Switzerland, and the proposer.

Mea Culpa

The solution to 5340 of Paolo Perfetti of the Mathematics Department at Tor Verga University in Rome, Italy, was inadvertently omitted by the editor from the list of those who had solved the problem. But on the other hand, Paolo also solved 5322, but he inadvertently forgot to send it to the editor on time. Paolo Perfetti should be credited with having solved both 5322 and 5340.