Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before December 15, 2009

- 5074: *Proposed by Kenneth Korbin, New York, NY*

  Solve in the reals:
  $$\sqrt{25+9x+30\sqrt{x}}-\sqrt{16+9x+30\sqrt{x-1}}=\frac{3}{x\sqrt{x}}.$$

- 5075: *Proposed by Kenneth Korbin, New York, NY*

  An isosceles trapezoid is such that the length of its diagonal is equal to the sum of the lengths of the bases. The length of each side of this trapezoid is of the form $a+b\sqrt{3}$ where a and b are positive integers.

  Find the dimensions of this trapezoid if its perimeter is $31+16\sqrt{3}$.

- 5076: *Proposed by M.N. Deshpande, Nagpur, India*

  Let a, b, and m be positive integers and let F_n satisfy the recursive relationship
  $$F_{n+2}=mF_{n+1}+F_n, \text{ with } F_0=a, F_1=b, n\geq 0.$$
  Furthermore, let $a_n=F_n^2+F_{n+1}^2, n\geq 0$. Show that for every a, b, m, and n,
  $$a_{n+2}=(m^2+2)a_{n+1}-a_n.$$

- 5077: *Proposed by Isabel Iriberri Díaz and José Luis Díaz-Barrero, Barcelona, Spain*

  Find all triplets (x, y, z) of real numbers such that
  $$\left.\begin{array}{r} xy(x+y-z)=3, \\ yz(y+z-x)=1, \\ zx(z+x-y)=1. \end{array}\right\}$$

- 5078: *Proposed by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy*

  Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that
  $$\frac{a}{\sqrt{b}(b+c)}+\frac{b}{\sqrt{c}(a+c)}+\frac{c}{\sqrt{a}(a+b)}\geq\frac{3}{2}\frac{1}{\sqrt{ab+ac+cb}}.$$

- 5079: *Proposed by Ovidiu Furdui, Cluj, Romania*

  Let $x \in (0,1)$ be a real number. Study the convergence of the series

$$\sum_{n=1}^{\infty} x^{\sin\frac{1}{1} + \sin\frac{1}{2} + \cdots + \sin\frac{1}{n}}.$$

Solutions

- 5056: *Proposed by Kenneth Korbin, New York, NY*

  A convex pentagon with integer length sides is inscribed in a circle with diameter $d = 1105$. Find the area of the pentagon if its longest side is 561.

  Solution by proposer

  The answer is 25284.

  The sides are $561, 169, 264, 105$, and 47 (in any order).

  Check: $\arcsin\left(\frac{561}{d}\right) = \arcsin\left(\frac{169}{d}\right) + \arcsin\left(\frac{264}{d}\right) + \arcsin\left(\frac{105}{d}\right) + \arcsin\left(\frac{47}{d}\right)$.

  Let $\overline{AB} = 561, \overline{BC} = 105, \overline{CD} = 47, \overline{DE} = 169, \overline{EA} = 264$. Then Diag $\overline{AC} = 468$.

  Check: $\arcsin\left(\frac{468}{d}\right) = \arcsin\left(\frac{47}{d}\right) + \arcsin\left(\frac{169}{d}\right) + \arcsin\left(\frac{264}{d}\right)$.

  Area $\triangle ABC = \sqrt{567 \cdot 99 \cdot 462 \cdot 6} = 12474$.

  Diag $\overline{AD} = 425$.

  Check: $\arcsin\left(\frac{425}{d}\right) = \arcsin\left(\frac{169}{d}\right) + \arcsin\left(\frac{264}{d}\right)$.

  Area $\triangle ACD = \sqrt{470 \cdot 45 \cdot 423 \cdot 2} = 4230$, and
  Area $\triangle ADE = \sqrt{429 \cdot 260 \cdot 165 \cdot 4} = 8580$.

  Area pentagon $= 12474 + 4230 + 8580 = 25284$.

  Editor's comments: Several solutions to this problem were received each claiming, at least initially, that the problem was impossible. I sent these individuals Ken's proof and some responded with an analysis of their errors. **Brian Beasley of Clinton, SC** responded as follows:

  "My assumption was that the inscribed pentagon was large enough to contain the center of the circle, so that I could subdivide the pentagon into five isosceles triangles, each with two radii as sides along with one side of the pentagon. But this pentagon is very small compared to the circle; it does not contain the center of the circle, and the ratio of its area to the area of the circle is only bout 2.64%. Attached is a rough diagram with two attempts to draw such an inscribed pentagon."

  "This has been a fascinating exercise! I found a Wolfram site and a Monthly paper with results about cyclic pentagons: <http://mathworld.wolfram.com/CyclicPentagon.html> and Areas of Polygons Inscribed in a Circle, by D. Robbins, American Mathematical Monthly, 102(6), 1995, 523-530."

"I salute Ken for creating this problem and for finding the arcsine identities to make it work."

David Stone and John Hawkins of Statesboro GA wrote: "Using MATLAB, we found the following four cyclic pentagons which have a side of length 561 and can be inscribed in a circle of diameter 1105. The first one has longest side 561, as required by the problem."

561	264	169	105	47	Area = 25284
817	663	663	561	520	Area = 705276
817	744	576	561	520	Area = 699984
817	744	663	561	425	Area = 692340

- 5057: *Proposed by David C. Wilson, Winston-Salem, N.C.*

We know that $1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k = \dfrac{1}{1-x}$ where $-1 < x < 1$.

Find formulas for $\sum_{k=1}^{\infty} kx^k$, $\sum_{k=0}^{\infty} k^2x^k$, $\sum_{k=0}^{\infty} k^3x^k$, $\sum_{k=0}^{\infty} k^4x^k$, and $\sum_{k=0}^{\infty} k^5x^k$.

Solution 1 by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX

By differentiating the geometric series when $|x| < 1$,

$$\begin{aligned}
\sum_{k=1}^{\infty} x^k &= \frac{1}{1-x} \\
&\Rightarrow \sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2} \\
&\Rightarrow \sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2} \qquad (1)
\end{aligned}$$

Similarly, by differentiating (1),

$$\begin{aligned}
\sum_{k=1}^{\infty} k^2x^{k-1} &= \frac{1+x}{(1-x)^3} \\
&\Rightarrow \sum_{k=1}^{\infty} k^2x^k = \frac{x(1+x)}{(1-x)^3}.
\end{aligned}$$

Continuing this technique, it can be shown that

$$\begin{aligned}
\sum_{k=1}^{\infty} k^3x^k &= \frac{x(x^2+4x+1)}{(1-x)^4} \\
\sum_{k=1}^{\infty} k^4x^k &= \frac{x(x^3+11x^2+11x+1)}{(1-x)^5} \\
\sum_{k=1}^{\infty} k^5x^k &= \frac{x(x^4+26x^3+66x^2+26x+1)}{(1-x)^6}
\end{aligned}$$

Solution 2 by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy

The sums are respectively:

$$\frac{x}{(1-x)^2},\quad \frac{x(x+1)}{(1-x)^3},\quad \frac{x(x^2+4x+1)}{(1-x)^4},$$

$$\frac{x(x^3+11x^2+11x+1)}{(1-x)^5},\quad \frac{x(x^4+26x^3+66x^2+26x+1)}{(1-x)^6}$$

One might invoke standard theorems about the differentiability of convergent power series, but we propose the following proof which we believe is attributed to Euler. We define

$S_p(x) \doteq \sum_{k=1}^{\infty} k^p x^k$, $p = 1, \ldots, 5$ and employ $\sum_{k=1}^{\infty} x^k = \left(\sum_{k=0}^{\infty} x^k\right) - 1 = \frac{1}{1-x} - 1 = \frac{x}{1-x}$.

To compute $\sum_{k=0}^{\infty} x^k - 1 = \frac{1}{1-x}$ we proceed as follows:

$$P \doteq \sum_{k=0}^{\infty} x^k = 1 + x(1 + x + x^2 + \ldots) = 1 + xP \quad \Longrightarrow \quad P = \frac{1}{1-x}.$$

$\mathbf{S_1(x)}$:

$$\sum_{k=1}^{\infty} kx^k = \sum_{k=2}^{\infty}(k-1)x^k + \sum_{k=0}^{\infty} x^k - 1 = x\sum_{n=1}^{\infty} nx^n + \frac{1}{1-x} - 1 \text{ or}$$

$$(1-x)\sum_{k=1}^{\infty} kx^k = \frac{x}{1-x} \quad \Longrightarrow \quad \sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2}.$$

$\mathbf{S_2(x)}$:

$$\sum_{k=1}^{\infty} k^2x^k = \sum_{k=2}^{\infty}(k-1)^2x^k + 2\sum_{k=1}^{\infty} kx^k - \sum_{k=1}^{\infty} x^k \text{ or}$$

$$\begin{aligned}\sum_{k=1}^{\infty} k^2x^k - x\sum_{n=1}^{\infty} n^2x^n &= 2\sum_{k=1}^{\infty} kx^k - \sum_{k=1}^{\infty} x^k \\ &= \frac{2x}{(1-x)^2} - \frac{x}{(1-x)} \Longrightarrow S_2(x) = \frac{x(x+1)}{(1-x)^3}.\end{aligned}$$

$\mathbf{S_3(x)}$:

$$\begin{aligned}\sum_{k=1}^{\infty} k^3x^k &= \sum_{k=2}^{\infty}(k-1)^3x^k + 3\sum_{k=1}^{\infty} k^2x^k - 3\sum_{k=1}^{\infty} kx^k + \sum_{k=1}^{\infty} x^k \\ &= x\sum_{k=1}^{\infty} k^3x^k + 3\sum_{k=1}^{\infty} k^2x^k - 3\sum_{k=1}^{\infty} kx^k + \sum_{k=1}^{\infty} x^k \text{ or}\end{aligned}$$

$$(1-x)\sum_{k=1}^{\infty} k^3x^k = 3S_2(x) - 3S_1(x) + \frac{x}{1-x} \Longrightarrow S_3(x) = \frac{x(x^2+4x+1)}{(1-x)^4}.$$

$\mathbf{S_4(x)}$:

$$\begin{aligned}\sum_{k=1}^{\infty} k^4x^k &= \sum_{k=2}^{\infty}(k-1)^4x^k + 4\sum_{k=1}^{\infty} k^3x^k - 6\sum_{k=1}^{\infty} k^2x^k + 4\sum_{k=1}^{\infty} kx^k - \sum_{k=1}^{\infty} x^k\\ &= x\sum_{k=1}^{\infty} k^4x^k + 4\sum_{k=1}^{\infty} k^3x^k - 6\sum_{k=1}^{\infty} k^2x^k + 4\sum_{k=1}^{\infty} kx^k - \sum_{k=1}^{\infty} x^k \text{ or}\end{aligned}$$

$$(1-x)\sum_{k=1}^{\infty} k^4x^k = 4S_3(x) - 6S_2(x) + 4S_1(x) - \frac{x}{1-x} \Longrightarrow S_4(x) = \frac{x(x^3+11x^2+11x+1)}{(1-x)^5.}$$

$\mathbf{S_5(x)}$:

$$\begin{aligned}\sum_{k=1}^{\infty} k^5x^k &= \sum_{k=2}^{\infty}(k-1)^5x^k + 5\sum_{k=1}^{\infty} k^4x^k - 10\sum_{k=1}^{\infty} k^3x^k + 10\sum_{k=1}^{\infty} k^2x^k - 5\sum_{k=1}^{\infty} kx^k + \sum_{k=1}^{\infty} x^k\\ &= x\sum_{k=1}^{\infty} k^5x^k + 5S_4(x) - 10\sum_{k=1}^{\infty} k^3x^k + 10\sum_{k=1}^{\infty} k^2x^k - 5\sum_{k=1}^{\infty} kx^k + \sum_{k=1}^{\infty} x^k \text{ or}\end{aligned}$$

$$\begin{aligned}(1-x)\sum_{k=1}^{\infty} k^5x^k &= 5S_4(x) - 10S_3(x) + 10S_2(x) - 5S_1(x) + \frac{x}{1-x}\\ &\Longrightarrow S_5(x) = \frac{x(x^4+26x^3+66x^2+26x+1)}{(1-x)^6}.\end{aligned}$$

Also solved by Matei Alexianu (student, St. George's School), Spokane,WA; Brian D. Beasley, Clinton, SC; Sully Blake (student, St. George's School), Spokane,WA; Michael Brozinsky, Central Islip, NY; Mark Cassell (student, St. George's School), Spokane,WA; Richard Caulkins (student, St. George's School), Spokane,WA; Pat Costello, Richmond, KY; Michael C. Faleski, University Center, MI; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; John Hawkins and David Stone (jointly), Statesboro, GA; David E. Manes, Oneonta, NY; John Nord, Spokane, WA; Nguyen Pham and Quynh Anh (jointly; students, Belarusian State University), Belarus; Boris Rays, Brooklyn, NY, and the proposer.

- 5058: *Proposed by Juan-Bosco Romero Márquez, Avila, Spain.*

  If p, r, a, A are the semi-perimeter, inradius, side, and angle respectively of an acute triangle, show that
$$r + a \le p \le \frac{p}{\sin A} \le \frac{p}{\tan \dfrac{A}{2}},$$
  with equality holding if, and only if, $A = 90^o$.

  Solution by Manh Dung Nguyen,(student, Special High School for Gifted Students) HUS, Vietnam

1) $\mathbf{r+a\le p}$:

$\tan\dfrac{A}{2}\le 1$ for all $A\in(0,\pi/2]$, so by the well known formula $\tan\dfrac{A}{2}=\dfrac{(p-b)(p-c)}{p(p-a)}$ we have $(p-b)(p-c)\le p(p-a)$. Letting S be the area of $\triangle ABC$ and using Heron's formula,

$$\begin{aligned} S^2=p^2r^2 &= p(p-a)(p-b)(p-c)\le p^2(p-a)^2. \text{ Thus}\\ r &\le p-a \text{ or } r+a\le p. \end{aligned}$$

2) $\mathbf{p\le \dfrac{p}{\sin A}}$:

We have $\sin A\le 1$ for all $A\in(0,\pi)$, so $p\le\dfrac{p}{\sin A}$.

3) $\mathbf{\dfrac{p}{\sin A}\le\dfrac{p}{\tan\dfrac{A}{2}}}$:

For $A\in(0,\pi/2]$ we have

$$\sin A-\tan\frac{A}{2}=\sin\frac{A}{2}\left(2\cos\frac{A}{2}-\frac{1}{\cos\frac{A}{2}}\right)=\frac{\sin\frac{A}{2}\cos A}{\cos\frac{A}{2}}\ge 0. \text{ Hence}$$

$$\frac{p}{\sin A}\le\frac{p}{\tan\frac{A}{2}}.$$

Equality holds if and only if $A=90^0$.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Scott H. Brown, Montgomery, AL; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; John Hawkins and David Stone (jointly), Statesboro, GA; Kee-Wai Lau, Hong Kong, China; Boris Rays, Brooklyn, NY, and the proposer.

- 5059: *Proposed by Panagiote Ligouras, Alberobello, Italy.*

  Prove that for all triangles ABC

$$\sin(2A)+\sin(2B)+\sin(2C)+\sin(A)+\sin(B)+\sin(C)+\sin\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)+\sin\left(\frac{C}{2}\right)\le\frac{6\sqrt{3}+1}{8}.$$

Editor's comment: Many readers noted that the inequality as stated in the problem is incorrect. It should have been $\dfrac{3(2\sqrt{3}+1)}{2}$.

Solution 1 by Kee-Wai Lau, Hong Kong, China

We need the following inequalities

$$\sin(A)+\sin(B)+\sin(C) \ge \sin(2A)+\sin(2B)+\sin(2C) \qquad (1)$$

$$\sin(A)+\sin(B)+\sin(A) \le \frac{3\sqrt{3}}{2} \qquad (2)$$

$$\sin(\frac{A}{2})+\sin(\frac{B}{2})+\sin(\frac{C}{2}) \quad \leq \quad \frac{3}{2} \qquad (3)$$

Inequalities (1), (2), (3) appear respectively as inequalities 2.4, 2.2(1),and 2.9 in Geometric Inequalities by O. Bottema, R.Z. Dordevic, R.R. Janic, D.S. Mitrinovic, and P.M. Vasic, (Groningen), 1969.

It follows from (1),(2),(3) that

$$\sin(2A)+\sin(2B)+\sin(2C)+\sin(A)+\sin(B)+\sin(C)+\sin\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)+\sin\left(\frac{C}{2}\right) \leq \frac{3(2\sqrt{3}+1)}{2}.$$

Solution 2 by John Hawkins and David Stone, Statesboro, GA

We treat this as a Lagrange Multiplier Problem: let

$$f(A,B,C)=\sin(2A)+\sin(2B)+\sin(2C)+\sin(A)+\sin(B)+\sin(C)+\sin\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)+\sin\left(\frac{C}{2}\right).$$

We wish to find the maximum value of this function of three variables, subject to the constraint $g(A,B,C): A+B+C=\pi$. That is, (A,B,C) lies in the closed, bounded, triangular region in the first octant with vertices on the coordinate axes: $(\pi,0,0),(0,\pi,0),(0,0,\pi)$.

By taking partial derivatives with respect to the variables A, B, and C and setting $\nabla f(A,B,C)=\lambda\nabla g(A,B,C)$ or $\Big\langle f_A, f_B, f_C\Big\rangle=\lambda\Big\langle g_A, g_B, g_C\Big\rangle=\lambda\langle 1,1,1\rangle$, we are lead to the system

$$\begin{cases} 2\cos(2A)+\cos(A)+\frac{1}{2}\cos\left(\frac{A}{2}\right)=\lambda \\ 2\cos(2B)+\cos(B)+\frac{1}{2}\cos\left(\frac{B}{2}\right)=\lambda \\ 2\cos(2C)+\cos(C)+\frac{1}{2}\cos\left(\frac{C}{2}\right)=\lambda \end{cases}$$

It is clear that one solution is to let $A=B=C$. We claim there are no others in our domain.

To show this, we investigate the fuction $h(\theta)=2\cos(2\theta)+\cos(\theta)+\dfrac{1}{2}\cos\left(\dfrac{\theta}{2}\right)$ on the interval $0\leq\theta\leq\pi$. Finding a solution to our system is equivalent to finding values A, B and C such that $h(A)=h(B)=h(C)=\lambda$.

We determine that $h(0)=3.5$; then the function h decreases, passing through height 1 at (0.802,1), reaching a minimum at $(1.72,-1.73)$, then rising to height 1 at π. No horizontal line crosses the graph three times, so we cannot find distinct A, B and C with $h(A)=h(B)=h(C)$. In fact, because the function is decreasing from 0 to 1.72, and increasing from 1.72 to π, any horizontal line crossing the graph more than once must do so after $\theta=0.802$. That is all of A, B and C would have to be greater than 0.802, and at least one of them greater than 1.72. Because $0.802+0.802+1.72=3.324>\pi$, this violates the condition that $A+B+C=\pi$.

Thus the maximum value occurs when $A = B = C = \dfrac{\pi}{3}$:

$$f\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right) = 3\sin\left(\frac{2\pi}{3}\right) + 3\sin\left(\frac{\pi}{3}\right) + 3\sin\left(\frac{\pi}{6}\right) = 6\frac{\sqrt{3}}{2} + \frac{3}{2} = \frac{6\sqrt{3}+3}{2}.$$

This method tells us that the only point on the plane $A + B + C = \pi$ (in the first octant) where the function f achieves a maximum value is the point we just found. We must check the boundaries for a minimum.

Note that $f(\pi, 0, 0) = 1 = f(0, \pi, 0) = f(0, 0, \pi)$. That is f achieves the lower bound 1 at the vertices of our triangular region.

We also consider the behavior of the function f along the edges of this region. For instance, in the AB-plane where $C = 0$, we have $A + B = \pi$. Then $f(A, \pi - A, 0) = 2\sin A + \sin\left(\dfrac{A}{2}\right) + \cos\left(\dfrac{A}{2}\right)$, which has value 1 (of course) at the endpoints $A = 0$ and $A = \pi$, and climbs to a local maximum value of $2 + \sqrt{2}$ when $A = \dfrac{\pi}{2}$. This value is less than $f\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$.

There is identical behavior along the other two edges.

In summary, the function f achieves an absolute maximum of $\dfrac{6\sqrt{3}+3}{2}$ at the interior point $A = B = C = \dfrac{\pi}{3}$, and f achieves its absolute minimum of 1 at the vertices.

However, for a non-degenerate triangle ABC

$$1 < \sin(2A)+\sin(2B)+\sin(2C)+\sin(A)+\sin(B)+\sin(C)+\sin\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)+\sin\left(\frac{C}{2}\right) \le \frac{6\sqrt{3}+3}{2},$$

and the lower bound is never actually achieved.

Solution 3 by Tom Leong, Scranton, PA

This inequality follows from summing the three known inequalities labeled (1), (2), and (3) below. Both $\sin x$ and $\sin\dfrac{x}{2}$ are concave down on $(0, \pi)$. Applying the AM-GM inequality followed by Jensen's inequality gives

$$\sin A \sin B \sin C \le \left(\frac{\sin A + \sin B + \sin C}{3}\right)^3 \le \sin^3\left(\frac{A+B+C}{3}\right) = \frac{3\sqrt{3}}{8} \qquad (1)$$

and

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \left(\frac{\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}}{3}\right)^3 \le \sin^3\left(\frac{A+B+C}{6}\right) = \frac{1}{8}. \qquad (2)$$

For the third inequality, we use the AM-GM inequality along with the identity

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$$

and (1):

$$\sin 2A \sin 2B \sin 2C \quad \le \quad \left(\frac{\sin 2A + \sin 2B + \sin 2C}{3}\right)^3 = \left(\frac{4\sin A \sin B \sin C}{3}\right)^3$$

$$\le \left(\frac{4}{3}\cdot\frac{3\sqrt{3}}{8}\right)^3 = \frac{3\sqrt{3}}{8}. \qquad (3)$$

Equality occurs if and only if $A = B = C = \pi/3$ as it does in every inequality used above.

Also solved by Brian D. Beasley, Clinton, SC; Scott H. Brown, Montgomery, AL; Michael Brozinsky, Central Islip, NY; Elsie Campbell, Dionne Bailey, and Charles Diminnie (jointly), San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Manh Dung Nguyen (student, Special High School for Gifted Students) HUS, Vietnam; Boris Rays, Brooklyn, NY, and the proposer.

- 5060: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Show that there exists $c \in (0, \pi/2)$ such that

$$\int_0^c \sqrt{\sin x}\, dx + c\sqrt{\cos c} = \int_c^{\pi/2} \sqrt{\cos x}\, dx + (\pi/2 - c)\sqrt{\sin c}$$

Solution 1 by Paul M. Harms, North Newton, KS

Let

$$f(x) = \int_0^x \sqrt{\sin t}\, dt + x\sqrt{\cos x} - \int_x^{\pi/2} \sqrt{\cos t}\, dt - (\frac{\pi}{2} - x)\sqrt{\sin x} \text{ where } x \in [0, \pi/2].$$

For $x \in [0, \pi/2]$, each term of $f(x)$ is continuous including the integrals of continuous functions. Then $f(x)$ is continuous for $x \in [0, \pi/2]$. For any $x \in [0, \pi/2]$, the two integrals of nonnegative functions are positive except when the lower limit equals the upper limit. We have

$$f(0) = -\int_0^{\pi/2} \sqrt{\cos t}\, dt < 0 \text{ and } f(\pi/2) = \int_0^{\pi/2} \sqrt{\sin t}\, dt > 0.$$

Since $f(x)$ is continuous for $x \in [0, \pi/2]$, $f(0) < 0$ and $f(\pi/2) > 0$, there is at least one $c \in (0, \pi/2)$ such that

$$f(c) = 0 = \int_0^c \sqrt{\sin t}\, dt + c\sqrt{\cos c} - \int_c^{\pi/2} \sqrt{\cos t} - (\pi/2 - c)\sqrt{\sin c}.$$

This last equation is equivalent to the equation in the problem.

Solution 2 by Michael C. Faleski, University Center, MI

The given equation will hold if the integrals and their constants of integration are the same on each side of the equality.
For the integral $\int_0^c \sqrt{\sin x} dx$ we substitute $x = \frac{\pi}{2} - y$ to obtain

$$\int_0^c \sqrt{\sin x} dx = \int_{\pi/2}^{\pi/2 - c} \sqrt{\sin\left(\frac{\pi}{2} - y\right)}(-dy) = \int_{\pi/2 - c}^{\pi/2} \sqrt{\cos y} dy.$$

We substitute this into the original statement of the problem and equate the integrals on each side of the equation.

$$\int_{\pi/2-c}^{\pi/2} \sqrt{\cos y}dy = \int_{c}^{\pi/2} \sqrt{\cos y}dy$$

For equality to hold the lower limits of integration must be the same; that is,
$\frac{\pi}{2} - c = c \Longrightarrow c = \frac{\pi}{4}$

We now check the constants of integration on each side of the equality when $c = \frac{\pi}{4}$, and we see that they are equal.

$$\frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right)^{1/2} = \frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right)^{1/2}$$

Hence, the value of $c = \frac{\pi}{4}$ satisfies the original equation.

Also solved by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Andrew Siefker (jointly), San Angelo, TX; Brian D. Beasley, Clinton, SC; Bruno Salgueiro Fanego, Viveiro, Spain; Ovidiu Furdui, Cluj, Romania; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Nguyen Pham and Quynh Anh (jointly; students, Belarusian State University), Belarus; Angel Plaza, Las Palmas, Spain; Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy; David Stone and John Hawkins (jointly) Statesboro, GA , and the proposer.

- 5061: *Michael P. Abramson, NSA, Ft. Meade, MD.*

Let $a_1, a_2, \ldots, a_n$ be a sequence of positive integers. Prove that

$$\sum_{i_m=1}^{n} \sum_{i_{m-1}=1}^{i_m} \cdots \sum_{i_1=1}^{i_2} a_{i_1} = \sum_{i=1}^{n} \binom{n-i+m-1}{m-1} a_i.$$

Solution by Tom Leong, Scranton, PA

We treat the a's as variables; they don't necessarily have to be integers. Fix an i, $1 \le i \le n$, and imagine completely expanding all the sums on the lefthand side. We wish to show that, in this expansion, the number of times that the term a_i appears is $\binom{n-i+m-1}{m-1}$. Now each term in this expansion corresponds to some m-tuple of indices in the set

$$I = \{(i_1, i_2, \ldots, i_m) : 1 \le i_1 \le i_2 \le \cdots \le i_m \le n\}.$$

We want to count the number of elements of I of the form $(i, i_2, \ldots, i_m)$. Equivalently, using the one-to-one correspondence between I and

$$J = \{(j_1, j_2, \ldots, j_m) : 1 \le j_1 < j_2 < \cdots < j_m \le n+m-1\}$$

given by

$$(i_1, i_2, \ldots, i_m) \leftrightarrow (j_1, j_2, \ldots, j_m) = (i_1, i_2+1, i_3+2 \ldots, i_m+m-1),$$

we wish to count the number elements of J of the form $(i, j_2, \ldots, j_m)$. This number is simply the number of $(m-1)$-element subsets of $\{i+1, i+2, \ldots, n+m-1\}$ which is just $\binom{n-i+m-1}{m-1}$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain and the proposer.