Problems Ted Eisenberg, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before December 15, 2017

- **5457:** *Proposed by Kenneth Korbin, New York, NY*

  Given angle A with $\sin A = \dfrac{12}{13}$. A circle with radius 1 and a circle with radius x are each tangent to both sides of the angle. The circles are also tangent to each other. Find x.

- **5458:** *Proposed by Michal Kremzer, Gliwice, Silesia, Poland*

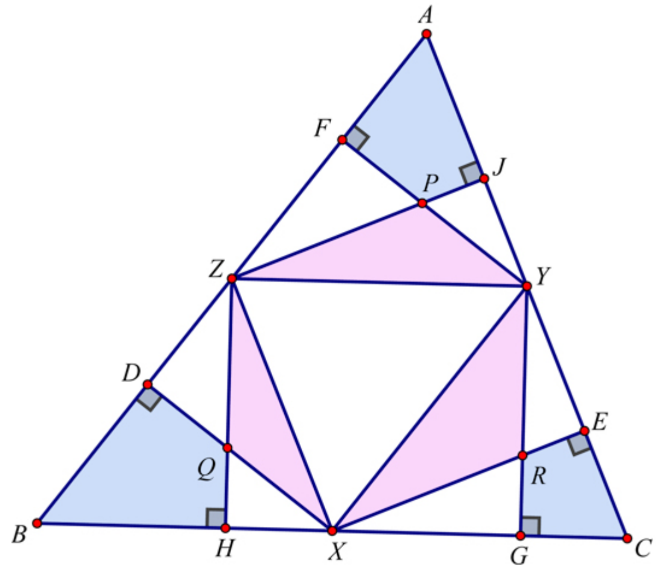
  Find two pairs of integers (a, b) from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that for all positive integers n, the number

  $$c = 537aaa\underbrace{b\ldots b}18403$$

  is composite, where there are $2n$ numbers b between a and 1 in the string above.

- **5459:** *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

  Triangle ABC is an arbitrary acute triangle. Points X, Y, and Z are midpoints of three sides of $\triangle ABC$. Line segments XD and XE are perpendiculars drawn from point X to two of the sides of $\triangle ABC$. Line segments YF and YG are perpendiculars drawn from point Y to two of the sides of $\triangle ABC$. Line segments ZJ and ZH are perpendiculars drawn from point Z to two of the sides of $\triangle ABC$. Moreover,
  $P = ZJ \cap FY$, $Q = ZH \cap DX$, and $R = YG \cap XE$. Three of the triangles, and three of the quadrilaterals in the figure are shaded. If the sum of the areas of the three shaded triangles is 5, find the sum of the areas of the three shaded quadrilaterals.



- **5460:** *Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

  If $a,b>0$ and $x,y>0$ then prove that

$$\frac{a^3}{ax+by}+\frac{b^3}{bx+ay}\geq\frac{a^2+b^2}{x+y}.$$

- **5461:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Compute the following sum:

$$\sum_{n=1}^{\infty}\frac{\cos(2n-1)}{(2n-1)^2}.$$

- **5462:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $n\geq 1$ be an integer. Calculate

$$\int_0^{\frac{\pi}{2}}\frac{\cos x}{\left(1+\sqrt{\sin(2x)}\right)^n}\mathrm{d}x.$$

Solutions

- **5439:** *Proposed by Kenneth Korbin, New York, NY*

  Express the roots of the equation $\dfrac{(x+1)^4}{(x-1)^2}=20x$ in closed form.
  "Closed form" means that the roots cannot be expressed in their approximate decimal equivalents.

  Solution 1 by David E. Manes, Oneonta, NY

The four roots of the equation are: $x = 4 + \sqrt{5} \pm 2\sqrt{5 + 2\sqrt{5}}$ and $x = 4 - \sqrt{5} \pm 2\sqrt{5 - 2\sqrt{5}}$. One verifies that each of these values is a solution of the equation.

With $x \neq 1$, the equation is equivalent to $(x+1)^4 = 20x(x-1)^2$. After expanding, this equation reduces to the reciprocal equation $x^4 - 16x^3 + 46x^2 - 16x + 1 = 0$. To solve it, define the polynomial function $f(x)$ as follows and note that

$$\begin{aligned} f(x) = x^4 - 16x^3 + 46x^2 - 16x + 1 &= x^2\left(x^2 - 16x + 46 - \frac{16}{x} + \frac{1}{x^2}\right) \\ &= x^2\left[\left(x^2 + \frac{1}{x^2}\right) - 16\left(x + \frac{1}{x}\right) + 46\right]. \end{aligned}$$

Let $z = x + \dfrac{1}{x}$. Then $\left(x + \dfrac{1}{x}\right)^2 = x^2 + \dfrac{1}{x^2} + 2$ implies $x^2 + \dfrac{1}{x^2} = z^2 - 2$. Therefore, $f(x) = x^2 \cdot g(z)$ where $g(z) = z^2 - 2 - 16z + 46 = z^2 - 16z + 44$. Then the roots of the reciprocal equation are the zeroes of $g(z)$ since $x = 0$ is not a solution of the equation. Using the quadratic formula, the roots of $z^2 - 16z + 44 = 0$ are $z = 8 \pm 2\sqrt{5}$. If $x + \dfrac{1}{x} = 8 + 2\sqrt{5}$, then $x^2 - (8 + 2\sqrt{5})x + 1 = 0$ with roots $x = 4 + \sqrt{5} \pm 2\sqrt{5 + 2\sqrt{5}}$. If $x + \dfrac{1}{x} = 8 - 2\sqrt{5}$, then $x^2 - (8 - 2\sqrt{5})x + 1 = 0$ with roots $x = 4 - \sqrt{5} \pm 2\sqrt{5 - 2\sqrt{5}}$. This completes the solution.

Solution 2 by Brian Bradie, Christopher Newport University, Newport News, VA

The equation

$$\frac{(x+1)^4}{(x-1)^2} = 20x$$

is equivalent to

$$x^4 + 4x^3 + 6x^2 + 4x + 1 = 20x^3 - 40x^2 + 20x,$$

or

$$x^4 - 16x^3 + 66x^2 - 16x + 1 = 20x^2.$$

Now,

$$x^4 - 16x^3 + 66x^2 - 16x + 1 = (x^2 - 8x + 1)^2,$$

so

$$(x^2 - 8x + 1)^2 - 20x^2 = \left[x^2 - (8 + 2\sqrt{5})x + 1\right]\left[x^2 - (8 - 2\sqrt{5})x + 1\right] = 0.$$

Thus, by the quadratic formula,

$$\begin{aligned} x &= \frac{8 + 2\sqrt{5} \pm \sqrt{(8 + 2\sqrt{5})^2 - 4}}{2} = \frac{8 + 2\sqrt{5} \pm \sqrt{80 + 32\sqrt{5}}}{2} \\ &= 4 + \sqrt{5} \pm 2\sqrt{5 + 2\sqrt{5}}, \end{aligned}$$

or

$$\begin{aligned} x &= \frac{8 - 2\sqrt{5} \pm \sqrt{(8 - 2\sqrt{5})^2 - 4}}{2} = \frac{8 - 2\sqrt{5} \pm \sqrt{80 - 32\sqrt{5}}}{2} \\ &= 4 - \sqrt{5} \pm 2\sqrt{5 - 2\sqrt{5}}. \end{aligned}$$

Solution 3 by Anna V. Tomova, Varna, Bulgaria

The possible values of the variable are those for which $x \neq 1$. The following equations are equivalent:

$$\frac{(x+1)^4}{(x-1)^2} = 20x \iff 20x(x+1)^4 - 20x(x-1)^2 = 0 \iff x^4 - 16x^3 + 46x^2 - 16x + 1 = 0.$$

Now we are looking for a representation of the left hand side of the equation as a product:

$$x^4 - 16x^3 + 46x^2 - 16x + 1 = \left(x^2 + ax + 1\right)\left(x^2 + bx + 1\right) = x^4 + (a+b)x^3 + (2+ab)x^2 + (a+b)x + 1.$$

Therefore we have to solve the system $\begin{cases} a+b = -16 \\ ab + 2 = 46 \end{cases} \iff \begin{cases} a+b = -16 \\ \quad ab = 44 \end{cases}$ or to solve the quadratic equation
$X^2 + 16X + 44 = 0 \iff X_{1,2} = -8 \pm \sqrt{64-44} = -8 \pm 2\sqrt{5}$. Then we have:

$$x^4 - 16x^3 + 46x^2 - 16x + 1 = \left(x^2 + \left(2\sqrt{5} - 8\right)x + 1\right)\left(x^2 - \left(2\sqrt{5} + 8\right)x + 1\right) = 0.$$

the solutions to the problem are then:

$$x^2 + \left(2\sqrt{5} - 8\right)x + 1 = 0 \iff x_{1,2} = 4 - \sqrt{5} \pm 2\sqrt{5 - 2\sqrt{5}};$$

$$x^2 - \left(2\sqrt{5} + 8\right)x + 1 = 0 \iff x_{3,4} = 4 + \sqrt{5} \pm 2\sqrt{5 + 2\sqrt{5}}.$$

Editor's Comment: **David Stone and John Hawkins of Georgia Southern University in Statesboro** made the following remark in their solution: "It's surprising that the line $y = 20x$ actually intersects the rational function four times. The line $y = 10x$, for instance, would not do so. So an interesting questions would be: *for which values of m does the equation $\dfrac{(x+1)^4}{(x-1)^2} = mx$ have four solutions?*" **Kenneth Korbin,** author of the problem, answered it as follows: "Possible values other than 20 would be any number 16 or greater."

Also solved by Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo State University, San Angelo, TX; Brian D. Beasley, Presbyterian College, Clinton, SC; Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND; Pat Costello, Eastern Kentucky University, Richmond, KY; Bruno Salgueiro Fanego (two solutions), Viveiro, Spain; Zhi Hwee Goh, Singapore, Singapore; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vergata, Rome, Italy; Henry Ricardo, Westchester Area Math Circle, NY; Toshihiro Shimizu, Kawasaki, Japan; Albert Stadler Herrliberg, Switzerland; Neculai Stanciu, "George Emil Palade" School, Buzău Romania and Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins of Georgia Southern University in Statesboro, GA, and the proposer.

- **5440:** *Proposed by Roger Izard, Dallas,TX*

The vertices of rectangle ABCD are labeled in clockwise order, and point F lines on line segment AB. Prove that $AD + AC > DF + FC$.

Solution 1 by Titu Zvonaru, Comănesti, Romania and Neculai Stanciu,"George Emil Palade" School Romania

We consider the ellipse with foci D and C which passes through the points A and B. Since the point F belongs to the segment AB, we know that F is inside the ellipse. Hence, $FD + FC < AD + AC$, and we are done.

Solution 2 by Kee-Wai Lau, Hong Kong,China

We first suppose that $AF \leq BF$. We produce CB to G such that $BG = BC$. It is easy to see that $AG = AC$ and $FG = FC$.

If $AF = BF$, then DFG is a straight line. By the triangle inequality, we have $AD + AC = AD + AG > DG = DF + FG = DF + FC$ as required.

If $AF < BF$, we produce DF to meet the line AG at H. Applying the triangle inequality to triangles DAH and FHG, we obtain respectively $AD + AH > DF + HF$ and $HF + HG > FG$. Adding up the lat two inequalities, we have $AD + AG > DF + FG$ or $AD + AC > DF + FC$.

Now suppose that $AF > BF$. We produce DA to I such that $DA = IA$. Similar to the case $AF < BF$, we obtain $BC + BD > CF + FD$. Since $AD = BC$ and $BD = AC$, so again $AD + AC > DF + FC$, and this completes the solution.

$Editor's\ Comment$: **David Stone and John Hawkins of Georgia Southern University in Statesboro** corrected the inequality to read: $AD + AC \geq DF + FC$, because equality occurs if F is either end point of the segment AB. They presented three different solution paths to the problem. In one of them they used the notion of reflection. They reflected the rectangle across the segment AB to include $AD'C'B$ as an upper rectangle, and then they reflected FC to FC'. They then argued that in triangle DAC' it is clear that $AD + AC' \geq DF + FC' \geq DC'$ because $AC' = AC$ and $FC' = FC$.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Bruno Salgueiro Fanego, Viveiro, Spain; Michael N. Fried, Ben-Gurion University of the Negev, Beer-Sheva, Israel; Paul M. Harms, North Newton, KS; Zhi Hwee Goh, Singapore, Singapore; Ed Gray, Highland Beach, FL; David A. Huckaby, Angelo State University, San Angelo, TX; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Sachit Misra, Delhi, India; Toshihiro Shimizu, Kawasaki, Japan; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins (three solutions), Georgia Southern University, Statesboro, GA, and the proposer.

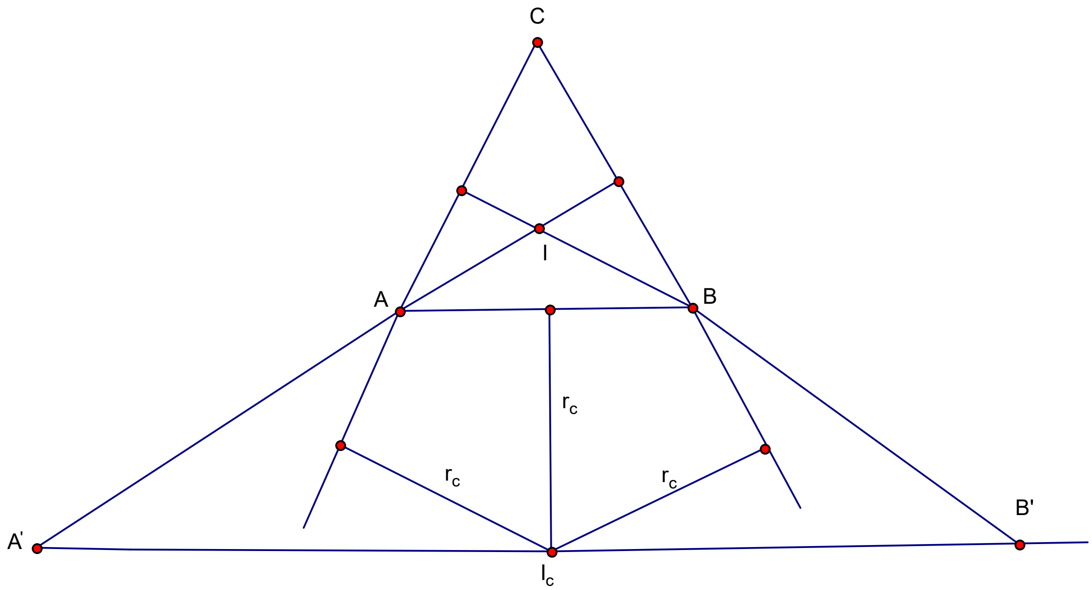
- **5441:** *Proposed by Larry G. Meyer, Fremont, OH*

In triangle ABC draw a line through the ex-center corresponding to side c so that it is parallel to side c. Extend the angle bisectors of A and B to meet the constructed lines at points A' and B' respectively. Find the length of $\overline{A'B'}$ if given either

(1) Angles A, B, C and the circumradius R
(2) Sides a, b, c

(3) The semiperimeter s, the inradius r and the exradius r_c
(4) Semiperimeter s and side c.

Solution 1 by Arkady Alt, San Jose, CA



Since $AB \parallel A'B'$ then $\triangle A'IB' \sim \triangle AIB$ and $r + r_c$ is length of height of the triangle $A'IB'$ from I to $A'B'$.

Hence, $\dfrac{A'B'}{c} = \dfrac{r+r_c}{r} \iff A'B' = \dfrac{c(r+r_c)}{r}$ and since

$[ABC] = rs = r_c(s-c) \implies \dfrac{r_c}{r} = \dfrac{s}{s-c}$ then

$$A'B' = c\left(1+\frac{r_c}{r}\right) = c\left(1+\frac{s}{s-c}\right) = \frac{c(2s-c)}{s-c} = \frac{c(a+b)}{s-c} = \frac{2c(a+b)}{a+b-c} =$$
$$\frac{8R^2\sin C(\sin A+\sin B)}{2R(\sin A+\sin B-\sin C)} = \frac{4R\sin C(\sin A+\sin B)}{\sin A+\sin B-\sin C}.$$

Also, since $rs = r_c(s-c) \iff c = \dfrac{(r_c-r)s}{r_c}$ we obtain

$$A'B' = c\left(1+\frac{r_c}{r}\right) = \frac{(r_c-r)s}{r_c}\cdot\frac{r+r_c}{r} = \frac{\left(r_c^2-r^2\right)s}{rr_c}.$$

So, $A'B' = \dfrac{4R\sin C(\sin A+\sin B)}{\sin A+\sin B-\sin C} = \dfrac{2c(a+b)}{a+b-c} = \dfrac{\left(r_c^2-r^2\right)s}{rr_c} = \dfrac{c(2s-c)}{s-c}$.

Solution 2 by Kee-Wai Lau, Hong Kong, China

Let the incenter of triangle ABC be I and the ex-center corresponding to side c be E_c so that CIE_c is a straight line cutting AB at D, say. Let the feet of the perpendiculars from I to AB and from E_c to AB be X and Y respectively. It is easy to see that triangle IAB and $I'A'B'$, triangles IAD and $IA'E_c$n triangles IDX and E_cDY are pairwise similar. Hence

$$A'B' = AB\frac{IA'}{IA} = AB\frac{IE_c}{ID} = AB\frac{ID+E_cD}{ID} = AB\left(1+\frac{E_cY}{IX}\right) = AB\left(1+\frac{r_c}{r}\right).$$

It is well known that $c = 2R\sin C, r = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$ and
$r_c = 4R\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2}$. Hence the answer to part (1) is

$$A'B' = 2R\sin C\left(1+\cot\frac{A}{2}\cot\frac{B}{2}\right).$$

It is also well-known that $r = \dfrac{[ABC]}{s}$ and $r_c = \dfrac{[ABC]}{s-c}$ were $[ABC]$ equals the area of triangle ABC. Hence the answer to part (2) is

$$A'B' = \frac{2c(a+b)}{a+b-c}.$$

and the answer to part (4) is

$$A'B' = \frac{c(2s-c)}{s-c}.$$

From $r = \dfrac{[ABC]}{s}$ and $r_c = \dfrac{ABC}{s-c}$, we obtain that $c = s\left(1-\dfrac{r}{r_c}\right)$. The answer to part (3) is then

$$A'B' = \frac{s\left(r_c^2-r^2\right)}{rr_c}.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Toshihiro Shimizu, Kawasaki, Japan; Zhi Hwee Goh, Singapore, Singapore; Neculai Stanciu, "George Emil Palade" School, Buzău Romania and Titu Zvonaru, Comănesti, Romania; and the proposer.

- **5442:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Let L_n be the n^{th} Lucas number defined by $L_0 = 2, L_1 = 1$ and for $n \geq 2, L_n = L_{n-1} + L_{n-2}$. Prove that for all $n \geq 0$,

$$\frac{1}{2}\begin{vmatrix} (L_n+2L_{n+1})^2 & L_{n+2}^2 & L_{n+1}^2 \\ L_{n+2}^2 & (2L_n+L_{n+1})^2 & L_n^2 \\ L_{n+1}^2 & L_n^2 & L_{n+2}^2 \end{vmatrix}$$

is the cube of a positive integer and determine its value.

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

The problem may be generalized as follows. For a, b positive numbers, then

$$\frac{1}{2}\begin{vmatrix} (a+2b)^2 & (a+b)^2 & b^2 \\ (a+b)^2 & (2a+b)^2 & a^2 \\ b^2 & a^2 & (a+b)^2 \end{vmatrix} = \left(a^2+3ab+b^2\right)^3$$

which may be checked by a symbolic calculus package like Mathematica. So for the proposed expression in the problem we have

$$\frac{1}{2}\begin{vmatrix} (L_n+2L_{n+1})^2 & L_{n+2}^2 & L_{n+1}^2 \\ L_{n+2}^2 & (2L_n+L_{n+1})^2 & L_n^2 \\ L_{n+1}^2 & L_n^2 & L_{n+2}^2 \end{vmatrix} = \left(L_n^2+3L_nL_{n+1}+L_{n+1}^2\right)^3.$$

Solution 2 by Moti Levy, Rehovot, Israel

Let A denote our matrix,

$$A := \begin{bmatrix} (L_n + 2L_{n+1})^2 & L_{n+2}^2 & L_{n+1}^2 \\ L_{n+2}^2 & (2L_n + L_{n+1})^2 & L_n^2 \\ L_{n+1}^2 & L_n^2 & L_{n+2}^2 \end{bmatrix}.$$

Using the identity related to Lucas and Fibonacci numbers,

$$L_{n+m} = L_{m+1}F_n + L_m F_{n-1},$$

the matrix A is expressed by Fibonacci numbers F_n and F_{n-1} only,

$$A = \begin{bmatrix} (7F_n + 4F_{n-1})^2 & (4F_n + 3F_{n-1})^2 & (3F_n + F_{n-1})^2 \\ (4F_n + 3F_{n-1})^2 & (5F_n + 5F_{n-1})^2 & (F_n + 2F_{n-1})^2 \\ (3F_n + F_{n-1})^2 & (F_n + 2F_{n-1})^2 & (4F_n + 3F_{n-1})^2 \end{bmatrix}.$$

From now on, our arguments do not relate to Fibonacci or Lucas numbers properties (any decent CAS, say Mathematica, can relieve us of tedious calculations). Let B be a symmetric matrix defined as follows:

$$B := \begin{bmatrix} (7x + 4y)^2 & (4x + 3y)^2 & (3x + y)^2 \\ (4x + 3y)^2 & (5x + 5y)^2 & (x + 2y)^2 \\ (3x + y)^2 & (x + 2y)^2 & (4x + 3y)^2 \end{bmatrix},$$

where x, y are real or complex numbers.
The determinant of B divided by 2 is

$$\frac{1}{2}\det B = \left(19x^2 + 31xy + 11y^2\right)^3.$$

It follows that the positive number we are seeking is $19F_n^2 + 31F_nF_{n-1} + 11F_{n-1}^2$.

Solution 3 by Albert Stadler, Herrliberg, Switzerland

We replace L_{n+2} by $L_n + L_{n+1}$, expand the determinate and factor to get

$$\begin{aligned} &\frac{1}{2}\Big((L_n + 2L_{n+1})^2(2L_n + L_{n+1})^2(L_n + L_{n+1})^2 + 2(L_n + L_{n+1})^2 L_n^2 L_{n+1}^2 \\ &-(2L_n + L_{n+1})^2 L_{n+1}^4 - (L_n + 2L_{n+1})^2 L_n^4 - (L_n + L_{n+1})^6\Big) \\ = &\left(L_n^2 + 3L_nL_{n+1} + L_{n+1}^2\right)^3. \end{aligned}$$

L_n can be represented as

$L_n = f^n + (-f)^n$ with $f = \dfrac{1+\sqrt{5}}{2}$ (see: https://en.wikipedia.org/wiki/ Lucas_number).

Therefore

$$L_n^2 + 3L_nL_{n+1} + L_{n+1}^2 = f^{2n} + (-f)^{-2n} + 2(-1)^n$$

$$
\begin{aligned}
&+3f^{2n+1}+3(-f)^{-2n-1}+3(-1)^n f+3(-1)^{n+1}f^{-1}\\
&+f^{2n+2}+(-f)^{-2n-2}+2(-1)^{n+1}\\
=\ & nL_{2n}+3L_{2n+1}+L_{2n+2}+3(-1)^n\left(f-\frac{1}{f}\right)\\
=\ & \underbrace{L_{2n}+L_{2n+1}}_{=L_{2n+2}}+2L_{2n+1}+L_{2n+2}+3(-1)^n\left(f-\frac{1}{f}\right)\\
=\ & 2L_{2n+3}+3(-1)^n.
\end{aligned}
$$

So the given determinant can be represented as $(2L_{2n+3}+3(-1)^n)^3$.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Zhi Hwee Goh, Singapore, Singapore; Kee-Wai Lau Hong Kong, China; Toshihiro Shimizu, Kawasaki, Japan; David Stone and John Hawkins (partial solution), Georgia Southern University, Statesboro, GA, and the proposer.

- **5443:** *Proposed by D.M. Băinetu– Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu "Geroge Emil Palade" General School, Buzău, Romania*

  Compute $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{x}{\sin 2x}dx.$

  Solution 1 by Ed Gray, Highland Beach, FL

  Letting $y=6x-\dfrac{3\pi}{2}$ and changing the limits we see that:

$$
\begin{aligned}
\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{x}{\sin 2x}dx &= \frac{1}{36}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{y+3+\pi/2}{\cos\frac{y}{3}}dy\\
&= \frac{1}{36}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{y}{\cos\frac{y}{3}}dy+\frac{1}{36}\cdot\frac{3\pi}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{1}{\cos\left(\frac{y}{3}\right)}dy.
\end{aligned}
$$

  The first integral is zero because the integrand is odd, while the second integral (with the help of Wolfram-Alpha) is $\ln(27)=3\ln(3)$. Therefore,
  $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{x}{\sin 2x}dx=\frac{1}{36}\cdot\frac{3\pi}{2}\cdot 3\cdot\ln 3=\frac{\pi\cdot\ln 3}{8}.$

Solution 2 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

Let $x=\arctan t$. The integral becomes

$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\arctan t}{2\frac{t}{\sqrt{1+t^2}\sqrt{1+t^2}}}\frac{dt}{1+t^2} = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{2}\frac{\arctan t}{t}dt = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{2}\frac{\frac{\pi}{2}-\arctan\frac{1}{t}}{t}dt$$

Moreover,

$$-\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{2}\frac{\arctan\frac{1}{t}}{t}dt \underbrace{=}_{t=1/y} \int_{\sqrt{3}}^{\frac{1}{\sqrt{3}}} \frac{1}{2}\frac{\arctan y}{y}dy = -\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{2}\frac{\arctan y}{y}dy$$

It follows,

$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{2}\frac{\arctan t}{t}dt = \frac{\pi}{8}\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{t}dt = \frac{\pi}{8}\left(\ln\sqrt{3} - \ln\frac{1}{\sqrt{3}}\right) = \frac{\pi}{8}\ln 3.$$

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Brian Bradie, Christopher Newport University, Newport News, VA; Pat Costello, Eastern Kentucky University, Richmond, KY; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Motti Levy, Rehovot, Israel; Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Toshihiro Shimizu, Kawasaki, Japan; Albert Stadler, Herrliberg, Switzerland; Students at Taylor University {Group 1: Ellie Grace Moore, Samantha Korn, and Gwyn Terrett; Group 2: Luke Wilson, California Drage, Jonathan DeHaan}, Upland, IN; Anna V. Tomova, Varna, Bulgaria, and the proposer.

- **5444:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Solve in $\Re$ the equation $\{(x+1)^2\} = 2x^2$, where $\{a\}$ denotes the fractional part of a.

Solution 1 by Jeremiah Bartz, University of North Dakota, Grand Forks, ND

Since $0 \le \{(x+1)^2\} < 1$, any solution x satisfies $0 \le 2x^2 < 1$ or equivalently $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$. Note that $\sqrt{2}-1 < \frac{1}{\sqrt{2}} < \sqrt{3}-1$ and observe

$$\{(x+1)^2\} = \begin{cases} (x+1)^2 & \text{if } -1 < x < 0 \\ (x+1)^2 - 1 & \text{if } 0 \le x < \sqrt{2}-1 \\ (x+1)^2 - 2 & \text{if } \sqrt{2}-1 \le x < \sqrt{3}-1. \end{cases}$$

We consider three cases.
For $-\frac{1}{\sqrt{2}} < x < 0$ the equation reduces to $(x+1)^2 = 2x^2$ or equivalent $x^2-2x-1=0$. Solving gives $x = 1 \pm \sqrt{2}$, however only $x = 1-\sqrt{2}$ lies in $\left(-\frac{1}{\sqrt{2}}, 0\right)$. Thus $x = 1-\sqrt{2}$ produces the only solution in this case.
For $0 \le x < \sqrt{2}-1$ the equation reduces to $(x+1)^2 - 1 = 2x^2$ or equivalent $x^2 - 2x = 0$. Solving gives $x = 0, 2$. We see only $x = 0$ lies in $\left[0, \sqrt{2}-1\right)$, so $x = 0$ produces the only solution in this case.
For $\sqrt{2}-1 \le x < \frac{1}{\sqrt{2}}$ the equation reduces to $(x+1)^2 - 2 = 2x^2$ or equivalent $x^2 - 2x + 1 = 0$. Solving gives $x = 1$, however this does not lie in $\left[\sqrt{2}-1, \frac{1}{\sqrt{2}}\right)$. This case yields no solution.

In summary, there are two solutions, namely $x = 1 - \sqrt{2}$ and $x = 0$.

Solution 2 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

Since $0 \le \{(x+1)^2\} < 1$ we must have $2x^2 < 1$ and so $|x| < 1/\sqrt{2}$. Hence

$$x^2 + 2x + 1 < \frac{1}{2} + \sqrt{2} + 1 < 3.$$

Thus $(x+1)^2 = x^2 + 2x + 1$ must be in $[0, 3)$.
If $x^2 + 2x + 1 \in [0, 1)$ then

$$2x^2 = \{x^2 + 2x + 1\} = x^2 + 2x + 1 \implies x^2 - 2x - 1 = 0$$

and so $x = 1 \pm \sqrt{2}$, but only $x = 1 - \sqrt{2}$ satisfies the original equation.
If $x^2 + 2x + 1 \in [1, 2)$ then

$$2x^2 = \{x^2 + 2x + 1\} = x^2 + 2x \implies x^2 - 2x = 0$$

and so $x = 0$ or $x = 2$, but only $x = 0$ satisfies the original equation.
Finally, if $x^2 + 2x + 1 \in [2, 3)$ then

$$2x^2 = \{x^2 + 2x + 1\} = x^2 + 2x - 1 \implies x^2 - 2x + 1 = 0$$

and so $x = 1$, but this does not satisfy the original equation.
Thus the only solutions are $x = 0$ and $x = 1 - \sqrt{2}$.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

Let $\lfloor a \rfloor$ denote the integer part of a.

Since $\lfloor a \rfloor$ is the only integer such that $a - 1 < \lfloor a \rfloor \le a$ and $\{a\} = a - \lfloor a \rfloor$, we have that $0 \le \{a\} < 1$. Thus, $0 \le \left\{(x+1)^2\right\} = 2x^2 < 1$, so $x^2 < \dfrac{1}{2}$ and hence $-\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}}$.

Moreover, $\left\{(x+1)^2\right\} = 2x^2 \iff \lfloor (x+1)^2 \rfloor = (x+1)^2 - 2x^2 \iff \lfloor (x+1)^2 \rfloor = 1 + 2x - x^2$.
But $(x+1)^2 \ge 0 \iff 1 + 2x - x^2 = \lfloor (x+1)^2 \rfloor \ge 0$; since the graph of $f(x) = 1 + 2x - x^2$ is a concave parabola which cuts the x-axis in $x = 1 \pm \sqrt{2}$ and with vertex (absolute maximum) at $(1, 2)$, then the last obtained inequality $f(x) \ge 0$ is equivalent to $1 - \sqrt{2} \le x \le 1 + \sqrt{2}$. or what is the same, taking into account that
$-\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}}$, 1-$\sqrt{2} \le x \le \dfrac{1}{\sqrt{2}}$.
On the other hand, $f(x) = \lfloor (x+1)^2 \rfloor \in Z$ and $0 \le f(x) \le 2$ implies $f(x) \in \{0, 1, 2\}$, that is, $x \in \{1 - \sqrt{2}, 1 + \sqrt{2}, 0, 1, 2\}$, which is equivalent, because $1 - \sqrt{2} \le x \le \dfrac{1}{\sqrt{2}}$ to x$\in \left\{1 - \sqrt{2}, 0\right\}$.

Since $\{(0+1)^2\} = 0 = 2 \cdot 0^2$ and $\left\{(1 - \sqrt{2} + 1)^2\right\} = \{6 - 4\sqrt{2}\} = 6 - 4\sqrt{2} = 2(1 - \sqrt{2})^2$, we conclude that the solutions $x \in \Re$ of the given equations $\{(x+1)^2\} = 2x^2$ are exactly $x = 1 - \sqrt{2}$ and $x = 0$.

Solution 4 by Toshihiro Shimizu, Kawasaki, Japan

From the given equation, $0 \le \left\{(x+1)^2\right\} = 2x^2 < 1$ or $0 \le x^2 < \frac{1}{2}$ or $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$. Then, we have $0 \le (x+1)^2 < \left(1 + \frac{1}{\sqrt{2}}\right)^2 = 1 + \frac{1}{2} + \sqrt{2} < 1.5 + 1.5 = 3$. Therefore, the integer part $k = \left\lfloor (x+1)^2 \right\rfloor$ is 0 or 1 or 2.

If $k=0$, we have $2x^2=(x+1)^2$ or $x^2-2x-1=0$ or $x=1\pm\sqrt{2}$. Only $x=1-\sqrt{2}$ is valid for $k=0$.
If $k=1$, we have $2x^2+1=(x+1)^2$ or $x^2-2x=0$ or $x=0,2$. Only $x=0$ is valid for $k=1$.
If $k=2$, we have $2x^2+2=(x+1)^2$ or $x^2-2x+1=0$ or $x=1$. This is not valid for $k=2$.
Therefore, $x=1-\sqrt{2},\ 0$.

Also solved by Arkady Alt, San Jose CA; Brian Bradie, Christopher Newport University, Newport News, VA; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Zhi Hwee Goh, Singapore, Singapore; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Albert Stadler, Herrliberg, Switzerland, and the proposers.

End Notes

The following should have been credited with having solved the problems below, but their names were inadvertently not listed; *mea culpa*.

Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy for problems 5433, 5437, and 5438.

Jeremiah Bartz, University of North Dakota, Grand Forks, ND for 5434.

David Stone and John Hawkins, Georgia Southern University, Statesboro, GA for 5433.