Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
December 15, 2018

5505: *Proposed by Kenneth Korbin, New York, NY*

Given a Primitive Pythagorean Triple (a, b, c) with $b^2 > 3a^2$. Express in terms of a and b the sides of a Heronian Triangle with area $ab(b^2 - 3a^2)$.

(A Heronian Triangle is a triangle with each side length and area an integer.)

5506: *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu-Severin, Mehedinti, Romania*

Find $\Omega = \det\left[\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}^{100} + \begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}^{100}\right]$.

5507: *Proposed by David Benko, University of South Alabama, Mobile, AL*

A car is driving forward on the real axis starting from the origin. Its position at time $0 \le t$ is $s(t)$. Its speed is a decreasing function: $v(t), 0 \le t$. Given that the drive has a finite path (that is $\lim\limits_{t\to\infty} s < \infty$), that $v(2t)/v(t)$ has a real limit c as $t \to \infty$, find all possible values of c.

5508: *Proposed by Pedro Pantoja, Natal RN, Brazil*

Let a, b, c be positive real numbers such that $a + b + c = 1$. Find the minimum value of

$$f(a, b, c) = \frac{a}{3ab + 2b} + \frac{b}{3bc + 2c} + \frac{c}{3ca + 2a}.$$

5509: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let x, y, z be positive real numbers that add up to one and such that
$0 < \dfrac{x}{y}, \dfrac{y}{z}, \dfrac{z}{x} < \dfrac{\pi}{2}$. Prove that

$$\sqrt{x\cos\left(\frac{y}{z}\right)} + \sqrt{y\cos\left(\frac{z}{x}\right)} + \sqrt{z\cos\left(\frac{x}{y}\right)} < \frac{3}{5}\sqrt{5}.$$

5510: *Proposed by Ovidiu Furdui and Alina Sîntămărian both at the Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate

$$\sum_{n=1}^{\infty}\left[4^n\left(\zeta(2n)-1\right)-1\right],$$

where ζ denotes the Riemann zeta function.

Solutions

5487: *Proposed by Kenneth Korbin, New York, NY*

Given that $\dfrac{(x+1)^4}{x(x-1)^2} = a$ with $x = \dfrac{b+\sqrt{b-\sqrt{b}}}{b-\sqrt{b-\sqrt{b}}}$. Find positive integers a and b.

Solution 1 by David E. Manes, Oneonta, NY

If $x = \dfrac{b+\sqrt{b-\sqrt{b}}}{b-\sqrt{b-\sqrt{b}}}$, then $x+1 = \dfrac{2b}{b-\sqrt{b-\sqrt{b}}}$ and $x-1 = \dfrac{2\sqrt{b-\sqrt{b}}}{b-\sqrt{b-\sqrt{b}}}$. Moreover, $(x+1)^4 = \dfrac{16b^4}{(b-\sqrt{b-\sqrt{b}})^4}$ and $(x-1)^2 = \dfrac{4(b-\sqrt{b})}{(b-\sqrt{b-\sqrt{b}})^2}$. Therefore,

$$\begin{aligned} a = \frac{(x+1)^4}{x(x-1)^2} &= \frac{\frac{16b^4}{(b-\sqrt{b-\sqrt{b}})^4}}{\frac{(b+\sqrt{b-\sqrt{b}})(4(b-\sqrt{b})}{(b-\sqrt{b-\sqrt{b}})^3}} \\ &= \frac{16b^4}{4(b-\sqrt{b})(b+\sqrt{b-\sqrt{b}})(b-\sqrt{b-\sqrt{b}})} \\ &= \frac{4b^4}{b^3-b^2-b^2\sqrt{b}+2b\sqrt{b}-b}. \end{aligned}$$

Note that the two terms with $\sqrt{b}$ have opposite signs and cancel off if $b=2$. Let $b=2$. Then $b^3-b^2-b^2\sqrt{b}+2b\sqrt{b}-b = 2$ and $a = 2^6/2 = 32$. Hence, $b=2$ and $a=32$ is the unique solution.

Solution 2 by Anthony J. Bevelacqua, University of North Dakota, Great Falls, ND

For notational convenience set $c = \sqrt{b - \sqrt{b}}$. We have $x = \dfrac{b+c}{b-c}$ so $x + 1 = \dfrac{2b}{b-c}$ and $x - 1 = \dfrac{2c}{b-c}$. Thus a is

$$\begin{aligned}\frac{(x+1)^4}{x(x-1)^2} &= \left(\frac{2b}{b-c}\right)^4 \cdot \frac{b-c}{b+c} \cdot \left(\frac{b-c}{2c}\right)^2 \\ &= \frac{4b^4}{(b^2-c^2)c^2}.\end{aligned}$$

and so $a(b^2 - c^2)c^2 = 4b^4$. Now

$$\begin{aligned}(b^2-c^2)c^2 &= (b^2 - b + \sqrt{b})(b - \sqrt{b}) \\ &= (b^3 - b^2 - b) + (2b - b^2)\sqrt{b}\end{aligned}$$

and so

$$a((b^2 - b - 1) + (2 - b)\sqrt{b}) = 4b^3.$$

Thus $(2 - b)\sqrt{b}$ is a rational number. Therefore either $b = 2$ or $b = d^2$ for some positive integer d.

In the first case our last displayed equation yields $a \cdot 1 = 4 \cdot 2^3$ and so $a = 32$. Thus $a = 32$ and $b = 2$ is a solution to our problem.

In the second case we have

$$(b^2 - b - 1) + (2 - b)\sqrt{b} = d^4 - d^3 - d^2 + 2d - 1.$$

Call this n. We have $an = 4b^3$. Since a and b are positive so is n. Since d and n are relatively prime we see that n must be a divisor of 4. If $n = 1$ we have

$$d^4 - d^3 - d^2 + 2d - 1 = 1 \text{ and so } d^4 - d^3 - d^2 + 2d - 2 = 0.$$

By the rational root theorem the only possible positive integer d would be 1 and 2, but neither of these are roots. Similarly $n = 2$ gives $d^4 - d^3 - d^2 + 2d - 3 = 0$ and $n = 4$ gives $d^4 - d^3 - d^2 + 2d - 5 = 0$, but, again, neither of these have positive integer roots. Thus the only solution to our problem is $a = 32$ and $b = 2$.

Solution 3 by Brian D. Beasley, Presbyterian College, Clinton, SC

Let $c = b - \sqrt{b - \sqrt{b}}$. Then $x + 1 = 2b/c$ and $x - 1 = 2(b - c)/c$, so

$$a = \frac{(x+1)^4}{x(x-1)^2} = \frac{16b^4}{c^4} \cdot \frac{c^3}{4(b-c)^2(b + \sqrt{b - \sqrt{b}})} = \frac{4b^4}{(b^2 - b + \sqrt{b})(b - \sqrt{b})}.$$

This in turn yields $a = 4b^4/(b^3 - b^2\sqrt{b} - b^2 + 2b\sqrt{b} - b)$. Since a is a positive integer, we must have either $b = n^2$ for some positive integer n or $-b^2 + 2b = 0$. If $b = n^2$, then

$$a = 4n^2 + 4n + 8 + \frac{4(n^3 + n^2 - 3n + 2)}{n^4 - n^3 - n^2 + 2n - 1};$$

the fraction in this latter expression is not an integer for $1 \leq n \leq 5$ and is strictly between 0 and 1 for $n > 5$, so a is not a positive integer. Thus $-b^2 + 2b = 0$, so $b = 2$ and hence $a = 32$.

Also solved by Michel Bataille, Rouen, France; Ed Gray, Highland Beach, FL; Khanh Hung Vu (Student), Tran Nghia High School, Ho Chi Minh,

Vietnam; Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrain University of Athens, Greece; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5488: *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta, Turnu-Severin, Mehedinti, Romania*

Let a, and b be complex numbers. Solve the following equation:

$$x^3 - 3ax^2 + 3(a^2 - b^2)x - a^3 + 3ab^2 - 2b^3 = 0.$$

Solution 1 by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith, Angelo State University, San Angelo, TX

To begin, we note that

$$x^3 - 3ax^2 + 3\left(a^2 - b^2\right)x - a^3 + 3ab^2 - 2b^3$$

can be re-written as

$$\left(x^3 - 3ax^2 + 3a^2x - a^3\right) - 3b^2x + 3ab^2 - 2b^3$$

or

$$(x-a)^3 - 3b^2(x-a) - 2b^3.$$

Hence, if we substitute $y = x - a$, the given equation becomes

$$y^3 - 3b^2y - 2b^3 = 0. \tag{1}$$

Next, the left side of equation (1) can be re-grouped to obtain

$$\begin{aligned} y^3 - 3b^2y - 2b^3 &= \left(y^3 + b^3\right) - 3b^2(y+b) \\ &= (y+b)\left[\left(y^2 - by + b^2\right) - 3b^2\right] \\ &= (y+b)\left(y^2 - by - 2b^2\right) \\ &= (y+b)^2(y-2b). \end{aligned}$$

Therefore, the solutions of (1) are $y = 2b$ and $y = -b$ (double solution).
Finally, since $y = x - a$, the solutions of the original equation are $x = a + 2b$ and $x = a - b$ (double solution).

Solution 2 by Michel Bataille, Rouen, France

Let $p(x)$ denote the polynomial on the left-hand side. Then, a short calculation gives

$$p(X+a) = X^3 - 3b^2X - 2b^3 = (X+b)^2(X-2b)$$

which has $2b$ as a simple root and $-b$ as a double one. It immediately follows that the solution of the given equation are $a-b, a-b, a+2b$.

Solution 3 by Paul M. Harms, North Newton, KS

The equation can be written as $(x-a)^3 - 3ab^2(x-a) - 2b^3 = 0$. If $b = 0$, the solution is $x = a$. If b is not zero, let $x - a = yb$. Then the equation become $b^3(y^3 - 3y - 2) = 0$. We have $y^3 - 3y - 2 = (y-2)(y+1)^2 = 0$. The y solutions are $2, -1$ and -1. The solutions of the equation in the problem are $x = a + 2b$ and $x = a - b$ as a double root.

Solution 4 by G. C. Greubel, Newport News,VA

$$
\begin{aligned}
0 &= x^3 - 3\,a\,x^2 + 3(a^2 - b^2)\,x - (a^3 - 3ab^2 + 2b^3) \\
&= x^3 - 3ax^2 + (a-b)(3a+3b)\,x - ((a^2 - 2ab + b^2)(a+2b) \\
&= x^3 - (2(a-b) + (a+2b))\,x^2 + (a-b)((a-b) + 2(a+2b))\,x \\
&\qquad - (a-b)^2(a+2b) \\
&= (x^2 - 2(a-b)\,x + (a-b)^2)(x - (a+2b)) \\
&= (x - (a-b))^2\,(x - (a+2b)).
\end{aligned}
$$

From this factorization the solutions of the cubic equation are

$$x \in \{a-b, a-b, a+2b\}.$$

Editor′s comment: **David Stone and John Hawkins** made an instructive comment in their solution that merits being repeated. They wrote: "We confess - we did not immediately recognize the factorization. We originally used Cardano's Formula to find the solutions.

However, there is a line of heuristic reasoning which would lead to the solution. If we consider $a = b$, the equation become $x^3 - 3ax^2 = 0$, which has $x = 0$ as a double root. Hence, the difference $a - b$ could be significant. Trying $x = a - b$ (via synthetic division) then proves to be productive."

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Anthony J. Bevelacqua, University of North Dakota, Great Falls, ND; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Ioannis D. Sfikas, National and Kapodistrain University of Athens, Greece; Neculai Stanciu "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania (two solutions); David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5489: *Proposed by D.M. Bătinetu-Giurgiu, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School Buzău, Romania*

If $a > 0$, compute $\displaystyle\int_0^a \left(x^2 - ax + a^2\right)\arctan(e^x - 1)dx$.

Solution by Soumitra Mandal, Chandar Nagore, India

Let $x = a - y \Rightarrow dx = -dy$, when $x = 0, y = a$; when $x = a; y = 0$.

$$\Omega \;=\; \int_0^a (x^2 - xa + a^2)\tan^{-1}(e^x - 1)dx$$

$$
\begin{aligned}
&= -\int_a^0 \{(a-y)^2 - a(a-y) + a^2\} \tan^{-1}(e^{a-y}-1)dy \\
&= \int_0^a (y^2 - ay + a^2) \tan^{-1}(e^{a-y}-1)dy, \text{therefore,} \\
2\Omega &= \int_0^a (x^2 - ax + a^2)\{\tan^{-1}(e^x - 1) + \tan^{-1}(e^{a-x}-1)\}dx \\
&= \int_0^a (x^2 - xa + a^2) \tan^{-1} \frac{e^x - 1 + e^{a-x} - 1}{1 - (e^x - 1)(e^{a-x}-1)} dx \\
&= \int_0^a (x^2 - ax + a^2) \tan^{-1}(1)dx = \frac{\pi}{4}\left(\frac{x^3}{3} - a\frac{x^2}{2} + a^2 x \Bigg|_{x=0}^{x=a} \right) = \frac{5\pi a^3}{24}. \\
\text{Therefore, } \Omega &= \frac{5\pi a^3}{48}.
\end{aligned}
$$

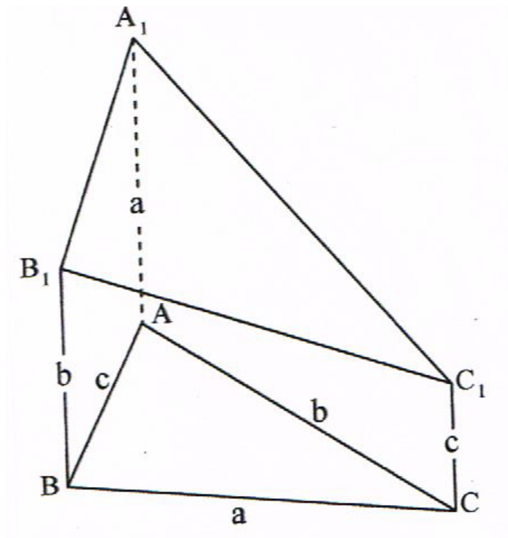
Also solved by Ioannis D. Sfikas, National and Kapodistrain University of Athens, Greece, and the proposers.

5490: *Proposed by Moshe Stupel, "Shaanan" Academic College of Education and Gordon Academic College of Education, and Avi Sigler, "Shaanan" Academic College of Education, Haifa, Israel*

Triangle ABC whose side lengths are a, b, and c lies in plane P. The segment$A_1A,\ BB_1,\ CC_1$ satisfy:

$$A_1A \perp P,\ B_1B \perp P,\ C_1C \perp P,$$

where $A_1A = a,\ B_1B = b$ and $C_1C = c$, as shown in the figure. Prove that $\triangle A_1B_1C_1$ is acute -angled.



Solution 1 by Michel Bataille, Rouen, France

We shall use the dot product, recalling that $\overrightarrow{U} \cdot \overrightarrow{V}$ has the same sign as $\cos(\angle(\overrightarrow{U}, \overrightarrow{V}))$.
We calculate

$$\begin{aligned}
\overrightarrow{A_1B_1} \cdot \overrightarrow{A_1C_1} &= (\overrightarrow{A_1A} + \overrightarrow{AB} + \overrightarrow{BB_1}) \cdot (\overrightarrow{A_1A} + \overrightarrow{AC} + \overrightarrow{CC_1}) \\
&= a^2 + 0 - ac + 0 + \overrightarrow{AB} \cdot \overrightarrow{AC} + 0 - ab + 0 + bc \\
&= \frac{1}{2}(a^2 + b^2 + c^2 - 2ac - 2ab + 2bc) \quad (\text{since } 2\overrightarrow{AB} \cdot \overrightarrow{AC} = b^2 + c^2 - a^2) \\
&= \frac{1}{2}(b + c - a)^2.
\end{aligned}$$

Thus, $\overrightarrow{A_1B_1} \cdot \overrightarrow{A_1C_1} > 0$ and so $\angle B_1A_1C_1$ is acute.

Similarly, we obtain $\overrightarrow{B_1C_1} \cdot \overrightarrow{B_1A_1} = \frac{1}{2}(c + a - b)^2 > 0$ and
$\overrightarrow{C_1A_1} \cdot \overrightarrow{C_1B_1} = \frac{1}{2}(a + b - c)^2 > 0$ and therefore $\angle C_1B_1A_1$ and $\angle A_1C_1B_1$ are acute as well.

Solution 2 by Muhammad Alhafi, Al Basel High School, Aleppo, Syria

We will prove that $\overline{B_1C_1}^2 < \overline{B_1A_1}^2 + \overline{A_1C_1}^2$.

If we draw a line through C_1 *parallel to* $\overline{BC}$ we will see that $a^2 + (b - c)^2 = \overline{B_1C_1}^2$.

In the same manner we have:

$\overline{A_1B_1}^2 = c^2 + (a-b)^2,\ \overline{A_1C_1}^2 = b^2 + (a-c)^2.$
So the inequality is equivalent to:

$$\begin{aligned} a^2 + (b-c)^2 \quad &< \quad c^2 + (a-b)^2 + b^2 + (a-c)^2 \\ &\iff \quad 2ab + 2ac < a^2 + b^2 + c^2 + 2ab \\ &\iff \quad 2a(b+c) < a^2 + (b+c)^2, \text{ which follows from the AM}-\text{GM inequality.} \end{aligned}$$

Following this line of reasoning we can prove: $\overline{B_1A_1}^2 < \overline{B_1C_1}^2 + \overline{A_1C_1^2}$ and that $\overline{A_1C_1}^2 < \overline{B_1A_1}^2 + \overline{B_1C_1}^2$. Hence, $\triangle A_1B_1C_1$ is acute.

Solution 3 by Michael N. Fried, Ben-Gurion University, Beer Sheva, Israel

Suppose we are given an arbitrary triangle such as ABC with sides $BC = a$, $AC = b$, and $AB = c$. Let the lines AA', BB', CC' with lengths a, b, and c, respectively, be drawn perpendicular to the plane of ABC (see figure 1). Then the triangle $A'B'C'$ with sides $B'C' = a'$, $A'C' = b'$, and $A'B' = c'$ is acute.

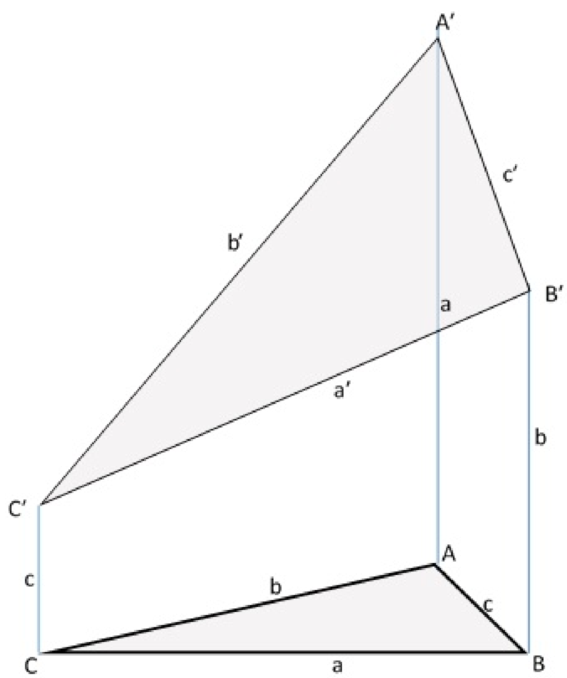


Fig.1

Let us consider first the special case when ABC is an isosceles triangle. First, it is obvious that if ABC is isosceles then also $A'B'C'$ will be isosceles. Moreover, if BC is the base and the angle at A is already acute then the angle at A' will also be acute since $a = a'$ and $c' = b' > b = c$ so that the angle at A' will be less than the angle at A. So we need only consider the case when A is obtuse. In that case, also $a > b = c$.
It makes life easier to consider $A'B'C'$ with respect to the plane UVW drawn through C' (or B') and parallel to ABC so that also $UVW \cong ABC$. In that case, VW coincides with $B'C'$ and $UA' = a - c$ (or $a - b$) (see figure 2).

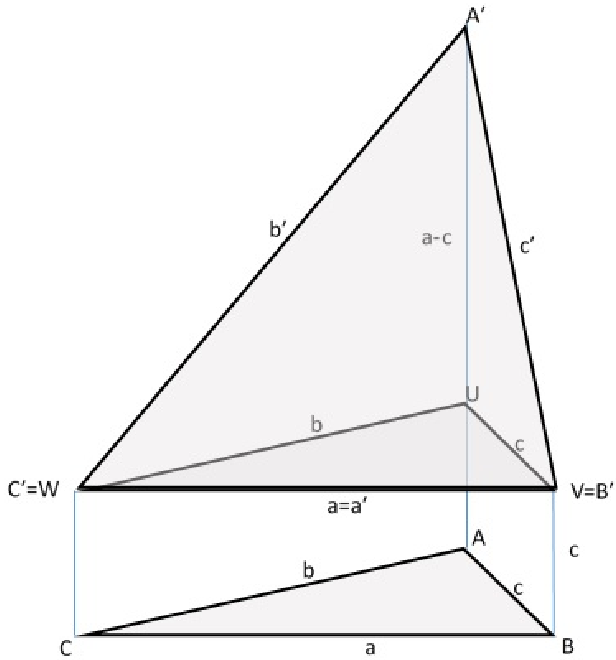


Fig.2

With that out of the way, we need to show that if α is the apex angle at A' then $\alpha < 90°$, or, by the law of cosines, that $2c'^2 \cos\alpha = 2c'^2 - a^2 > 0$. Or since $c'^2 = c^2 + (a-c)^2$:

$$2c^2 + 2(a-c)^2 - a^2 > 0$$

Or, opening parentheses and rearranging:

$$4c^2 - a(4c - a) > 0$$

Note that by the triangle inequality, $2c - a > 0$ so that certainly $4c - a > 0$. By the arithmetic/geometric mean inequality, then, we have (keeping in mind that $a \neq 4c - a$ since otherwise $2c = a$ which is impossible):

$$4c^2 = \left(\frac{a + (4c - a)}{2}\right)^2 > a(4c - a)$$

So, indeed, $4c^2 - a(4c - a) > 0$ and $\alpha < 90°$.
Now, let us consider the case in which ABC is not isosceles. Let us assume that $a > b > c$. As before, consider $A'B'C'$ with respect to the plane UVW drawn through C' and parallel to ABC. Then we have $WB' = b - c$ and $UA' = a - c$ (see figure 3).

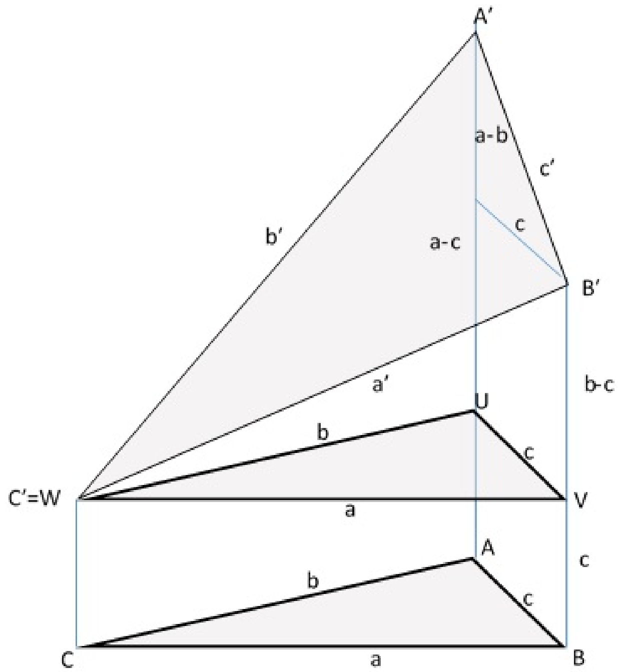


Fig.3

We have then:

$$a'^2 = a^2 + (b-c)^2$$
$$b'^2 = b^2 + (a-c)^2$$
$$c'^2 = c^2 + (a-b)^2$$

Observe that as $a > b > c$, also $a' > b' > c'$, for consider $a'^2 - b'^2$:

$$a'^2 - b'^2 = a^2 + (b-c)^2 - b^2 - (a-c)^2 = (a-b)2c > 0$$

so that $a'^2 > b'^2$. Similarly, we can show that $b'^2 > c'^2$. Since a' is thus the longest side of $A'B'C'$, the angle at A', which we call α', is the largest angle. Therefore, it suffices to show that $\alpha' < 90°$. Again, by the law of cosines this means we must show:

$$2b'c' \cos\alpha' = b'^2 + c'^2 - a'^2 > 0$$

Substituting the expressions above for a', b', and c', we have to show:

$$b^2 + (a-c)^2 + c^2 + (a-b)^2 - a^2 - (b-c)^2 > 0$$

After some algebra, the expression on the left-hand side can be rewritten as follows:

$$c^2 - (a-b)(2c - (a-b))$$

Notice that $a - b > 0$ since we are assuming that a is the longest side of ABC. Also since by the triangle inequality we have $c - (a-b) = b + c - a > 0$, it is certainly true that $2c - (a-b) > 0$. Therefore, again by the arithmetic/geometric-mean inequality, we have:

$$c^2 = \left(\frac{(a-b) + (2c - (a-b)}{2}\right)^2 > (a-b)(2c - (a-b))$$

So, indeed,

$$b'^2 + c'^2 - a'^2 = c^2 - (a-b)(2c - (a-b)) > 0$$

From which we have $\alpha' < 90°$.

Also solved by Yagub N. Aliyev, Problem Solving Group of ADA University, Baku Azerbeaijan; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Ioannis D. Sfikas, National and Kapodistrain University of Athens, Greece; Titu Zvonaru, Comănesti, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposers.

5491: *Proposed by Roger Izard,Dallas, TX*

Let O be the orthocenter of isosceles triangle $ABC,\ AB = AC$. Let OC meet the line segment AB at point F. If $m = FO$, prove that $c^4 \geq m^4 + 11m^2c^2$.

Solution 1 by Ed Gray, Highland Beach, FL

We assume that c is one of the two equal legs. We re-write the inequality by dividing by c^4, so:

1) $1 \geq \left(\dfrac{m}{c}\right)^4 + 11\left(\dfrac{m}{c}\right)^2$. We attempt to prove the inequality by finding the maximum value of $\dfrac{m}{c}$. We shall use the following notation: vertex A is the apex (top) with angle $2t$. We note that $2t < 90$, otherwise $O = A$, or O is external to the triangle. Vertex B is at lower left, and has value $90 - t$. Vertex C is at lower right, also having a value of $90 - t$. Let P be the mid-point of $BC,\ y = BF,\ c - y = AF,\ m = OF$, and the base, $BC = s$, so that $BP = PC = \dfrac{s}{2}$. We note that $\triangle FAC$ is a right triangle, so $\angle ACF = 90 - 2t$. Since $\angle ACB = 90 - t$, by subtraction,

2) $\angle FCB = t$. From $\triangle AOF$,

3) $\tan(t) = \dfrac{m}{c-y}$. From $\triangle FCB$,

4) $\sin(t) = \dfrac{y}{s}$, or $y = s \cdot \sin(t)$. From $\triangle ABP$,

5) $\sin(t) = \dfrac{s}{2c}$, or $c = \dfrac{s}{2\sin(t)}$. Substituting (4) and (5) into (3),

6) $m = \tan(t)\dfrac{s}{(2\sin(t)) - s\cdot\sin(t)}$. Dividing (6) by (5),

7) $\dfrac{m}{c} = \dfrac{\sin(t)}{\cos(t)} \cdot \dfrac{s}{2\sin(t)} - s\cdot\sin(t)\cdot 2\sin\dfrac{t}{s}$, or

8) $\dfrac{m}{c} = \dfrac{\sin(t) - 2\sin^3(t)}{\cos(t)}$

9) $\dfrac{d}{dt}\dfrac{m}{c} = \dfrac{\left(\cos(t)\left(\cos(t) - 6\cos(t)\sin^2(t)\right) - \left(\sin(t) - 2\sin^3(t)\right)(-\sin(t))\right)}{\cos^2(t)}$. Simplifying,

10) $8\sin^4(t) - 6\sin^2(t) + 1 = 0$. This is a quadratic equation in $\sin^2(t)$ with roots:

11) $16\sin^2(t) = 6 \pm \sqrt{(36-32)}$, or

12) $\sin^2(t) = \dfrac{1}{2}$, or $\sin^2(t) = \dfrac{1}{4}$. The former is impossible, since $t = 45$, and $2t = 90$, which would put $O = A$. Therefore, $\sin(t) = \frac{1}{2}$, and $t = 30,\ 2t = 60$, and we have an equilateral triangle. Then $c = s, y = \frac{c}{2}$, and from (3)

13) $\tan(30) = \frac{m}{\frac{c}{2}}$, and

14) $\frac{m}{c} = \frac{1}{2}\tan(30) = \frac{\sqrt{3}}{6},\ \left(\frac{m}{c}\right)^2 = \frac{3}{36} = \frac{1}{12},\ \left(\frac{m}{c}\right)^4 = \frac{1}{144}$, so

15) $\left(\frac{m}{c}\right)^4 + 11\left(\frac{m}{c}\right)^2 = \frac{1}{144} + \frac{11}{12} < 1$, and the conjecture is proved. Q.E.D.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

The angle α at the vertex A is $\le \frac{\pi}{2}$, because OC meets the line segment AB. Clearly $AF = AC\cos\alpha$ and $OF = AF\tan\left(\frac{\alpha}{2}\right) = AC\cos\alpha\tan\left(\frac{\alpha}{2}\right)$. Furthermore $\frac{OF}{AC} = \frac{m}{c}$. Therefore we need to prove that

$$\cos^4\alpha\tan^4\frac{\alpha}{2} + 11\cos^2\alpha\tan^2\frac{\alpha}{2} \le 1,\ \text{for } 0 \le \alpha \le \frac{\alpha}{2}. \qquad (1)$$

We note that

$$y = \cos\alpha\tan\left(\frac{\alpha}{2}\right) = \left(2\cos^2\frac{\alpha}{2} - 1\right)\tan\frac{\alpha}{2} = \left(\frac{2}{1+\tan^2\frac{\alpha}{2}} - 1\right)\tan\frac{\alpha}{2} = \tan\frac{\alpha}{2}\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = x\frac{1-x^2}{1+x^2},$$

where we have put $x = \tan\frac{\alpha}{2}$. Clearly the function $x = \tan\frac{\alpha}{2}$ maps the interval $\left[0, \frac{\alpha}{2}\right]$ to the interval $[0,1]$. We claim that

$$\max_{0\le x\le 1} x\frac{1-x^2}{1+x^2} = \sqrt{\sqrt{5}-2}\frac{\sqrt{5}-1}{2}.$$

Indeed,

$$\frac{d}{dx}x\frac{1-x^2}{1+x^2} = \frac{1-4x^2-x^4}{(1+x^2)^2} = \frac{-\left(x^2+2x+\sqrt{5}\right)\left(x-\sqrt{\sqrt{5}-2}\right)\left(x+\sqrt{\sqrt{5}-2}\right)}{(1+x^2)^2},$$

so the maximum of $x\frac{1-x^2}{1+x^2}$ in the interval $[0,1]$ is assumed at $\sqrt{\sqrt{5}-2}$ and equals

$$\sqrt{\sqrt{5}-2}\frac{3-\sqrt{5}}{\sqrt{5}-1} = \sqrt{\sqrt{5}-2}\frac{\sqrt{5}-1}{2}.$$

Therefore

$$\cos^4\alpha\tan^4\frac{\alpha}{2} + 11\cos^2\alpha\tan^2\frac{\alpha}{2} \le \left(\sqrt{\sqrt{5}-2}\frac{\sqrt{5}-1}{2}\right)^4 + 11\left(\sqrt{\sqrt{5}-2}\frac{\sqrt{5}-1}{2}\right)^4 = 1,$$

and (1) is proven.

Solution 3 by Kee-Wai Lau, Hong Kong, China

Without loss of generality, let $b = c = 1$. Let $AB = AC$ and AO is perpendicular to BC so AO bisects $\angle BAC$. Let $\angle BAC = 2\theta$, where $0 < \theta \le \frac{\pi}{4}$.

By considering triangles AOF and ACF, we obtain respectively $m = AF\tan\theta$ and $AF = \cos 2\theta$, so that $m = \tan\theta\cos 2\theta$. Let $t = \tan\theta$, so that $0 < t \le 1$. Then $m = \dfrac{t(1-t^2)}{1+t^2}$. We have $\dfrac{dm}{dt} = \dfrac{1-4t^2-t^4}{(1+t^2)^2}$, which vanishes when $t = \sqrt{\sqrt{5}-2}$, at which m attains its maximum value of $\sqrt{\dfrac{5\sqrt{5}-11}{2}}$. Hence

$$m^4 + 11m^2 \le \frac{123-55\sqrt{5}}{2} + \frac{55\sqrt{5}-121}{2} = 1,$$

and this completes the solution.

Also solved by Ioannis D. Sfikas, National and Kapodistrain University of Athens, Greece, David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5492: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let a, b, c, d be four positive numbers such that $ab + ac + ad + bc + bd + cd = 6$. Prove that

$$\sqrt{\frac{abc}{a+b+c+3d}} + \sqrt{\frac{bcd}{b+c+d+3a}} + \sqrt{\frac{cda}{c+d+a+3b}} + \sqrt{\frac{dab}{d+a+b+3c}} \le 2\sqrt{\frac{2}{3}}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

By the inequality of Cauchy-Schwarz, the left side of the inequality of the problem does not exceed $2\sqrt{\dfrac{abc}{a+b+c+3d} + \dfrac{bcd}{b+c+d+3a} + \dfrac{cda}{c+d+a+3b} + \dfrac{dab}{d+a+b+3c}}$.

From the given relation, we have $d = \dfrac{6-ab-bc-ca}{a+b+c}$, so that

$$\frac{abc}{a+b+c+3d} = \frac{2abc(a+b+c)}{(a-b)^2+(b-c)^2+(c-a)^2+36} \le \frac{abc(a+b+c)}{18}.$$

Similarly,

$$\begin{aligned}
\frac{bcd}{b+c+d+3a} &\le \frac{bcd(b+c+d)}{18} \\
\frac{cda}{c+d+a+3b)} &\le \frac{cda(c+d+a)}{18} \\
\frac{(dab}{d+a+b+3c)} &\le \frac{dab(d+a+b)}{18}.
\end{aligned}$$

Hence the inequality of the problem will follow from

$$abc(a+b+c) + bcd(b+c-d) + cda(c+d+a) + dab(d+a+b) \le 12. \qquad (1)$$

Now it can be checked readily that the left side of (1) equals

$$\frac{2(ab+ac+ad+bc+bd+cd)^2-(a-b)^2(c-d)^2-(b-c)^2(d-a)^2-(c-a)^2(b-d)^2}{6},$$

which does not exceed $\dfrac{(ab+ac+ad+bc+bd+cd)^2}{3}=12.$

This completes the solution.

Solution 2 by Ed Gray, Highland Beach, FL

1) Let $n=a+b+c+d$. Then:

2) $n^2=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd=a^2+b^2+c^2+d^2+12$

3) Suppose that $a=b=c=d=a$. Then (2) becomes:

4) $(4a)^2=4a^2=12$, and $a=1$.

The left side of the inequality becomes:

5) $4\sqrt{1/6)}=2\sqrt{4/6}=2\sqrt{2/3}$, and we see that the inequality becomes an equality. We need show that the expression is a maximum when $a=b=c=d$. We do this by leaving $a=b=1, c=.99, d=1.01$ so that the constant $n=a+b+c+d$ is maintained.

Substituting the new values into the left side,

6) $\sqrt{.99/6.02}+\sqrt{.9999/6}+\sqrt{.9999}+\sqrt{1.01/5.98}=$

7) $.405526605+.408227878+.407549194+.410969976=1.632273698<1.632993162=2\sqrt{2/3}$.

Hence the function is a maximum for $a=b=c=d$, and the inequality is proven.

Solution 3 by Neculai Stanciu "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania

With the Cauchy-Buniakovski-Schwarz inequality we have

$$\frac{abc}{a+b+c+3d}+\frac{bcd}{b+c+d+3a}+\frac{cda}{c+d+a+3b}+\frac{dab}{d+a+b+3c}$$

$$\leq\quad 4\left(\frac{abc}{a+b+c+3d}+\frac{bcd}{b+c+d+3a}+\frac{cda}{c+d+a+3b}+\frac{dab}{d+a+b+3c}\right).$$

With the AM−HM inequality we have

$$\frac{abc}{a+b+c+3d}=\frac{abc}{a+d+b+d+c+d}\leq\frac{1}{9}\left(\frac{abc}{a+d}+\frac{abc}{b+d}+\frac{abc}{c+d}\right)$$

$$\frac{abc}{a+b+c+3d}+\frac{bcd}{b+c+d+3a}+\frac{cda}{c+d+a+3b}+\frac{dab}{d+a+b+3c}\leq$$

$$\leq\quad\frac{1}{9}\left(\frac{abc}{a+d}+\frac{bcd}{a+d}+\frac{abc}{b+d}+\frac{acd}{b+d}+\frac{abc}{c+d}+\frac{abd}{c+d}+\frac{bcd}{a+b}+\frac{acd}{a+b}+\frac{bcd}{a+c}+\frac{abd}{a+c}+\frac{abd}{b+c}+\frac{acd}{b+c}\right)$$

$$=\quad\frac{1}{9}\left(bc+ac+ab+cd+bdf+ad\right)=\frac{2}{3}.$$

Hence, by the inequalities from above we obtain the desired inequality!

Solution 4 by Marian Ursărescu, National College "Roman-Voda," Roman, Romania

Cauchy's Inequality implies

$$4\sum \frac{abc}{a+b+c+3d} \geq \left(\sum \sqrt{\frac{abc}{a+b+c+3d}}\right)^2 \Rightarrow$$

$$\sum \sqrt{\frac{abc}{a+b+c+3d}} \leq 2\sqrt{\sum \frac{abc}{a+b+c+3d}} \Rightarrow$$

$$\sum \sqrt{\frac{abc}{a+b+c+3d}} \leq 2\sqrt{\sum \frac{abc}{(a+d)+(b+d)+(c+d)}} \qquad (1)$$

But, $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9 \;\Rightarrow\; \frac{1}{x+y+z} \leq \frac{1}{9}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$, which implies

$$\frac{1}{(a+d)+(b+d)+(c+d)} \leq \frac{1}{9}\left(\frac{1}{a+d}+\frac{1}{b+d}+\frac{1}{c+d}\right) \qquad (2)$$

From (1) and(2) we obtain,

$$\sum \sqrt{\frac{abc}{a+b+c+3d}} \leq \frac{2}{3}\sqrt{\sum abc\left(\frac{1}{a+d}+\frac{1}{b+d}+\frac{1}{c+d}\right)}. \qquad (3)$$

But

$$\sum abc\left(\frac{1}{a+d}+\frac{1}{b+d}+\frac{1}{c+d}\right) =$$

$$\frac{abc}{a+d}+\frac{abc}{b+d}+\frac{abc}{c+d}+\frac{bcd}{a+b}+\frac{bcd}{a+c}+\frac{bcd}{a+d}+$$

$$+\frac{cda}{b+a}+\frac{cda}{b+c}+\frac{cda}{b+d}+\frac{dab}{c+a}+\frac{dab}{c+b}+\frac{dab}{c+d} =$$

$$= \frac{bc(a+d)}{a+d}+\frac{ac(b+d)}{b+c}+\frac{ab(c+d)}{c+d}+\frac{bc(a+b)}{a+b}+\frac{4d(a+c)}{a+c}+\frac{ad(4+d)}{4+d} =$$

$$ab + ac + ad + bc + 4d + cd = 6. \tag{4}$$

Equations (3) and (4) implies that

$$\sum \sqrt{\frac{abc}{a+b+c+3d}} \le \frac{2}{3}\sqrt{6} = 2\sqrt{\frac{2}{3}}$$

Solutions 5 and 6 by Paolo Perfetti, Department of Mathematics, Tor Vergatta University, Rome, Italy

First Proof The first step uses the concavity of the function $\sqrt{x}$ yielding

$$\sum_{\text{cyc}} \sqrt{\frac{abc}{a+b+c+3d}} \le 2\sqrt{\sum_{\text{cyc}} \frac{abc}{a+b+c+3d}} \le 2\sqrt{\frac{2}{3}}$$

that is

$$\sum_{\text{cyc}} \frac{abc}{a+b+c+3d} \le \frac{2}{3}$$

Cauchy–Schwarz reversed yields

$$\frac{1}{a+d} + \frac{1}{b+d} + \frac{1}{c+d} \ge \frac{9}{a+b+c+3d}$$

so it suffices to prove

$$\frac{1}{9}\Bigg(\frac{abc}{a+d} + \frac{abc}{b+d} + \frac{abc}{c+d} + \frac{bcd}{d+a} + \frac{bcd}{b+a} + \frac{bcd}{c+a} + \\ + \frac{cda}{a+b} + \frac{cda}{c+b} + \frac{cda}{d+b} + \frac{dab}{a+c} + \frac{dab}{b+c} + \frac{dab}{d+c}\Bigg) \le \frac{2}{3}\frac{ab+bc+ca+ad+bd+cd}{6}$$

We can rewrite it as

$$\frac{abc}{a+d} + \frac{bcd}{d+a} + \frac{abc}{b+d} + \frac{cda}{d+b} + \frac{abc}{c+d} + \frac{dab}{d+c} + \frac{bcd}{b+a} + \frac{cda}{a+b} + \frac{bcd}{c+a} + \frac{dab}{a+c} + \\ + \frac{cda}{c+b} + \frac{dab}{c+b} \le ab+bc+ca+ad+bd+cd$$

This is actually an equality since

$$\frac{abc}{a+d} + \frac{bcd}{d+a} = bc$$

and so on for the other five cases. This concludes the proof.

Proof 6 (Computer assisted) The first step uses the concavity of the function $\sqrt{x}$ yielding

$$\sum_{\text{cyc}} \sqrt{\frac{abc}{a+b+c+3d}} \le 2\sqrt{\sum_{\text{cyc}} \frac{abc}{a+b+c+3d}} \le 2\sqrt{\frac{2}{3}}$$

that is

$$\sum_{\text{cyc}} \frac{abc}{a+b+c+3d} \le \frac{2}{3}$$

First case $d = 0$. The inequality is

$$\frac{abc}{a+b+c} \le \frac{2}{3} \tag{1}$$

We know that

$$\left(\sqrt{3}\sqrt{abc(a+b+c)}\right)^2 = 3abc(a+b+c) \le (ab+bc+ca)^2$$
$$\iff \quad abc(a+b+c) \le (ab)^2 + (bc)^2 + (ca)^2$$

and this holds true by the AGM $(ab)^2 + (ac)^2 \ge a^2bc$ and cyclic. Based on this we can write

$$6 = ab + bc + ca \ge \sqrt{3}\sqrt{abc(a+b+c)} \quad \iff \quad abc(a+b+c) \le 12$$

which inserted in (1) gives

$$3\frac{12}{a+b+c}\frac{1}{a+b+c} \le 2 \quad \iff \quad (a+b+c)^2 \ge 18$$

This follows easily by

$$(a+b+c)^2 \ge 3(ab+bc+ca) = 18$$

Second case $d = 1$ which is allowed by the homogeneity of the inequality after writing

$$\sum_{\text{cyc}} \frac{abc}{a+b+c+3d} \le \frac{2}{3}\frac{ab+bc+cd+da+ac+bd}{6}$$

For $d = 1$ the above inequality becomes

$$\frac{abc}{a+b+c+3} + \frac{bc}{b+c+1+3a} + \frac{ca}{c+1+a+3b} + \frac{ab}{1+a+b+3c} \le \frac{2}{3} \tag{2}$$

This is a algebraic symmetric inequality in three variables and we employ the so called "UVW" theory. Thus we change variables

$$a+b+c = 3u, \quad ab+bc+ca = 3v^2, \quad abc = w^3$$

By expanding (2) we get

$$A(a,b,c)\sum_{\text{cyc}}\Big(8ab + 3a + 16a^2 + 26a^3 + 16a^4 - 150a^2b^2 + 8a^4bc + 36a^2b^2c +$$
$$+42a^3bc + 36a^2bc - 150a^2b^2 + 26a^3b^3 + 3a^5\Big) +$$
$$+A(a,b,c)\sum_{\text{sym}}\Big(-11a^3b^2c - 11a^3b + 3a^5b + 16a^4b^2 - 11a^3b^2 + 8a^4b\Big) +$$
$$+A(a,b,c)(-150a^2b^2c^2 + 42abc) \doteq A(a,b,c)B(a,b,c)$$

$$A(a,b,c) = -9(a+b+c+3)(b+c+1+3a)(c+1+a+3b)(1+a+b+3c)$$

Now we prove the **Lemma that:** *The polynomial $B(a,b,c)$ is a concave parabola in the variable w^3.*

Proof of the Lemma We concentrate on the terms of order six, the only terms containing w^6.

$$\sum_{\text{cyc}}(8a^4bc + 26a^3b^3) + \sum_{\text{sym}}(-11a^3b^2c + 3a^5b + 16a^4b^2) - 150(abc)^2 \tag{3}$$

and once introduced the new variables (u, v, w), we are interested in those terms containing w^6. We have,

$$\sum_{\text{cyc}} a^4bc = abc\sum_{\text{cyc}} a^3 = w^3(3w^3 + 27u^3 - 27uv^2)$$

$$\sum_{\text{cyc}} a^3b^3 = 27v^6 - 27uv^2w^3 + 3w^6, \quad \sum_{\text{sym}} a^3b^2c = w^3(9uv^2 - 3w^3),$$

Moreover,

$$\sum_{\text{sym}} a^5b = \sum_{\text{cyc}} a \sum_{\text{cyc}} a^5 - \sum_{\text{cyc}} a^6, \quad \sum_{\text{cyc}} a^5 = \sum_{\text{cyc}} a^3 \sum_{\text{cyc}} a^2 - 2\sum_{\text{sym}} a^3b^2$$

Since $a^2 + b^2 + c^2 = 9u^2 - 6v^2$, in $\sum_{\text{sym}} a^5b$ only $\sum_{\text{sym}} a^6$ contains w^6 and precisely

$$\sum_{\text{cyc}} a^6 = 729u^6 - 1458u^4v^2 + 729u^2v^4 + 162u^3w^3 - 54v^6 - 108uv^2w^3 + 3w^6$$

$$\sum_{\text{sym}} a^4b^2 = \sum_{\text{cyc}} a^2 \sum_{\text{cyc}} a^4 - \sum_{\text{cyc}} a^6$$

The coefficient of the term w^6 of (3) is

$$24 + 26 \cdot 3 + 11 \cdot 3 - 3 \cdot 3 - 16 \cdot 3 - 150 = -72$$

and so the Lemma has been proved.

Since $A(a,b,c) < 0$, $-B(a,b,c)$ is a convex parabola whose maximum is attained at one or both the extreme points of variations of w^3. The "UVW" theory states that once fixed the values of u and v, the minimum value of w occurs when $abc = 0 = w^3$ or when $b = c$ (or cyclic) while the maximum value occurs when $b = c$ (or cyclic). So we need to study two cases.

First case. $c = 0$.

$$ab + bc + cd + da + ac + bd = 6 \iff a = (6 - b)/(1 + b)$$

Inequality (2) becomes

$$-\frac{(5b^2 - 16b + 14}{3(7 + b + b^2)} \leq 0$$

which evidently holds true.

Second case. $c = b$.

$$ab + bc + cd + da + ac + bd = 6 \quad \Longleftrightarrow \quad a = (6 - b^2 - 2b)/(1 + 2b)$$

Inequality (2) becomes

$$-\frac{(b^2 - 7)(7b^4 - 18b^3 - 27b^2 - 64b - 114)(b - 1)^2}{3(4b + 7b^2 + 7)(-2b + b^2 + 19)(3 + b^2 + 2b)} \le 0 \tag{4}$$

Clearly $a \ge 0$ so $b \le \sqrt{7} - 1$ and then $b^2 - 7 \le 0$. Moreover

$$7b^4 - 18b^3 - 27b^2 \le 0 \quad \Longleftrightarrow \quad b \le (9 + \sqrt{270})/7$$

and thus $7b^4 - 18b^3 - 27b^2 - 64b - 114 \le 0$. The conclusion is that (4) holds true and this completes the proof.

Also solved by Michel Bataille, Rouen, France; Ioannis D. Sfikas, National and Kapodistrain University of Athens, Greece, and the proposer.

Mea Culpa

Brian D. Beasley of Presbyterian College in Clinton, SC should have been credited with having solved problem 5510.