Problems

Ted Eisenberg, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before December 15, 2019

- **5553:** *Proposed by Kenneth Korbin, New York, NY*

  A triangle with sides $(x, x, 57)$ has the same area as a triangle with sides $(x+1, x+1, 55)$. Find x.

- **5554:** *Proposed by Michel Bataille, Rouen, France*

  Find all pairs of complex numbers (a, b) such that the polynomial $x^5 + x^2 + ax + b$ has two roots of multiplicity 2.

- **5555:** *Proposed by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy*

  Show that $x^x - 1 \leq x^{1-x^2} e^{x-1}(x-1)$ for $0 < x \leq 1$.

- **5556:** *Proposed by Pedro Jesús Rodríguez de Rivera (student) and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

  Let $\alpha_k = \dfrac{k + \sqrt{k^2+4}}{2}$. Evaluate $\displaystyle\lim_{k\to\infty} \frac{\prod_{n=1}^{\infty}\left(1 + \dfrac{(k-1)\alpha_k + 1}{\alpha_k^n + \alpha_k}\right)}{\alpha_k}$.

- **5557:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $n \geq 2$ be an integer. If for all $k \in \{1, 2, \ldots, n\}$ we have

  $$A_k = \begin{pmatrix} k+1 & k \\ k+3 & k+2 \end{pmatrix},$$

  compute the value of $\displaystyle\sum_{1\leq i<j\leq n} \det(A_i + A_j)$.

- **5558:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Find all continuous functions $f : \Re \to \Re$ such that

$$\int_{-x}^{0} f(t)dt + \int_{0}^{x} tf(x-t)dt = x, \forall x \in \Re.$$

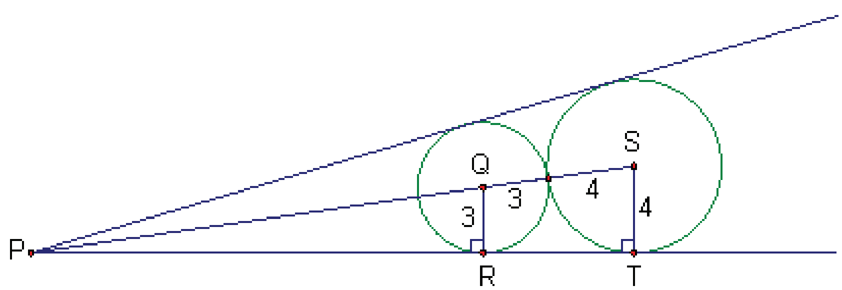
Solutions

- **5535:** *Proposed by Kenneth Korbin, New York, NY*

Given positive angles A and B with $A + B = 180^o$. A circle with radius 3 and a circle of radius 4 are each tangent to both sides of $\angle A$. The circles are also tangent to each other Find $\sin A$.

Solution 1 by David A. Huckaby, Angelo State University, San Angelo, TX

See the figure below, in which angle QPR is $\dfrac{A}{2}$.



We have $\dfrac{QR}{PQ} = \dfrac{ST}{PS}$, that is, $\dfrac{3}{PQ} = \dfrac{4}{PQ+7}$, whence $PQ = 21$. So $\sin\left(\dfrac{A}{2}\right) = \dfrac{1}{7}$ and $\cos\left(\dfrac{A}{2}\right) = \sqrt{1 - \left(\dfrac{1}{7}\right)^2} = \dfrac{4\sqrt{3}}{7}$.

So $\sin A = 2\sin\left(\dfrac{A}{2}\right)\cos\left(\dfrac{A}{2}\right) = 2\left(\dfrac{1}{7}\right)\left(\dfrac{4\sqrt{3}}{7}\right) = \dfrac{8\sqrt{3}}{49}$.

Solution 2 by David E. Manes, Oneonta, NY

The value of $\sin A$ is $8\sqrt{3}/49$.

Let X, Y denote the centers of the circles with radii 3 and 4, respectively. From vertex A, draw the line through the centers X and Y. This line splits the circles and the angle into two equal parts so that it is the angle bisector of $\angle A$. Construct the radius vector XR from the center of the circle with radius 3 to the point of tangency R with angle A. Similarly, YS is the radius vector from the circle of radius 4 to the point of tangency S of angle A. Then triangles AXR and AYS are similar right triangles with right angles at points R and S, respectively. If x denotes the hypotenuse AX of $\triangle AXR$, then $x+7$ is the hypotenuse AY of $\triangle AYS$. By the similarity of the two right triangles, it follows that the ratio of corresponding sides are equal. Therefore, $AX/XR = AY/YS$ or $x/3 = (x+7)/4$

implies $x = 21$. Let s denote the side length AR. Then $s^2 + 3^2 = 21^2$ or $s = 12\sqrt{3}$. Therefore,

$$\sin(A/2) = XR/AX = 3/21 = 1/7 \quad \text{and} \quad \cos(A/2) = AR/AX = 12\sqrt{3}/21 = 4\sqrt{3}/7.$$

Hence,

$$\sin A = 2\sin(A/2)\cos(A/2) = 2(1/7)(4\sqrt{3}/7) = 8\sqrt{3}/49 = \sin B.$$

Solution 3 by Ed Gray, Highland Beach, FL

Let:

$\angle A = \angle DAC$, where AC lies on the x-axis, and the coordinates of vertex $A = (0,0)$.

Let O = center of circle with radius 3, O' = center of circle with radius 4. The angle bisector passes through both circle centers. Let OP be perpendicular to AC, and $AP = x$. The coordinates of $O = (x, 3)$. Let $O'Q$ be perpendicular to AC, and $PQ = y$.

$AQ = x+y$, and the coordinates of $O' = (x+y, 4)$. The distance from O to $O' = 3+4 = 7$.

(1) $\tan(A/2) = 3/x = 4/(x+y)$.

(2) $4x = 3x + 3y$, and $x = 3y$.

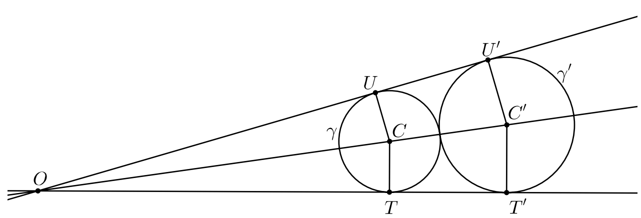
(3) Let T have coordinates $(x+y, 3)$, so that OT is parallel to AC.

(4) OTO' is a right triangle with legs of 1 and y, and hypotenuse of 7.

(5) Then $y^2 + 1 = 49, y^2 = 48$, and $y = 4\sqrt{3}, x = 3y = 12\sqrt{3}$.

(6) $\sin(A) = \sin[2(A/2)] = 2\sin(A/2)\cos(A/2) = 2\left(\frac{1}{7}\right)\left(\frac{y}{7}\right) = \frac{2y}{49} = \frac{8\sqrt{3}}{49}$.

Solution 4 by Michel Bataille, Rouen, France



Let γ and γ' be the circles with radii 3 and 4, respectively. The circle γ (resp. γ') is tangent to the sides of $\angle A$ at T and U (resp. at T' and U') [see figure]. Note that the centres C and C' of γ and γ' lie on the internal bisector of $\angle A$. Let O be the vertex of $\angle A$. The homothety with centre O and scale factor $\frac{4}{3}$ transforms γ into γ' and C into C'. Thus, we have $\frac{OC'}{OC} = \frac{4}{3}$ and, since γ and γ' are tangent to each other, $CC' = 4 + 3 = 7$. It follows that

$$\frac{OC'}{4} = \frac{OC}{3} = \frac{OC' - OC}{4 - 3} = \frac{CC'}{1} = 7.$$

As a result, we obtain $OC = 21$ and so $\sin\frac{A}{2} = \frac{CT}{OC} = \frac{3}{21} = \frac{1}{7}$. In addition, since $0 < \frac{A}{2} < 90°$, we have $\cos\frac{A}{2} > 0$ hence $\cos\frac{A}{2} = \sqrt{1 - \sin^2\frac{A}{2}} = \frac{4\sqrt{3}}{7}$. We can now conclude that

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = \frac{8\sqrt{3}}{49}.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5536:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

If $a \in (0,1)$ then calculate $\lim\limits_{n\to\infty} \sqrt[n]{(2n-1)!!}\left(\sin\left(\dfrac{a\cdot \sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right) - \sin a\right)$.

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA

By Stirling's approximation,

$$n! \sim \frac{n^n}{e^n},$$

so

$$\sqrt[n]{n!} \sim \frac{n}{e} \quad \text{and} \quad \sqrt[n+1]{(n+1)!} \sim \frac{n+1}{e}.$$

Moreover,

$$(2n-1)!! = \frac{(2n)!}{2^n n!} \sim \frac{(2n)^{2n}/e^{2n}}{2^n n^n/e^n} = \frac{2^n n^n}{e^n},$$

so

$$\sqrt[n]{(2n-1)!!} \sim \frac{2n}{e}.$$

It follows that

$$\sqrt[n]{(2n-1)!!}\left(\sin\frac{a\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} - \sin a\right) \sim \frac{2n}{e}\left(\sin a\left(1+\frac{1}{n}\right) - \sin a\right).$$

Using the identity

$$\sin A - \sin B = 2\sin\frac{A-B}{2}\cos\frac{A+B}{2}$$

with

$$A = a\left(1+\frac{1}{n}\right) \quad \text{and} \quad B = a,$$

we find

$$\sin a\left(1+\frac{1}{n}\right) - \sin a = 2\sin\frac{a}{2n}\cos\left(a+\frac{1}{2n}\right).$$

Thus,

$$\begin{aligned}
\sqrt[n]{(2n-1)!!}\left(\sin\frac{a\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} - \sin a\right) \quad &\sim \quad \frac{4n}{e}\sin\frac{a}{2n}\cos\left(a+\frac{1}{2n}\right) \\
&= \quad \frac{2a}{e}\frac{\sin\frac{a}{2n}}{\frac{a}{2n}}\cos\left(a+\frac{1}{2n}\right).
\end{aligned}$$

Finally,

$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{(2n-1)!!}\left(\sin\frac{a\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}-\sin a\right) &= \lim_{n\to\infty}\frac{2a}{e}\frac{\sin\frac{a}{2n}}{\frac{a}{2n}}\cos\left(a+\frac{1}{2n}\right)\\ &= \frac{2a}{e}\cos a.\end{aligned}$$

Note the restriction $a\in(0,1)$ is not necessary.

Solution 2 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

The solution is $\lim\limits_{n\to\infty}\sqrt[n]{(2n-1)!!}\left(\sin\left(\dfrac{a\cdot\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)-\sin a\right)=\dfrac{2a\cos a}{e}$.

Note first that by Stirling formula $\sqrt[n]{(2n-1)!!}\sim\dfrac{2n}{e}$, and also that $\dfrac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\to 1$, for $n\to\infty$, and therefore, by Taylor expassion of $\sin ax$ at $x=1$, it follows that the proposed limit, say L, is

$$\begin{aligned}L &= \frac{2}{e}\lim_{n\to\infty}\frac{-\frac{1}{2}(x-1)^2\left(a^2\sin(a)\right)+a(x-1)\cos(a)+\sin(a)-\sin(a)}{\frac{1}{n}}\\ &= \frac{2}{e}\lim_{n\to\infty}\frac{-\frac{1}{2}(x-1)^2\left(a^2\sin(a)\right)+a(x-1)\cos(a)}{\frac{1}{n}}\\ &= \frac{2a\cos a}{e},\end{aligned}$$

where we have used $x=\dfrac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}$, so, by the Stolz-Cezaro Lemma,

$$\lim_{n\to\infty}\frac{x-1}{\frac{1}{n}}=\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}}{\frac{\sqrt[n]{n!}}{n}}=e\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{n+1}=1$$

and consequently $\lim\limits_{n\to\infty}\dfrac{(x-1)^2}{\frac{1}{n}}=0$, and the conclusion follows.

Solution 3 by Michel Bataille, Rouen, France

The required limit is $\dfrac{2a\cos a}{e}$.

Recall the well-known asymptotic expansion of $\ln(n!)$ as $n\to\infty$:

$$\ln(n!)=n\ln(n)-n+o(n)\qquad (1).$$

From (1), we deduce $\sqrt[n]{n!}\sim\frac{n}{e}$ as $n\to\infty$ [because $\sqrt[n]{n!}=e^{\frac{\ln(n!)}{n}}=e^{\ln(n)-1+o(1)}=\frac{n}{e}\cdot e^{o(1)}$ so that $\lim\limits_{n\to\infty}\frac{e}{n}\cdot\sqrt[n]{n!}=1$]. It follows that

$$\sqrt[n]{(2n-1)!!}\sim\frac{2n}{e}$$

as $n \to \infty$. Indeed, since $(2n-1)!! = (2n-1)(2n-3)\cdots 3\cdot 1 = \frac{(2n)!}{2^n n!}$, we have

$$\sqrt[n]{(2n-1)!!} = \frac{\left(\sqrt[2n]{(2n)!}\right)^2}{2\sqrt[n]{n!}} \sim \frac{1}{2}\cdot\frac{(2n/e)^2}{n/e} = \frac{2n}{e}.$$

To address the second factor, we first remark that $u_n = \frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}$ satisfies $u_n \sim \frac{n+1}{e}\cdot\frac{e}{n} = \frac{n+1}{n}$ so that $\lim\limits_{n\to\infty} u_n = 1$. Since $\ln(x) \sim x-1$ as $x \to 1$, it follows that

$$\begin{aligned} u_n - 1 \sim \ln(u_n) &= \frac{1}{n+1}\left(\ln(n+1)+\ln(n!)\right) - \frac{1}{n}\ln(n!) \\ &= \frac{1}{n+1}\left(\ln(n+1) - \frac{1}{n}\ln(n!)\right) \\ &= \frac{1}{n}\left(1+\frac{1}{n}\right)^{-1}\left(\ln\left(1+\frac{1}{n}\right) + \ln(n) - \frac{1}{n}\ln(n!)\right) \end{aligned}$$

and so $u_n - 1 \sim \frac{1}{n}$ as $n\to\infty$ (note that (1) gives $\lim\limits_{n\to\infty}\left(\ln(n) - \frac{1}{n}\ln(n!)\right) = 1$).
Now, since $\sin x \sim x$ as $x \to 0$, we obtain

$$\sin(au_n) - \sin a = 2\sin\frac{a(u_n-1)}{2}\cos\frac{a(u_n+1)}{2} \sim (2\cos a)\cdot\frac{a(u_n-1)}{2} \sim (a\cos a)\cdot\frac{1}{n}$$

as $n \to \infty$ and deduce that the desired limit is

$$\lim_{n\to\infty}\frac{2n}{e}\cdot(a\cos a)\cdot\frac{1}{n} = \frac{2a\cos a}{e}.$$

Editor′s comment: The statement that there is no need to restrict a to $(0,1)$ was also noted in the solution submitted by **Moti Levy of Rehovot Israel.** Indeed, the result is valid for $a \in \mathbb{C}$.

Also solved by Ed Gray, Highland Beach, FL; Moti Levy, Rehovot, Israel; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland and the proposer.

- **5537:** *Proposed by Mohsen Soltanifar, Dalla Lana School of Public Health, University of Toronto, Canada*

Let X, Y be two real-valued continuous random variables on the real line with associated mean, median and mode $\overline{x}, \tilde{x}, \hat{x}$, and $\overline{y}, \tilde{y}, \hat{y}$, respectively. For each of the following conditions, show that there are variables X,Y satisfying them or prove such random variables do not exist.

$$\begin{array}{llllllll} (i) & \overline{x}\le\overline{y}, & \tilde{x}\le\tilde{y}, & \hat{x}\le\hat{y}, & (v) & \overline{x}>\overline{y}, & \tilde{x}\le\tilde{y}, & \hat{x}\le\hat{y} \\ (ii) & \overline{x}\le\overline{y}, & \tilde{x}\le\tilde{y}, & \hat{x}>\hat{y}, & (vi) & \overline{x}>\overline{y}, & \tilde{x}\le\tilde{y}, & \hat{x}>\hat{y} \\ (iii) & \overline{x}\le\overline{y}, & \tilde{x}>\tilde{y}, & \hat{x}\le\hat{y}, & (vii) & \overline{x}>\overline{y}, & \tilde{x}>\tilde{y}, & \hat{x}\le\hat{y} \\ iv) & \overline{x}\le\overline{y}, & \tilde{x}>\tilde{y}, & \hat{x}>\hat{y}, & (viii) & \overline{x}>\overline{y}, & \tilde{x}>\tilde{y}, & \hat{x}>\hat{y} \end{array}$$

Solution 1 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

We provide examples that satisfy each of the 8 conditions. They can all happen.

Note that examples satisfying conditions (i) through (iv) with strict inequality satisfy conditions $(viii), (vii), (vi)$, and (v) respectively, if X and Y are reversed. For example, if X and Y satisfy condition (i), then Y and X satisfy condition $(viii)$; (ii) and (vii), (iii) and (vi), and (iv) and (v). So we only need four examples.

We'll define the random variables, X, Y_1, Y_2, Y_3, Y_4.

The probability density function for X:

$$f_X(t) = \begin{cases} 0, & t < 0; \\ 2.5t, & 0 \le t \le .8; \\ 10(1-t), & .8 \le t \le 1; \\ 0, & 1 < t. \end{cases}$$

It is straightforward to verify that $\int_{-\infty}^{\infty} f_X(t)\, dt = 1$.

Then the cumulative distribution function is $F_X(x) = \int_{-\infty}^{x} f_x(t)\, dt$.

The <u>mean</u> of X is
$\overline{X} = \int_{-\infty}^{\infty} t f_X(t) dt = \int_0^{.8} t(2.5t) dt + \int_{.8}^{1} 10(1-t) t dt = \frac{32}{75} + \frac{13}{75} = .6.$
To find the <u>median</u> of X, we must find the value for x which makes
$F_X(x) = \int_{-\infty}^{x} f_X(t) dt = \frac{1}{2}$.
By the definition of the pdf, this spot must occur before $x = .8$. So either by geometry or solving $\int_0^x 2.5t dt = \frac{1}{2}$, we find that the median is $\tilde{x} = \sqrt{\frac{2}{5}} = \frac{\sqrt{10}}{6} \approx .63246$.
The maximum value of the pdf f_X is .4, which occurs at $x = .8$.
That is the <u>mode</u> of X is $\hat{x} = .8$.

Conditions $(viii)$ and (i).

We define Y_1 by the density function

$$f_{Y_1}(t) = \begin{cases} 0, & t < -.5; \\ 4(t+.5), & -.5 \le t \le 0; \\ 4(.5-t), & 0 \le t \le .5; \\ 0, & .5 < t. \end{cases}$$

As above, we calculate our three measures:
Mean of $Y_1 : \overline{y_1} = 0$.
Median of $Y_1 : \tilde{y_1} = 0$.
Mode of $Y_1 : \hat{y_1} = 0$.
We see that
$\overline{x} = .6 > \overline{y_1} = 0$.
$\tilde{x} = .6325 > \tilde{y_1} = 0$.
$\hat{x} = .8 > \hat{y_1} = 0$.
Thus, X and Y_1 satisfy condition $(viii)$. Reversing X and Y_1 gives an example which satisfies condition (i)

Conditions (iv) and (v).

We define Y_2 by the density function

$$f_{Y2}(t) = \begin{cases} 0, & t < .12; \\ 4(t - .12), & .12 \leq t \leq .62; \\ 4(1.12 - t), & .62 \leq t \leq 1.12; \\ 0, & 1.12 < t. \end{cases}$$

As above, we calculate our three measures:
Mean of $Y_2 : \overline{y_2} = .62$.
Median of $Y_2 : \tilde{y_2} = .62$.
Mode of $Y_2 : \hat{y_2} = .62$
We see that
$\overline{x} = .6 < \overline{y_2} = .62$
$\tilde{x} = .6325 > \tilde{y_2} = .62$
$\hat{x} = .8 > \hat{y_2} = .62$.
Thus, X and Y_2 satisfy condition (iv),

Reversing X and Y_2 gives an example which satisfies condition (v)

Conditions (ii) and (vii).

We define Y_3 by the density function

$$f_{Y_3}(t) = \begin{cases} 0, & t < .2; \\ 4(t - .2), & .2 \leq t \leq .7 \\ 4(1.2 - t), & .7 \leq t \leq 1.2; \\ 0, & 1.2 < t. \\ \end{cases}$$

As above, we calculate our three measures:
Mean of $Y_3 : \overline{y_3} = .7$.
Median of $Y_3 : \tilde{y_3} = .7$.
Mode of $Y_3 : \hat{y_3} = .7$.

We see that
$\overline{x} = .6 < \overline{y_3} = .7$
$\tilde{x} = .6325 < \tilde{y_3} = .7$
$\hat{x} = .8 > \hat{y_3} = .7$.
Thus, X and Y_3 satisfy condition (ii), Reversing X and Y_3 gives an example which satisfies condition (vii)

Conditions (iii) and (vi).

We define Y_4 by the density function, which is piecewise continuous and defined for all real numbers. Thus the cumulative distribution function for Y_4 is also continuous and defined everywhere. Thus Y_4 is a continuous random variable.

$$f_{Y_4}(t) = \begin{cases} 0, & t < .47; \\ 10/3, & .47 \leq t \leq .62 \\ \\ 8.33/3, & .62 \leq t \leq .8 \\ \\ 2000000(t-8), & .8 \leq t \leq .80001 \\ \\ 2000000(.80002-t), & .80001 \leq t \leq .80002 \\ \\ 0, & .80002 < t. \\ \end{cases}$$

It is more tedious, but we calculate our three measures:
Mean of $Y_4 : \overline{y_4} = .62736$.
Median of $Y_4 : \tilde{y_4} = .62$.
Mode of $Y_4 : \hat{y_4} = .80001$

We see that
$\overline{x} = .6 < \overline{y_4} = .62736$
$\tilde{x} = .6325 > \tilde{y_4} = .62$
$\hat{x} = .8 < \hat{y_4} = .80001$.
Thus, X and Y_4 satisfy condition (iii)

Reversing X and Y_4 gives an example which satisfies condition (vi).

So for each condition $(i) \ldots (viii)$, we have an example satisfying it.

Note: If a random variable that has a continuous probability density function is desired, the following can be used for the definition of Y_4, (but the mathematics to compute its mean and median is much more tedious):

$$f_{Y_4}(t) = \begin{cases} 0, & t < .45; \\ \frac{29167}{9303}(t-.45), & .45 \leq t < .46 \\ \\ \frac{29167}{9303}, & .46 \leq t \leq .61 \\ \\ \left(\frac{833}{3} - \frac{2916700}{9303}\right)(t-.61) + \frac{29167}{9303}, & .61 < t < .62 \\ \\ \frac{8.33}{3}, & .62 < t \leq .8 \\ \\ \frac{5167000}{3}(t-.8) + \frac{8.33}{3}, & .8 \leq t \leq .80001 \\ \\ \frac{60}{.000025835}(.80001-t) + 20, & .8000 \leq t \leq .80001 + \frac{.000025835}{3} \\ \\ 0, & t > .80001 + \frac{.000025835}{3} \\ \end{cases}$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

Let $0 < r < s < 1, u > 0, v > 0, w > 0$ such that

$$\begin{array}{l}(i)\ \ ur + v(s-r)/2 = 1,\\ (ii)\ \ v(s-r)/2 + w(1-s) = 1.\end{array}$$

Let $p(x)$ be the continuous probability density function that is zero in $]-\infty, 0] \cup [1, \infty[$ and piecewise linear in $[0,1]$ such that the graph of $p(x)$ consists of line segments joining the points $(0,0)$ and $(r/2, u), (r/2, u)$ and $(r, 0), (r, 0)$ and $((r+s)/2, v), ((r+s)/2, v)$ and $(s, 0), (s, 0)$ and $((s+1)/2, w), ((s+1)/2, w)$ and $(1, 0)$. $p(x)$ is a probability density function with three "peaks."

Let X be the random variable whose probability density function is $p(x)$. Let r, s, v be given. We solve (i) and (ii) for u and w and find:

$$u = \frac{2 - v(s-r)}{2r},$$

$$w = \frac{2 - v(s-r)}{2(1-s)}.$$

Clearly, $v < \dfrac{2}{s-r}$, since both $u > 0$ and $v > 0$.

We next calculate the mean, median and mode of X, and express these quantities in terms of r, s, v:

$$\begin{aligned}\overline{x} \;&=\; \int_0^1 xp(x)dx = \int_0^{r/2} x\left(\frac{2u}{r}x\right)dx + \int_{r/2}^{r} x\left(-\frac{2u}{r}(x-r)\right)dx + \int_r^{(r+s)/2} x\left(\frac{2v}{s-r}(x-r)\right)dx +\\ &+\; \int_{(r+s)/2}^{s} x\left(-\frac{2v}{s-r}(x-s)\right)dx + \int_s^{(s+1)/2} x\left(\frac{2w}{1-s}(x-s)\right)dx + \int_{(s+1)/2}^{1} x\left(-\frac{2w}{1-s}(x-1)\right)dx =\\ &=\; \frac{1}{4}\left(r^2u - r^2v + s^2v + w - s^2w\right) = \frac{1+r+s}{4} + v\frac{(s-r)(r+s-1)}{8}.\end{aligned}$$

Clearly, $\tilde{x} = \dfrac{r+s}{2}$, since $\displaystyle\int_0^{\tilde{x}} p(x)dx = \frac{1}{2}$ by (i).

The mode $\hat{x}$ is defined as the value $\hat{x}$ for which we have $p(\hat{x}) = \max(u, v, w)$.

We have

$\hat{x} = \dfrac{r}{2}$, if $u > v$ and $u > w$ which is equivalent to $v < \dfrac{2}{r+s}$ and $r + s < 1$.

$\hat{x} = \dfrac{r+s}{2}$, if $v > w$ and $v > u$ which is equivalent to $v > \dfrac{2}{r+s}$ and $v > \dfrac{2}{2-r-s}$.

$\hat{x} = \dfrac{s+1}{2}$, if $w > u$ and $w > v$ which is equivalent to $v < \dfrac{2}{2-r-s}$ and $r + s > 1$.

We have three free parameters at our disposal, namely r, s, v, we can play with. It turns out that by a suitable choice of these parameters all 8 variants can be realized as is evidenced by the subsequent table:

$case$	r	s	u	v	w	$Mean$	$Median$	$Mode$
(i)	0.296	0.615	0.528	5.289	0.406	0.459	0.456	0.456
	0.301	0.728	0.513	3.961	0.567	0.513	0.515	0.515
(ii)	0.301	0.509	0.547	8.031	0.336	0.413	0.405	0.405
	0.143	0.824	2.459	1.904	1.998	0.486	0.484	0.072
(iii)	0.407	0.881	2.214	0.418	7.571	0.579	0.644	0.941
	0.299	0.944	1.449	1.757	7.739	0.595	0.622	0.972
(iv)	0.502	0.953	1.652	0.758	17.640	0.633	0.728	0.977
	0.536	0.849	0.234	5.590	0.829	0.680	0.693	0.693
(v)	0.155	0.720	2.970	1.910	1.644	0.452	0.438	0.078
	0.260	0.630	0.462	4.756	0.325	0.448	0.445	0.445
(vi)	0.364	0.845	1.224	2.306	2.874	0.581	0.605	0.923
	0.485	0.744	1.603	1.719	3.037	0.570	0.615	0.872
(vii)	0.250	0.619	2.460	2.087	1.614	0.455	0.435	0.125
	0.269	0.452	3.507	0.620	1.721	0.426	0.361	0.135
$(viii)$	0.595	0.763	0.879	5.675	2.208	0.632	0.679	0.679
	0.371	0.752	1.401	2.521	2.096	0.546	0.562	0.562

The table was generated by a computer program that selected values for r, s, v randomly, thereby creating instances of the random variables X and Y, until a pair of random variables was found for each of the eight cases.

$Editor's\ comment$: This problem asked us to determine if certain relationships can exist between the mean, median, and mode in two sets of data that are subject to certain constrains. If the constraints on the data are relaxed, and by focusing on the mean, median, and mode on small finite sets of data, one can easily determine the validity of the relationships in this question.

Also solved by the proposer.

- **5538:** *Proposed by Seyran Brahimov, Baku State University, Masalli, Azerbaijan*

Solve for all real numbers $x \neq \dfrac{\pi}{2}(2k+1)$, $k \in \mathbb{Z}$.

$$2 - 2019x = e^{\tan x} + 3^{\sin x} + \tan^{-1} x.$$

Solution 1 by Michel Bataille, Rouen, France

For $k \in \mathbb{Z}$, let I_k denote the open interval $\left(\frac{\pi}{2}(2k-1), \frac{\pi}{2}(2k+1)\right)$. We first show that the equation has no solution in I_k for $k \geq 1$.
If $t \in I_k$ is a solution to the equation, then we have $2 - 3^{\sin t} - \tan^{-1} t = e^{\tan t} + 2019t$ and so

$$2019t < 2019t + e^{\tan t} = 2 - 3^{\sin t} - \tan^{-1} t < 2 - \frac{1}{3} + \frac{\pi}{2}$$

(since $3^{\sin t} \geq 3^{-1}$ and $\tan^{-1} t > -\frac{\pi}{2}$). It follows that $t < \frac{1}{2019}\left(\frac{5}{3} + \frac{\pi}{2}\right) < \frac{\pi}{2}$ and so we must have $k \leq 0$.

Now, we consider the function f defined by $f(x) = e^{\tan x} + 3^{\sin x} + \tan^{-1} x + 2019x$ whose derivative is $f'(x) = (1 + \tan^2 x)e^{\tan x} + (\ln 3)(\cos x)3^{\sin x} + \frac{1}{1+x^2} + 2019$.

Since $|(\ln 3)(\cos x)3^{\sin x}| \leq (\ln 3)3^{\sin x} \leq 3\ln 3$, we have $(\ln 3)(\cos x)3^{\sin x} + 2019 > 0$, hence $f'(x) > 0$. It follows that the restriction f_k of f to the interval I_k, which is continuous and strictly increasing, is a bijection from I_k onto the interval (α_k, β_k) where $\alpha_k = \lim\limits_{x \to \frac{\pi}{2}(2k-1)} f_k(x)$ and $\beta_k = \lim\limits_{x \to \frac{\pi}{2}(2k+1)} f_k(x)$. Since $e^{\tan x}$ tends to 0 when $\tan x$ tends to $-\infty$ and to ∞ when $\tan x$ tends to ∞, it is readily seen that for $k \leq 0$, $\alpha_k < 0$ and $\beta_k = \infty$. Thus, the equation $f_k(x) = 2$ has a unique solution x_k in I_k for $k \leq 0$; in particular $x_0 = 0$. Therefore the given equation has infinitely many solutions, the numbers $x_k = f_k^{-1}(2)$ for $k \leq 0$.

Solution 2 by Brian D. Beasley, Presbyterian College, Clinton, SC

We show that there are infinitely many solutions, one of which is zero and the rest of which are negative real numbers.

For each integer k, let D_k be the interval $(\pi(2k-1)/2, \pi(2k+1)/2)$. Let $D = \cup_{k=-\infty}^{\infty} D_k$. Define $f(x) = 3^{\sin x} + \tan^{-1} x + 2019x - 2$ for each x in $\mathbb{R}$, and define $g(x) = f(x) + e^{\tan x}$ for each x in D. Then

$$f'(x) = (\cos x)3^{\sin x}(\ln 3) + 1/(1 + x^2) + 2019$$

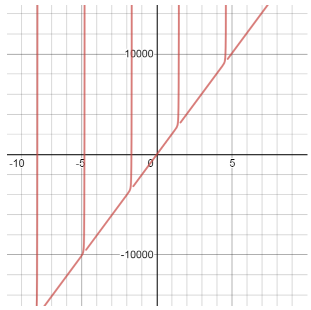
and $g'(x) = f'(x) + (\sec^2 x)e^{\tan x}$. Since $|(\cos x)3^{\sin x}(\ln 3)| \leq 3\ln 3$ for all real numbers x, we have $f'(x) > 0$ on $\mathbb{R}$ and $g'(x) > 0$ on D. Thus f is increasing on $\mathbb{R}$, while g is increasing on each D_k.

Next, we note that on each D_k,

$$\lim_{x \to \frac{\pi}{2}(2k-1)^+} g(x) = f\left(\frac{\pi}{2}(2k-1)\right) \quad \text{and} \quad \lim_{x \to \frac{\pi}{2}(2k+1)^-} g(x) = \infty.$$

Then there is exactly one zero of $g(x)$ in D_k if and only if $f\left(\frac{\pi}{2}(2k-1)\right) < 0$. Since $f(-\pi/2) < 0$ and $f(\pi/2) > 0$, we have exactly one zero x_k of $g(x)$ in D_k if and only if k is a non-positive integer. In particular, $x_0 = 0$, $x_{-1} \approx -1.693068317$, $x_{-2} \approx -4.820854357$, $x_{-3} \approx -7.956873841$, etc.

Graph of $g(x) = e^{\tan x} + 3^{\sin x} + \tan^{-1} x + 2019x - 2$:



Also solved by Ed Gray, Highland Beach, FL; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5539:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let α, β, γ be nonzero real numbers. Find the minimum value of

$$\left(\sum_{cyclic}\left(\frac{1+\sin^2\alpha\sin^2\beta}{\sin^2\alpha}\right)^3\right)^{1/3}$$

Solution 1 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

If we let: $a = \tan\alpha$, $b = \tan\beta$, and $c = \tan\gamma$ then,

$$\sin^2\alpha = \frac{\tan^2\alpha}{1+\tan^2\alpha} = \frac{a^2}{1+a^2}$$

$$\sin^2\beta = \frac{\tan^2\beta}{1+\tan^2\beta} = \frac{b^2}{1+b^2} \qquad\qquad \sin^2\gamma = \frac{\tan^2\gamma}{1+\tan^2\gamma} = \frac{c^2}{1+c^2}.$$

Since $0 \le \sin^2 x \le 1$, then $0 \le \dfrac{\tan^2 x}{1+\tan^2 x} \le 1$ for $x \in \{\alpha, \beta, \gamma\}$. So, we have:

$$\frac{1+\sin^2\alpha\sin^2\beta}{\sin^2\alpha} = \frac{(1+a^2)(1+b^2)+a^2b^2}{a^2(1+b^2)} = \frac{1+a^2}{a^2} + \frac{b^2}{1+b^2} \ge 1 + \frac{1+a^2}{a^2}.$$

Since $\displaystyle\lim_{a\to\pm\infty}\left(1+\frac{1+a^2}{a^2}\right) = 2$, then $\dfrac{1+\sin^2\alpha\sin^2\beta}{\sin^2\alpha} \ge 3$, and:

$$\left[\sum_{cyclic}\left(\frac{1+\sin^2\alpha\sin^2\beta}{\sin^2\alpha}\right)^3\right]^{1/3} \ge 3\sqrt[3]{3} \approx 4.32674871.$$

Solution 2 by Kee-Wai Lau, Hong Kong, China

Denote the expression of the problem by E. We show that the minimum of E is $2\sqrt[3]{3}$.

Since

$$\sum_{cyclic}\frac{1+\sin^2\alpha\sin^2\beta}{\sin^2\alpha} \quad = \quad \sum_{cyclic}\frac{1}{\sin^2\alpha} + \sum_{cyclic}\sin^2\beta$$

$$
\begin{aligned}
&= \sum_{cyclic}\left(\frac{1}{\sin^2\alpha}+\sin^2\alpha\right)\\
&= \sum_{cyclic}\left(\left(\frac{1}{\sin\alpha}-\sin\alpha\right)^2+2\right)\\
&\geq 6,
\end{aligned}
$$

so by Hölder's inequality, we have $E \geq 3^{-2/3}\displaystyle\sum_{cyclic}\frac{1+\sin^2\alpha\sin^2\beta}{\sin^2\alpha} \geq 2\sqrt[3]{3}$.

When $\alpha=\beta=\gamma=\dfrac{\pi}{2}$, we obtain $E = 2\sqrt[3]{3}$ and hence our claimed minimum.

Solution 3 by Albert Stadler, Herrliberg, Switzerland

We claim that the minimum value equals $2\sqrt[3]{3}$ and is assumed for $\alpha=\beta=\gamma=\dfrac{\pi}{2}$.

Let $u=\sin\alpha, v=\sin\beta, w=\sin\gamma$. The by the AM-GM inequality,

$$\sqrt[3)]{\sum_{cyclic}\left(\frac{1+\sin^2\alpha^2\beta}{\sin^2\alpha}\right)} = \sqrt[3]{\left(\frac{1}{u}+v\right)^3+\left(\frac{1}{v}+w\right)^3+\left(\frac{1}{w}+u\right)^3} \geq$$

$$\geq \sqrt[3]{\left(2\sqrt{\frac{v}{u}}\right)^3+\left(2\sqrt{\frac{w}{v}}\right)^3+\left(2\sqrt{\frac{u}{w}}\right)^3} \geq 2\sqrt[3]{3}\sqrt[3]{\frac{v}{u}\cdot\frac{w}{v}\cdot\frac{u}{w}} = 2\sqrt[3]{3}.$$

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; Michel Bataille, Rouen, France; Ed Gray, Highland Beach, FL; Moti Levy Rehovot, Israel; David E. Manes, Oneonta, NY, and the proposer.

- **5540:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $A \in M_2(\Re)$ be a matrix which has real eigenvalues. Prove that if $\sin A$ is similar to A then $\sin A = A$.

Solution 1 by Moti Levy, Rehovot, Israel

The matrix $A \in M_2(\mathbb{R})$, with real eigenvalues must be similar (according to the Jordan canonical form) to $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, or to $\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$.
If $A = P^{-1} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P$ then $A^n = P^{-1} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} P$ and it follows that

$$\sin A = \sum_{n=1}^{\infty} a_n A^n = P^{-1} \begin{bmatrix} \sum_{n=1}^{\infty} a_n \lambda_1^n & 0 \\ 0 & \sum_{n=1}^{\infty} a_n \lambda_2^n \end{bmatrix} P = P^{-1} \begin{bmatrix} \sin \lambda_1 & 0 \\ 0 & \sin \lambda_2 \end{bmatrix} P. \quad (1)$$

If $A = P^{-1} \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} P$ then $A^n = P^{-1} \begin{bmatrix} \lambda_1^n & n\lambda_1^{n-1} \\ 0 & \lambda_2^n \end{bmatrix} P$ and it follows that

$$\sin A = \sum_{n=1}^{\infty} a_n A^n = P^{-1} \begin{bmatrix} \sum_{n=1}^{\infty} a_n \lambda_1^n & \sum_{n=1}^{\infty} n a_n \lambda_1^{n-1} \\ 0 & \sum_{n=1}^{\infty} a_n \lambda_2^n \end{bmatrix} P = P^{-1} \begin{bmatrix} \sin \lambda_1 & \cos \lambda_1 \\ 0 & \sin \lambda_1 \end{bmatrix} P. \quad (2)$$

Similar matrices have the same eigenvalues, hence from (1)

$$\sin \lambda_1 = \lambda_1,$$
$$\sin \lambda_2 = \lambda_2,$$

which implies $\lambda_1 = \lambda_2 = 0$. In this case $A = \sin A = P^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
Similarly, it follows from (2) that

$$\sin \lambda_1 = \lambda_1,$$
$$\cos \lambda_1 = 1,$$

which implies $\lambda_1 = 0$. In this case $A = \sin A = P^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Solution 2 by Michel Bataille, Rouen, France

Let λ_1, λ_2 be the eigenvalues of A. First, we suppose that $\lambda_1 \neq \lambda_2$ and we show that $\sin A$ cannot be similar to A in that case. Since its eigenvalues are distinct, the matrix A is diagonalizable, that is, $A = PDP^{-1}$ where $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and $P \in GL_2(\mathbb{R})$. Then

$$\sin A = P(\sin D)P^{-1} = P \begin{pmatrix} \sin(\lambda_1) & 0 \\ 0 & \sin(\lambda_2) \end{pmatrix} P^{-1}$$

so that the eigenvalues of $\sin A$ are $\sin(\lambda_1)$ and $\sin(\lambda_2)$. If $\sin A$ were similar to A, then we would have $\{\lambda_1, \lambda_2\} = \{\sin(\lambda_1), \sin(\lambda_2)\}$. However, $\sin(\lambda_1) = \lambda_1$, $\sin(\lambda_2) = \lambda_2$ implies $\lambda_1 = \lambda_2 (= 0)$ contradicting $\lambda_1 \neq \lambda_2$. Nor can the remaining possibility $\sin(\lambda_1) = \lambda_2$, $\sin(\lambda_2) = \lambda_1$ occur; indeed, in that case $\lambda_1, \lambda_2 \in [-1, 1]$ and $\sin(\sin(\lambda_1)) = \lambda_1$, $\sin(\sin(\lambda_2)) = \lambda_2$. But the function $\phi : x \mapsto x - \sin(\sin x)$ is strictly increasing, hence injective, on $[-1, 1]$ (its derivative $x \mapsto 1 - (\cos x)\cos(\sin x)$ is nonnegative and vanishes only at 0 since $0 < \cos x < 1$ for $x \in [-1, 1]$, $x \neq 0$). Thus, from $\phi(\lambda_1) = \phi(\lambda_2)$ we deduce $\lambda_1 = \lambda_2$, again a contradiction.
Suppose now that A has a unique eigenvalue λ.
If A is diagonalizable, then $A = \lambda I_2$ and so $\sin A = (\sin \lambda)I_2$. If $\sin A$ is similar to A, then

$\sin\lambda = \lambda$, hence $\lambda = 0$ and we conclude that $\sin A = A (= O_2)$.
If A is not diagonalizable, the A is similar to its Jordan form $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$: $A = QJQ^{-1}$ for some matrix $Q \in GL_2(\mathbb{R})$. Since $J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$ for any positive integer n (easy induction), we obtain that

$$\sin A = Q\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, J^{2n+1}\right) Q^{-1} = Q\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \begin{pmatrix} \lambda^{2n+1} & (2n+1)\lambda^{2n} \\ 0 & \lambda^{2n+1} \end{pmatrix}\right] Q^{-1},$$

that is,

$$\sin A = Q \begin{pmatrix} \sin(\lambda) & \cos(\lambda) \\ 0 & \sin(\lambda) \end{pmatrix} Q^{-1}.$$

Now, if $\sin A$ is similar to A, then $\sin\lambda = \lambda$, hence $\lambda = 0$ and therefore $A = Q\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Q^{-1} = \sin A$.
We conclude that $\sin A = A$ whenever $\sin A$ is similar to A.

Solution 3 by Kee-Wai Lau, Hong Kong, China

Let λ_1 and λ_2 be the real eigenvalues of A, so that the eigenvalues of $\sin A$ are $\sin\lambda_1$ and $\sin\lambda_2$. Since $\sin A$ and A are similar, they have the same eigenvalues. Thus either a) $\sin\lambda_1 = \lambda_1, \sin\lambda_2 = \lambda_2$ or b) $\sin\lambda_1 = \lambda_2, \sin\lambda_2 = \lambda_1$.

For case a) let $f(x) = x - \sin x$, where x is any real number. We have $f'(x) = 1 - /cosx$, so that $f(x)$ is strictly increasing for $x \in (-1, 0) \cup (0, 1)$. Since $f(0) = 0$ and $f(x)$ is nondecreasing in general, so $f(x) = 0$ if and only if $x = 0$. It follows that $\lambda_1 = \lambda_2 = 0$.

For case b), we have $\sin(\sin\lambda_1) = \lambda_1$ and $\sin(\sin\lambda_2) = \lambda_2$. For real numbers x let $g(x) = x = \sin(\sin x)$ so that $g'(x) = 1 - \cos x \cos(\sin x)$. Similar to a), we see that $g(x) = 0$ if and only if $x = 0$. Again $\lambda_1 = \lambda_2 = 0$.

It is known ([1] p.200, Theorem 4.11) that if A has equal eigenvalues λ, then $= (\cos\lambda)A + (\sin\lambda - \lambda\cos\lambda)I_2$, where I_2 is the identity matrix of order 2.

Since $\lambda = 0$, so $\sin A = A$, as desired.

Reference 1. V. Pop, O. Furdui: Square Matrices of Order 2, Springer, 2017

Solution 4 by Albert Stadler, Herrliberg, Switzerland

Let a, b be the eigenvalues of A which are assumed to be real. Any matrix A (with real or complex entries) is similar to an upper triangular matrix whose diagonal entries are the eigenvalues of A, i.e. there is an invertible 2 by 2 matrix T such that

$$T^{-1}AT = \begin{pmatrix} a & * \\ 0 & b \end{pmatrix}.$$

We conclude that

$$T^{-1}\sin AT = T^{-1}\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} A^k\right) T = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(T^{-1}AT\right)^k = \begin{pmatrix} \sin a & * \\ 0 & \sin b \end{pmatrix}.$$

By assumption A and $\sin A$ are similar. Similar matrices have the same eigenvalues. Therefore $\{a, b\} = \{\sin a, \sin b\}$. So either $a = \sin a$ and $b = \sin b$ or $a = \sin b$ and $b = \sin a$.

In the first case we have $a = b = 0$, since a and b are real. In the second case we have $a = \sin \sin a$ and $b = \sin \sin b$ which implies again $a = b = 0$. (Note that for $x \neq 0 |\sin x| < |x|$.)

Thus $A = T \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} T^{-1}$. which implies that A^k is the null-matrix for all $k > 1$ and therefore

$$\sin A = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} A^k = A.$$

Also solved by the proposer.