Problems

Ted Eisenberg, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before April 15, 2020

- **5577:** *Proposed by Kenneth Korbin, New York, NY*

  Convex quadrilateral $ABCD$ with integer length sides is inscribed in a circle with diameter $\overline{AD} = 625$. Find the perimeter if $\left(\overline{AB}, \overline{BC}, \overline{CD}\right) = 1$.

- **5578:** *Proposed by Roger Izard, Dallas, TX*

  In triangle ABC points F, E, and D lie on lines segments AB, BC, and AC respectively, such that $\dfrac{\overline{AF}}{\overline{BA}} = \dfrac{\overline{BE}}{\overline{BC}} = \dfrac{\overline{DC}}{\overline{AC}}$ and $\angle BAE = \angle CBD = \angle ACF$. Prove or disprove: Triangle ABC must be an equilateral triangle.

- **5579:** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu", Mehedinti, Romania*

  Prove: If $a, b \in \Re, a \leq b$, then $\log 5 \cdot \displaystyle\int_a^b 5^{x^2}dx + \log 5 \cdot \int_a^b 5^{x^4}dx \geq 5^b - 5^a$.

- **5580:** *Proposed by D.M. Bătinetu-Giurgiu "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

  Compute: $\displaystyle\lim_{n\to\infty} \frac{1}{\left(\sqrt[n]{(2n-1)!!}\right)^2} \sum_{k=1}^{n} \left[\left(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!}\right)^2\right]$ where $[x]$ denotes the integer part of x.

- **5581:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let a, b, c be the lengths of the sides of an acute triangle ABC. Prove that

$$\sqrt{\frac{a^2+2bc}{b^2+c^2-a^2}}+\sqrt{\frac{b^2+2ca}{c^2+a^2-b^2}}+\sqrt{\frac{c^2+2ab}{a^2+b^2-c^2}}\geq 3\sqrt{3}.$$

- **5582:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Calculate

$$\lim_{n\to\infty}\sqrt[n]{\int_0^1\int_0^1\left(\frac{x+y^2+x^3+\cdots+x^{2n-1}+y^{2n}}{n}\right)^n \mathrm{d}x\mathrm{d}y}.$$

Solutions

- **5559:** *Proposed by Kenneth Korbin, New York, NY*

  For every positive integer N there are two Pythagorean triangles with area $(N)(N+1)(2N+1)(2N-1)(4N+1)(4N^2+2N+1)$. Find the sides of the triangles if $N=4$.

  Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

  If a and b are the sides and c is the hypotenuse of a Pythagorean triangle $\triangle$, then a, b, and c are positive integers for which $c^2=a^2+b^2$. It is well-known that one way to generate such triangles is to set

$$\begin{aligned} a &= k\left(m^2-n^2\right) \\ b &= 2kmn \\ c &= k\left(m^2+n^2\right) \end{aligned} \tag{1}$$

  for positive integers m, n, and k such that $m>n$, $mn \pmod 2$, and $\gcd(m,n)=1$. Then,

$$\begin{aligned} Area\left(\triangle\right) &= \frac{1}{2}ab \\ &= k^2mn\left(m^2-n^2\right). \end{aligned} \tag{2}$$

  For this problem, we are given that

$$Area\left(\triangle\right)=(4)(5)(9)(7)(17)(73).$$

  Therefore, by (2), we must find positive integers m, n, and k such that $m>n$, $mn \pmod 2$, $\gcd(m,n)=1$, and

$$k^2mn\left(m^2-n^2\right)=(4)(5)(9)(7)(17)(73).$$

Solution 1. Choose $k = 1$, $m = (9)(5) = 45$, and $n = (7)(4) = 28$. Then, m and n have the required properties and additionally (2) is satisfied because

$$m^2 - n^2 = 45^2 - 28^2 = 1241 = (17)(73).$$

The resulting values of a, b, and c are

$$\begin{aligned} a &= (1)\left(45^2 - 28^2\right) = 1241, \\ b &= (2)(1)(45)(28) = 2520, \end{aligned}$$

and

$$c \;=\; (1)\left(45^2 + 28^2\right) = 2809.$$

As a check, we note that

$$a^2 + b^2 = 1241^2 + 2520^2 = 7,890,481 = 2809^2 = c^2$$

and

$$\begin{aligned} Area\,(\triangle) &= \frac{1}{2}ab \\ &= \frac{1}{2}(1241)(2520) \\ &= 1,563,660 \\ &= (4)(5)(9)(7)(17)(73). \end{aligned}$$

Solution 2. Choose $k = 1$, $m = (4)(17) = 68$, and $n = 5$. Then, once again, m and n have the required properties and additionally, (2) is satisfied because

$$m^2 - n^2 = 68^2 - 5^2 = 4599 = (9)(7)(73).$$

The resulting values of a, b, and c are

$$\begin{aligned} a &= (1)\left(68^2 - 5^2\right) = 4599, \\ b &= (2)(1)(68)(5) = 680, \end{aligned}$$

and

$$c \;=\; (1)\left(68^2 + 5^2\right) = 4649.$$

Then,

$$a^2 + b^2 = 4599^2 + 680^2 = 21,613,201 = 4649^2 = c^2$$

and

$$\begin{aligned} Area\,(\triangle) &= \frac{1}{2}ab \\ &= \frac{1}{2}(4599)(680) \\ &= 1,563,660 \\ &= (4)(5)(9)(7)(17)(73). \end{aligned}$$

Remark: In both solutions, gcd $(a, b) = 1$. Hence, both solutions give primitive Pythagorean triangles for this case.

Solution 2 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

One of the problems in Pythagorean triangles, which have engaged the attention of many mathematicians throughout the centuries is to find two or more Pythagorean triangles having equal areas (Beiler, 1966, p. 109).

In the work of Guy [2], he wondered how many primitive Pythagorean triangles can have the same area. A triple of such, with generators $(77, 38), (78, 55)$ and $(138, 5)$ was found by Charles L. Shedd in 1945. In 1986, Rathbun found three more: $(1610, 869), (2002, 1817), (2622, 143)$; $(2035, 266), (3306, 61), (3422, 55)$ and $(2201, 1166), (2438, 2035), (3565, 198)$. A fifth triple, $(7238, 2465), (9077, 1122), (10434, 731)$, was found independently on consecutive days by Dan Hoey and Rathbun. Is there an infinity of triples? Are there quadruples?

For the question of the smallest number that is the area of n distinct Pythagorean triangles, then 71831760 is area of 5 Pythagorean triangles: $(2415, 59488, 59537)$, $(2640, 54418, 54482), (5070, 28336, 28786), (7280, 19734, 21034), (10010, 14352, 17498)$ [see 3]. Furthermore, 210 is the smallest area common to 2 primitive Pythagorean triangles [viz. triples $(20, 21, 29), (12, 35, 37)$]; followed by 2730 [triples (60,91,109), (28,195,197)]; 7980 [triples (95,168,193), (40,399,401)]; 71610 [triples $(341, 420, 541), (132, 1085, 1093)$]; [see 4].

Fermat used a simple method for obtaining two Pythagorean triangles with equal areas. If a and b are the two legs and c the hypotenuse of a Pythagorean triangle, so that $a^2 + b^2 = c^2$, he used $m = c^2$ and $n = 2ab$ as the generators of a new Pythagorean triangle with legs: $m^2 - n^2 = c^4 - 4a^2,\ b^2 = (a^2 - b^2)^2$, and $2mn = 4c^2ab$ and hypotenuse $m^2 + n^2 = c^4 + 4a^2b^2$. Its area is $2c^2ab(c^4 - 4a^2b^2) = 2c^2ab(a^2 - b^2)^2$. This triangle has the same area as the one obtained when the sides of triangle a, b, c are each multiplied by $2c(a^2 - b^2)$. This is easily proved: The two legs of the magnified triangle are $a \cdot 2c(a^2 - b^2)$ and $b \cdot 2c(a^2 - b^2)$, and the area is $2c^2ab(a^2 - b^2)^2$, the same as above (Beiler, 1966, pp. 126-127).

Taking $a = 4, b = 3, c = 5$, the generators become $m = 25, n = 24$, forming the triangle $49, 1200$, and 1201. Multiplying 4, 3, and 5 by $2 \cdot 4 \cdot (4^2 - 3^2) = 70$, the magnified triangle becomes $280, 210$, and 350. The area of both of these triangles is 29400 (Beiler, 1966, p. 127).

If $A = (N)(N + 1)(2N + 1)(2N - 1)(4N + 1)(4N^2 + 2N + 1)$, where A is the area of the Pythagorean triangle, then for $N = 4 : A = 1563660 = 2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 17 \cdot 73$. If a and b are the two legs of the Pythagorean triangle, then: $\dfrac{ab}{2} = A$, or:

$$ab = 3127320 = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 17 \cdot 73. \qquad (1)$$

If d denotes the divisors of 3127320, then:
$d = (3 + 1)(2 + 1)(1 + 1)(1 + 1)(1 + 1)(1 + 1) = 192$.

The 192 divisors are presented below:

1	2	3	4	5	6	7	8	9	10
12	14	15	17	18	20	21	24	28	30
34	35	36	40	42	45	51	56	60	63
68	70	72	73	84	85	90	102	105	119
120	126	136	140	146	153	168	170	180	204
210	219	238	252	255	280	292	306	315	340
357	360	365	408	420	438	476	504	510	511
584	595	612	630	657	680	714	730	765	840
876	952	1020	1022	1071	1095	1190	1224	1241	1260
1314	1428	1460	1530	1533	1752	1785	2049	2044	2142
2190	2389	2482	2520	2555	2628	2856	2920	3060	3066
3285	3570	3723	4088	4284	4380	4599	4760	4964	5110
5256	5355	6120	6132	6205	6570	7140	7446	7665	8568
8687	8760	9198	9928	10220	10710	11169	12264	12410	13140
14280	14892	15330	17374	18396	18635	20440	211420	22338	22995
24820	26061	26280	29784	30660	34748	36792	37230	42840	43435
44676	45990	49640	52122	55845	61320	69496	74460	78183	86870
89352	91980	104244	111690	130305	148920	156366	173740	183960	208488
223380	260610	312732	347480	390915	446760	521220	625464	781830	1042440
1563660	3127320								

Since a and b are integers, then they have to satisfy equation (1) and the Pythagorean theorem $a^2 + b^2 = c^2$, that means the sum of the squares of a and b must be a perfect square. By some calculations, we may find:

$$(a, b, c) = (680, 4599, 4649) \text{ and } (a, b, c) = (1241, 2520, 2809).$$

[1] Beiler, Albert H. (1966). "The Eternal Triangle," Ch. 14. In Recreations in the Theory of Numbers: The Queen of Mathematics Entertains. New York: Dover.

[2] Guy, Richard K. (1994). Unsolved problems in intuitive mathematics. New York: Springer-Verlag, pp. 188-190.
[3] http://oeis.org/A055193
[4] http://oeis.org/A093536
[5] https://math.stackexchange.com/questions/1272064/quadruple-of-pythagorean-triples-with-same-area/12742241274224
[6] https://math.stackexchange.com/questions/2433492/is-there-a-general-formula-for-three-pythagorean-triangles-which-share-an-area [7] https://math.stackexchange.com/questions/2448242/is-there-a-formula-for-this-specific-pattern-of-pythagorean-triangles-sharing-an [8] Rathbun, Randall L. (1994). Table of Equal Area Pythagorean Triangles, from Co-primitive sets of Integer Generator Pairs. Mathematics of Computation, 62 (205):440.

Solution 3 by Michel Bataille, Rouen, France

If m, n are positive integers with $m > n$, then $(2mn)^2 + (m^2 - n^2)^2 = (m^2 + n^2)^2$, hence $2mn,\ m^2-n^2,\ m^2+n^2$ are the sides of a Pythagorean triangle whose area is $\frac{1}{2}(2mn)(m^2-n^2) = mn(m-n)(m+n)$.
Let $A = N(N+1)(2N+1)(2N-1)(4N+1)(4N^2+2N+1)$ where $N \in N$.
First, if we take $m = N(4N+1)$ and $n = N+1$, it is readily checked that $mn(m-n)(m+n) = A$. Therefore A is the area of the Pythagorean triangle with sides

$$2N(N+1)(4N+1),\ N^2(4N+1)^2 - (N+1)^2,\ N^2(4N+1)^2 + (N+1)^2.$$

Second, if we take $m = (N+1)(2N+1)$ and $n = N(2N-1)$, then $mn(m-n)(m+n) = A$ again, hence A is the area of the Pythagorean triangle with sides

$$2N(N+1)(4N^2-1),\ (N+1)^2(2N+1)^2 - N^2(2N-1)^2,\ (N+1)^2(2N+1)^2 + N^2(2N-1)^2.$$

We remark that in both cases the first side is the only even side of the triangle; in addition, each of these sides, namely $2N(N+1)(4N+1)$ and $2N(N+1)(4N^2-1)$, are distinct (since $4N+1 = 4N^2-1$ rewrites as $(2N-1)^2 = 3$, which does not hold if N is an integer). Thus, the two triangles found above are distinct.

In the case when $N = 4$, a simple calculation gives the sides of the two triangles: in the first case, the sides are $680,\ 4599,\ 4649$ and in the second case $2520,\ 1241,\ 2809$.

Solution 4 by Albert Natian, Los Angeles Valley College, Valley Glen, CA

All Pythagorean Triplets are given by $\left(a = 2xy,\ b = x^2 - y^2,\ c = x^2 + y^2\right)$ where $x,\ y \in N$ and where sides a and b are the legs of the right triangle and side c is the hypotenuse. The area of a Pythagorean Triangle with legs a and b is given by $\mathcal{A} = xy\,(x+y)\,(x-y)$. The two choices for the two Pythagorean triangles are given by

$$\textbf{Choice One}\left\{\begin{array}{c} x = (N+1)\,(2N+1) = 2N^2+3N+1, \quad y = N\,(2N-1) = 2N^2 - N \\ \\ x+y = 4N^2+2N+1, \quad x-y = 4N+1 \\ \\ a = 2\,(N+1)\,(2N+1)\,N\,(2N-1)\,, \quad b = \left(4N^2+2N+1\right)(4N+1) \end{array}\right\},$$

$$\textbf{Choice Two} \left\{ \begin{array}{c} x = N\,(4N+1) = 4N^2 + N, \quad y = N+1 \\ \\ x + y = 4N^2 + 2N + 1, \quad x - y = 4N^2 - 1 = (2N-1)\,(2N+1) \\ \\ a = 2N\,(4N+1)\,(N+1)\,, \quad b = \left(4N^2+2N+1\right)(2N-1)\,(2N+1) \end{array} \right\}.$$

For $N = 4$:

$$\textbf{Choice One} \left\{\; a = 2520, \quad b = 1241, \quad c = 2809 \;\right\},$$

$$\textbf{Choice Two} \left\{\; a = 680, \quad b = 4599, \quad c = 4649 \;\right\}.$$

Solution 5 by Brian D. Beasley, Presbyterian College, Clinton, SC

For each positive integer N, let $A_N = N(N+1)(2N+1)(2N-1)(4N+1)(4N^2+2N+1)$ and define a_1, b_1, a_2, and b_2 as follows:

$$a_1 = 2N(N+1)(2N+1)(2N-1) \quad \text{and} \quad b_1 = (4N+1)(4N^2+2N+1)$$
$$a_2 = 2N(N+1)(4N+1) \quad \text{and} \quad b_2 = (2N+1)(2N-1)(4N^2+2N+1)$$

Then straightforward algebraic calculations show that for $i \in \{1,2\}$, we obtain $(1/2)a_ib_i = A_N$ and $a_i^2 + b_i^2 = c_i^2$, where

$$c_1 = 8N^4 + 8N^3 + 14N^2 + 6N + 1 \quad \text{and} \quad c_2 = 16N^4 + 8N^3 + 2N^2 + 2N + 1.$$

Hence both (a_1, b_1, c_1) and (a_2, b_2, c_2) are Pythagorean triangles with area A_N. In particular, when $N = 4$, the triangles are $(2520, 1241, 2809)$ and $(680, 4599, 4649)$.

Solution 6 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

The two triangles are $(680, 4599, 4649)$ and $(1241, 2520, 2809)$

Let $P = N(N+1)(2N+1)(2N1)(4N+1)(4N2+2N+1)$. We seek legs a and b such that $P = \text{area} = 1/2ab$, where $a^2 + b^2$ is a square. That is, $ab = 2P$.
So we can just test factorizations of $2P = ab$, which satisfy the condition that $a^2 + b^2$ is an integer.

First, with some help from Excel, we find the two triangles for $N = 1$, in which case $P = 210:$ $(a, b, c) = (12, 35, 37)$ and $(a, b, c) = (20, 21, 29)$.

Then we find the two triangles for $N = 2$, in which case $P = 17010:$ $(a, b, c) = (108, 315, 333)$ and $(a, b, c) = (180, 189, 261)$.

In each case, a and b have opposite parity, so we narrow the search somewhat. With $N = 4, 2P = 2(1,563,660) = 2232 \cdot 5 \cdot 7 \cdot 17 \cdot 73$. Knowing the prime factorization allows us to easily count and find all 144 divisors, and it is easy to check them.

We present the two triangles for $N = 1, 2, 3, 4,$ and $5:$

N	P	a_1	b_1	c_1	a_2	b_2	c_2
1	210	12	35	37	20	21	29
2	17,010	108	315	333	180	189	261
3	234,780	312	1,505	1,537	559	840	1,009
4	1,563,660	680	4,599	4,649	1,241	2,520	2,809
5	6,923,070	1,260	10,989	11,061	2,331	5,940	6,381

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; Ed Gray, Highland Beach, FL; Pat Costello, Eastern Kentucky University Richmond, KY; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Albert Stadler, Herrliberg, Switzerland; Daniel Văcaru, Pitesti, Romania, and the proposer.

- **5560:** *Proposed by Michael Brozinsky, Central Islip, NY*

Square ABCD (in clockwise order) with all sides equal to x has point E as the midpoint of side AB. The right triangle EBC is folded along segment EC so that what was previously corner B is now at point B' which is at a distance d from side AD. Find d and also the distance of B' from AB.

Solution 1 by Michael N. Fried, Ben-Gurion University of the Negev, Beer-Sheva, Israel

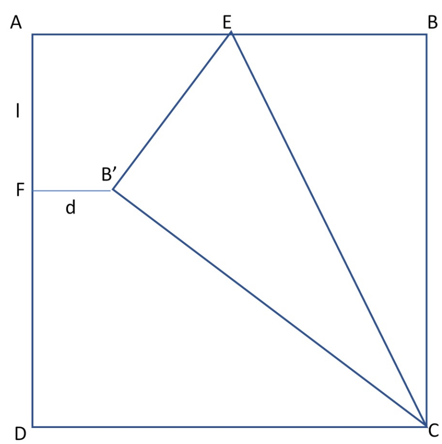


Figure 1

First, note that the area of the quadrilateral $EBCB'$ is exactly half the area of the square. Therefore, the areas of the two trapezoids, $AEB'F$ and $FB'CD$ taken together are also equal to half the area of the square. In other words:

$$l\frac{\frac{x}{2}+d}{2}+(x-l)\frac{d+x}{2}=\frac{x^2}{2}$$

From which, after multiplying and simplifying, we find:

$$d-\frac{l}{2}=0$$

Or,

$$l=2d$$

This means the triangle AFB' is similar to triangle CBE, so that $AB'||EG$. But because quadrilateral $B'EBC$ is a kite, $EG \perp EC$. Therefore $AB' \perp B'B$, so that triangle $BB'A$ is also similar to triangle CBE. Hence, $AB' = \frac{x}{\sqrt{5}}$.

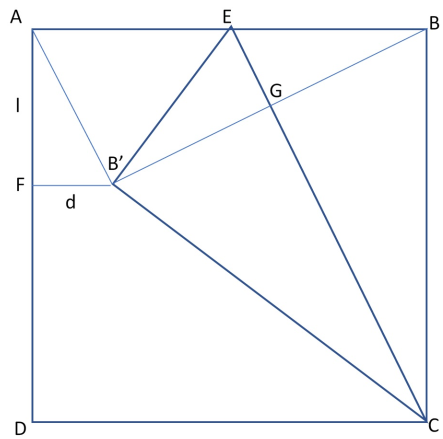


Figure 2

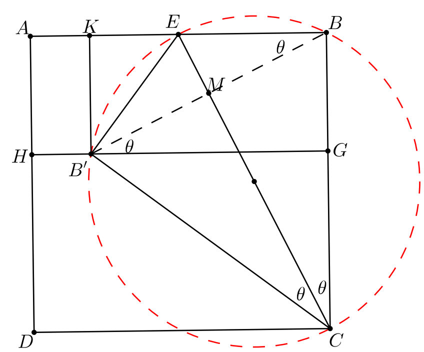
But also, $(AB')^2 = d^2 + l^2$. So, since also $l = 2d$, we have $\frac{x^2}{5} = d^2 + 4d^2 = 5d^2$, so that

$$d = \frac{1}{5}x$$

and

$$l = \frac{2}{5}x$$

Solution 2 by Michel Bataille, Rouen, France



Let $\theta = \angle BCE$. Then $\tan\theta = \frac{EB}{BC} = \frac{x/2}{x} = \frac{1}{2}$ and $\angle B'CE = \theta$ (since B' is the reflection of B in EC). Since E, B, C, B' lie on the circle with diameter EC, we also have $\angle EBB' = \theta$. If G is the orthogonal projection of B' onto BC (see figure), we deduce that $\angle BB'G = \theta$ (since $B'G$ is parallel to EB).

Now, let H, K denote the orthogonal projections of B' onto AD, AB, respectively, and let M be the midpoint of BB'. Then,

$$\cos\theta = \frac{BM}{BE} = \frac{2BM}{x} = \frac{BB'}{x}$$

and so

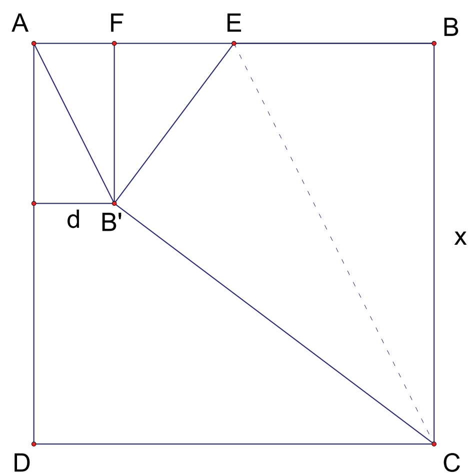
$$BB' = x\cos\theta = x \cdot \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{2x}{\sqrt{5}}.$$

Since $B'G = BB'\cos\theta = \frac{4x}{5}$, we readily obtain

$$d = B'H = x - \frac{4x}{5} = \frac{x}{5}, \qquad B'K = BG = \sqrt{BB'^2 - B'G^2} = \sqrt{\frac{4x^2}{5} - \frac{16x^2}{25}} = \frac{2x}{5}.$$

Solution 3 by David A. Huckaby, Angelo State University, San Angelo, TX

See the figure below, in which F is the point on AB such that $B'F$ is the distance from B' to AB.



Since $EB = \frac{x}{2}$, $\angle BEC = \tan^{-1} 2$, so that $\angle B'EC = \tan^{-1} 2$. Since $AE = \frac{x}{2}$ and $B'E = \frac{x}{2}$, triangle AEB' is isoceles. Along with the fact that exterior angle $\angle B'EB = 2\tan^{-1} 2$, this implies that $\angle EAB' = \angle EB'A = \tan^{-1} 2$.

By the Law of Sines, $\dfrac{AB'}{\sin(180^\circ - 2\tan^{-1} 2)} = \dfrac{\frac{x}{2}}{\sin(\tan^{-1} 2)}$.

Now $\sin\left(180^\circ - 2\tan^{-1} 2\right) = \sin 180^\circ \cdot \cos\left(2\tan^{-1} 2\right) - \cos 180^\circ \cdot \sin\left(2\tan^{-1} 2\right) = 0 + 2 \cdot \sin\left(\tan^{-1} 2\right) \cdot \cos\left(\tan^{-1} 2\right) = 2 \cdot \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} = \frac{4}{5}$.

So $\dfrac{AB'}{\frac{4}{5}} = \dfrac{\frac{x}{2}}{\frac{2}{\sqrt{5}}}$, whence $AB' = \frac{x}{\sqrt{5}}$.

So $d = AB' \cdot \cos\left(\tan^{-1} 2\right) = \frac{x}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} = \frac{1}{5}x$, and the distance of B' from AB is $B'F = AB' \cdot \sin\left(\tan^{-1} 2\right) = \frac{x}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}} = \frac{2}{5}x$.

Solution 4 by David E. Manes, Oneonta, NY

Introduce the following coordinates: $A(0,0)$, $B(0,x)$, $C(x,x)$, $D(x,0)$ and $E(0,x/2)$. Then $ABCD$ is a square in clockwise order such that all sides have length x and E is the midpoint of AB. Moreover, denote the coordinates of point B' by $B'(a,b)$. Then the distance from B' to side AD is $d = b$ and the value of a represents the distance from B' to side AB. We will show that $d = b = (1/5)x$ and $a = (2/5)x$.

Since the right triangle EBC is folded along the segment EC, it follows that side $EB' = EB =$

$x/2$ and $B'C = BC = x$. Therefore,

$$EB' = \sqrt{(a-0)^2 + \left(b - \frac{x}{2}\right)^2} = \frac{x}{2} \Longrightarrow a^2 + b^2 - bx + \frac{x^2}{4} = \frac{x^2}{4} \Longrightarrow a^2 = bx - b^2$$

and

$$B'C = \sqrt{(a-x)^2 + (b-x)^2} = x \Longrightarrow a^2 - 2ax + x^2 + b^2 - 2bx + x^2 = x^2.$$

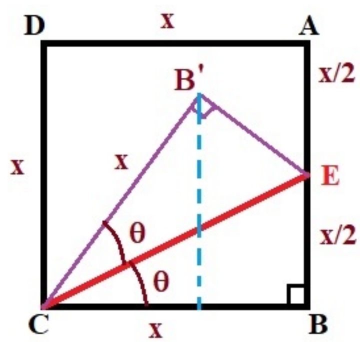
Using the substitution $a^2 = bx - b^2$, this equation reduces to $bx = x^2 - 2ax$. Therefore, $b = x - 2a$ since $x \neq 0$. Then

$$a^2 = bx - b^2 = (x-2a)x - (x-2a)^2 \implies 5a^2 - 2ax = 0 \implies 5a - 2x = 0$$

since $a \neq 0$. Hence, $a = \dfrac{2}{5}x$ so that $b = x - 2a = x - 2\left(\dfrac{2x}{5}\right) = \dfrac{1}{5}x$. Therefore, $a = \dfrac{2}{5}x$ and $b = \dfrac{1}{5}x$.

Solution 5 by Michael C. Faleski, Delta College, University Center, MI

For simplicity of considering the problem on a coordinate system, we are choosing C to be located at the origin as in the figure.



From the right triangle formed by BCE, $\tan(\theta) = \frac{1}{2}$, and after folding over the segment CE, we have point B' located at $(x\cos(2\theta), x\sin(2\theta))$ By the trig identity that $\tan(2\theta) = \frac{\tan(\theta)+\tan(\theta)}{1-\tan(\theta)\tan(\theta)}$, we find $\tan(2\theta) = \frac{\frac{1}{2}+\frac{1}{2}}{1-\frac{1}{2}\frac{1}{2}} = \frac{4}{3}$ This means that $\cos(2\theta) = \frac{3}{5}$ and $\sin(2\theta) = \frac{4}{5}$. Hence, the point B' is located at $(\frac{3}{5}x, \frac{4}{5}x)$.

This makes d, the distance of B' to AD, a length of $x - \frac{4}{5}x = \frac{1}{5}x$ and the distance of B' to AB equal to $x - \frac{3}{5}x = \frac{2}{5}x$

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Brian Bradie, Christopher Newport University, Newport News,VA; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Daniel Văcaru, Pitesti, Romania; Albert Stadler, Herrliberg, Switzerland; Seán M. Stewart, Bomaderry, NSW, Australia; David Stone and John Hawkins, Georgia Souther University, Statesboro, GA, and the proposer.

- **5561:** *Proposed by Pedro Pantoja, Natal/RN, Brazil*

Calculate the *exact value* of:

$$\cos\frac{5\pi}{28} + \cos\frac{13\pi}{28} - \cos\frac{17\pi}{28}.$$

Solution 1 by Brian Bradie, Christopher Newport News, VA

We start with some preliminary results.

- First,

$$\begin{aligned}\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7} &= \operatorname{Re}\left(e^{i\pi/7}+e^{3i\pi/7}+e^{5i\pi/7}\right)\\ &= \operatorname{Re}\left(\frac{e^{i\pi/7}+1}{1-e^{2i\pi/7}}\right)\\ &= -\operatorname{Re}\left(\frac{1+e^{-i\pi/7}}{2i\sin\pi/7}\right)\\ &= -\operatorname{Im}\left(\frac{1+e^{-i\pi/7}}{2\sin\pi/7}\right)\\ &= \frac{1}{2}.\end{aligned}$$

- Second,

$$\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}=-\left(\cos\frac{5\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{\pi}{7}\right)=-\frac{1}{2}.$$

- Third,

$$\begin{aligned}&\left(\cos\frac{\pi}{14}+\cos\frac{3\pi}{14}-\cos\frac{5\pi}{14}\right)^2\\ &\quad= \cos^2\frac{\pi}{14}+\cos^2\frac{3\pi}{14}+\cos^2\frac{5\pi}{14}+2\cos\frac{\pi}{14}\cos\frac{3\pi}{14}\\ &\qquad -2\cos\frac{\pi}{14}\cos\frac{5\pi}{14}-2\cos\frac{3\pi}{14}\cos\frac{5\pi}{14}\\ &\quad= \frac{3}{2}+\frac{1}{2}\left(\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}\right)+\cos\frac{2\pi}{7}+\cos\frac{\pi}{7}\\ &\qquad -\cos\frac{3\pi}{7}-\cos\frac{2\pi}{7}-\cos\frac{4\pi}{7}-\cos\frac{\pi}{7}\\ &\quad= \frac{3}{2}+\frac{1}{2}\cdot\frac{1}{2}=\frac{7}{4}.\end{aligned}$$

Because

$$\cos\frac{\pi}{14}>0 \quad\text{and}\quad \cos\frac{3\pi}{14}>\cos\frac{5\pi}{14},$$

it follows that

$$\cos\frac{\pi}{14}+\cos\frac{3\pi}{14}-\cos\frac{5\pi}{14}>0$$

Therefore,

$$\cos\frac{\pi}{14}+\cos\frac{3\pi}{14}-\cos\frac{5\pi}{14}=\frac{\sqrt{7}}{2}.$$

We now return to the original question. We find

$$\begin{aligned}
&\left(\cos\frac{5\pi}{28}+\cos\frac{13\pi}{28}-\cos\frac{17\pi}{28}\right)^2\\
&\quad= \left(\cos\frac{5\pi}{28}+\cos\frac{13\pi}{28}+\cos\frac{11\pi}{28}\right)^2\\
&\quad= \cos^2\frac{5\pi}{28}+\cos^2\frac{13\pi}{28}+\cos^2\frac{11\pi}{28}+2\cos\frac{5\pi}{28}\cos\frac{13\pi}{28}\\
&\qquad +2\cos\frac{5\pi}{28}\cos\frac{11\pi}{28}+2\cos\frac{13\pi}{28}\cos\frac{11\pi}{28}\\
&\quad= \frac{3}{2}+\frac{1}{2}\left(\cos\frac{5\pi}{14}+\cos\frac{11\pi}{14}+\cos\frac{13\pi}{14}\right)+\cos\frac{4\pi}{7}+\cos\frac{3\pi}{14}\\
&\qquad +\cos\frac{2\pi}{7}+\cos\frac{9\pi}{14}+\cos\frac{6\pi}{7}+\cos\frac{\pi}{14}\\
&\quad= \frac{3}{2}+\frac{1}{2}\left(\cos\frac{\pi}{14}+\cos\frac{3\pi}{14}-\cos\frac{5\pi}{14}\right)+\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}\\
&\quad= \frac{3}{2}+\frac{1}{2}\cdot\frac{\sqrt{7}}{2}-\frac{1}{2}=1+\frac{\sqrt{7}}{4}\\
&\quad= \frac{8+2\sqrt{7}}{8}=\left(\frac{1+\sqrt{7}}{2\sqrt{2}}\right)^2.
\end{aligned}$$

Because

$$\cos\frac{5\pi}{28}>0 \quad \cos\frac{13\pi}{28}>0, \quad \text{and} \quad \cos\frac{17\pi}{28}<0,$$

it follows that

$$\cos\frac{5\pi}{28}+\cos\frac{13\pi}{28}-\cos\frac{17\pi}{28}>0.$$

Therefore,

$$\cos\frac{5\pi}{28}+\cos\frac{13\pi}{28}-\cos\frac{17\pi}{28}=\frac{1+\sqrt{7}}{2\sqrt{2}}.$$

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

$$\cos\frac{5\pi}{28}+\cos\frac{13\pi}{28}-\cos\frac{17\pi}{28}=\cos\frac{5\pi}{28}+\sin\frac{\pi}{2}-\frac{13\pi}{28}+\cos\left(\pi-\frac{17\pi}{28}\right)$$

$$= \cos\left(\frac{3\pi}{7}-\frac{\pi}{4}\right)+\sin\left(\frac{2\pi}{7}-\frac{\pi}{4}\right)+\cos\frac{\pi}{7}+\frac{\pi}{4}=\cos\frac{3\pi}{7}\cos\frac{\pi}{4}+\sin\frac{3\pi}{7}\sin\frac{\pi}{4}$$

$$= \sin\frac{2\pi}{7}\cos\frac{\pi}{4}-\cos\frac{2\pi}{7}\sin\frac{\pi}{4}+\cos\frac{\pi}{7}\cos\frac{\pi}{4}-\sin\frac{\pi}{7}\sin\frac{\pi}{4}$$

$$= \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{7}-\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}-\sin\frac{\pi}{7}+\sin\frac{2\pi}{7}+\sin\frac{3\pi}{7}\right).$$

The complex roots of the polynomial z^7+1 are the seventh roots of -1, that is, $e^{i\frac{\pi+2k\pi}{7}}, k \in N, 0 \le k \le 6$ or equivalently,

$$\cos\frac{\pi}{7}+i\sin\frac{\pi}{7},\ \cos\frac{3\pi}{7}+i\sin\frac{3\pi}{7},\ -\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7},\ -1,$$
$$-\cos\frac{2\pi}{7}-i\sin\frac{2\pi}{7}, \cos\frac{3\pi}{7}-i\sin\frac{3\pi}{7},\ \text{and}\ \cos\frac{\pi}{7}-i\sin\frac{\pi}{7}.$$

By the first formula of Cardano-Viéte, the sum of these seven seven roots is equal to the opposite of the coefficient of z^6 of the polynomial z^7+1, which is 0, so in particular (in fact, equivalently) the real part of the sum is 0 , that is,

$$\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}-\cos\frac{2\pi}{7}-1-\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{\pi}{7}=0$$

or equivalently

$$\cos\frac{\pi}{7}-\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}=\frac{1}{2}.$$

Now, since

$$\left(-\sin\frac{\pi}{7}+\sin\frac{2\pi}{7}+\sin\frac{3\pi}{7}\right)^2=\sin^2\frac{\pi}{7}+\sin^2\frac{2\pi}{7}+\sin^2\frac{3\pi}{7}-2\sin\frac{\pi}{7}\sin\frac{2\pi}{7}+$$
$$+2\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}-2\sin\frac{3\pi}{7}\sin\frac{\pi}{7}=\frac{1}{2}\left(1-\cos\frac{2\pi}{7}\right)+\frac{1}{2}\left(1-\cos\frac{4\pi}{7}\right)+\frac{1}{2}\left(1-\cos\frac{6\pi}{7}\right)+$$
$$+\cos\left(\frac{\pi}{7}+\frac{2\pi}{7}\right)-\cos\left(\frac{\pi}{7}-\frac{2\pi}{7}\right)+\cos\left(\frac{2\pi}{7}-\frac{3\pi}{7}\right)-\cos\left(\frac{2\pi}{7}+\frac{3\pi}{7}\right)-\cos\left(\frac{3\pi}{7}-\frac{\pi}{7}\right)+\cos\left(\frac{3\pi}{7}+\frac{\pi}{7}\right)$$
$$=\frac{3}{2}+\frac{1}{2}\left(-\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{\pi}{7}\right)+\cos\frac{3\pi}{7}-\cos\frac{\pi}{7}+\cos\frac{\pi}{7}+\cos\frac{2\pi}{7}-\cos\frac{2\pi}{7}-\cos\frac{3\pi}{7}=\frac{3}{2}+\frac{1}{2}\frac{1}{2}=\frac{7}{4}.$$

Then

$$-\sin\frac{\pi}{7}+\sin\frac{2\pi}{7}+\sin\frac{3\pi}{7}=\frac{\sqrt{7}}{2}.$$

So the required value is

$$\cos\frac{5\pi}{28}+\cos\frac{13\pi}{28}-\cos\frac{17\pi}{28}=\frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{7}-\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}-\sin\frac{\pi}{7}+\sin\frac{2\pi}{7}+\sin\frac{3\pi}{7}\right)$$
$$=\frac{\sqrt{2}}{2}\left(\frac{1}{2}+\frac{\sqrt{7}}{2}\right)=\frac{1}{4}\left(\sqrt{2}+\sqrt{14}\right).$$

Equivalently, $\sqrt{\frac{1}{16}\left(\sqrt{2}+\sqrt{14}\right)^2}=\sqrt{\frac{1}{16}\left(16+2\sqrt{28}\right)}=\sqrt{\frac{1}{4}\left(4+\sqrt{7}\right)}=\frac{1}{2}\left(\sqrt{4+\sqrt{7}}\right).$

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that

$$\cos\frac{5\pi}{28}+\cos\frac{13\pi}{28}-\cos\frac{17\pi}{28}=\frac{\sqrt{2}+\sqrt{14}}{4}. \qquad (1)$$

Let $a=2\cos\frac{5\pi}{28}+2\cos\frac{11\pi}{28}+2\cos\frac{13\pi}{28}+\cos\frac{3\pi}{4}$, so that (1) will follow from

$$a=\frac{\sqrt{14}}{2} \qquad (2)$$

Since $a > \cos\frac{5\pi}{28} + \cos\frac{3\pi}{4} = 2\cos\frac{2\pi}{7}\cos\frac{13\pi}{28} > 0$, so (2) will follow from

$$a^2 = \frac{13}{2} - 6\left(\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7}\right) \tag{3}$$

and the well-known result that $\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$. To prove (3), we first note that a^2 is of the form $\sum c_i \cos\alpha_i \cos\beta_i$, where c_i are constants.

By using the formulas $2\cos x \cos y = \cos(x-y) + \cos(x+y)$ and $\cos(\pi \pm x) = -\cos x$, we then transform $\sum c_i \cos\alpha_i \cos\beta_i$ to the form $\sum k_j \cos\theta_j$, where k_j are constants and $\theta_j \in \left[\frac{\pi}{2}\right]$. In this way we arrive at (3), and this completes the solution.

Solution 4 by Seán M. Stewart, Bomaderry, NSW, Australia

The exact value of the expression

$$\cos\frac{5\pi}{28} + \cos\frac{13\pi}{28} - \cos\frac{17\pi}{28},$$

will be shown to be equal to $\dfrac{\sqrt{4+\sqrt{7}}}{2}$.

Our analysis will be greatly aided by the observation

$$\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}.$$

In proving this result, since $2\sin\frac{\pi}{7} \neq 0$, then

$$\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{2\sin\frac{\pi}{7}\cos\frac{\pi}{7} - 2\sin\frac{\pi}{7}\cos\frac{2\pi}{7} + 2\sin\frac{\pi}{7}\cos\frac{3\pi}{7}}{2\sin\frac{\pi}{7}}.$$

Making use of the product to sum identity $2\sin\theta\cos\varphi = \sin(\theta+\varphi)+\sin(\theta-\varphi)$ and the reduction formula $\sin(\pi-\theta) = \sin\theta$ allows us to rewrite the above result as

$$\begin{aligned}
\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} &= \frac{\sin\left(\frac{2\pi}{7}\right) - \sin\left(\frac{3\pi}{7}\right) - \sin\left(-\frac{\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(-\frac{2\pi}{7}\right)}{2\sin\frac{\pi}{7}} \\
&= \frac{\sin\frac{2\pi}{7} - \sin\frac{3\pi}{7} + \sin\frac{\pi}{7} + \sin\frac{3\pi}{7} - \sin\frac{2\pi}{7}}{2\sin\frac{\pi}{7}} \\
&= \frac{\sin\frac{\pi}{7}}{2\sin\frac{\pi}{7}} = \frac{1}{2},
\end{aligned}$$

as required.

To find the exact value of the desired expression, let

$$S = \cos\frac{5\pi}{28} + \cos\frac{13\pi}{28} - \cos\frac{17\pi}{28}.$$

In finding its value, the following results will be used when needed:

(i) the product to sum identity of $2\cos\theta\cos\varphi = \cos(\theta-\varphi)+\cos(\theta+\varphi)$,

(ii) the double angle formula of $2\cos^2\theta = 1+\cos 2\theta$,

(iii) the reduction formula of $\cos(\pi-\theta) = -\cos\theta$, and

(iv) the half period shift formula of $\cos(\theta+\pi) = -\cos\theta$.

Squaring S we have

$$\begin{aligned}
S^2 &= \left(\cos\frac{5\pi}{28}+\cos\frac{13\pi}{28}-\cos\frac{17\pi}{28}\right)^2 \\
&= \cos^2\frac{5\pi}{28}+\cos^2\frac{13\pi}{28}+\cos^2\frac{17\pi}{28} \\
&\quad + 2\cos\frac{5\pi}{28}\cos\frac{13\pi}{28} - 2\cos\frac{5\pi}{28}\cos\frac{17\pi}{28} - 2\cos\frac{13\pi}{28}\cos\frac{17\pi}{28} \\
&= \frac{1}{2}\left(1+\cos\frac{10\pi}{28}\right)+\frac{1}{2}\left(1+\cos\frac{26\pi}{28}\right)+\frac{1}{2}\left(1+\cos\frac{34\pi}{28}\right) \\
&\quad + \cos\frac{8\pi}{28}+\cos\frac{18\pi}{28}-\cos\frac{12\pi}{28}-\cos\frac{22\pi}{28}-\cos\frac{4\pi}{28}-\cos\frac{30\pi}{28} \\
&= \frac{3}{2}+\frac{1}{2}\cos\frac{5\pi}{14}+\frac{1}{2}\cos\frac{13\pi}{14}+\frac{1}{2}\cos\frac{17\pi}{14} \\
&\quad + \cos\frac{2\pi}{7}+\cos\frac{9\pi}{14}-\cos\frac{3\pi}{7}-\cos\frac{11\pi}{14}-\cos\frac{\pi}{7}-\cos\frac{15\pi}{14} \\
&= \frac{3}{2}+\frac{1}{2}\cos\frac{5\pi}{14}+\frac{1}{2}\cos\frac{13\pi}{14}-\frac{1}{2}\cos\frac{3\pi}{14} \\
&\quad + \cos\frac{2\pi}{7}+\cos\frac{9\pi}{14}-\cos\frac{3\pi}{7}-\cos\frac{11\pi}{14}-\cos\frac{\pi}{7}+\cos\frac{\pi}{14} \\
&= \frac{3}{2}-\left(\cos\frac{\pi}{7}-\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}\right)+\frac{1}{2}\left(\cos\frac{\pi}{14}+\cos\frac{3\pi}{14}-\cos\frac{5\pi}{14}\right).
\end{aligned}$$

But as $\cos\frac{\pi}{7}-\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}=\frac{1}{2}$, we have

$$S^2 = 1+\frac{1}{2}\left(\cos\frac{\pi}{14}+\cos\frac{3\pi}{14}-\cos\frac{5\pi}{14}\right),$$

or

$$2\left(S^2-1\right) = \cos\frac{\pi}{14}+\cos\frac{3\pi}{14}-\cos\frac{5\pi}{14}.$$

Squaring again gives

$$\begin{aligned}
4\left(S^2-1\right)^2 &= \left(\cos\frac{\pi}{14}+\cos\frac{3\pi}{14}-\cos\frac{5\pi}{14}\right)^2\\
&= \cos^2\frac{\pi}{14}+\cos^2\frac{3\pi}{14}+\cos^2\frac{5\pi}{14}\\
&\quad +2\cos\frac{3\pi}{14}\cos\frac{\pi}{14}-2\cos\frac{5\pi}{14}\cos\frac{\pi}{14}-2\cos\frac{5\pi}{14}\cos\frac{3\pi}{14}\\
&= \frac{1}{2}\left(1+\cos\frac{2\pi}{14}\right)+\frac{1}{2}\left(1+\cos\frac{6\pi}{14}\right)+\frac{1}{2}\left(1+\cos\frac{10\pi}{14}\right)\\
&\quad +\cos\frac{4\pi}{14}+\cos\frac{2\pi}{14}-\cos\frac{6\pi}{14}-\cos\frac{4\pi}{14}-\cos\frac{8\pi}{14}-\cos\frac{2\pi}{14}\\
&= \frac{3}{2}+\frac{1}{2}\cos\frac{\pi}{7}+\frac{1}{2}\cos\frac{3\pi}{7}+\frac{1}{2}\cos\frac{5\pi}{7}\\
&\quad +\cos\frac{2\pi}{7}+\cos\frac{\pi}{7}-\cos\frac{3\pi}{7}-\cos\frac{2\pi}{7}-\cos\frac{4\pi}{7}-\cos\frac{\pi}{7}\\
&= \frac{3}{2}+\frac{1}{2}\cos\frac{\pi}{7}+\frac{1}{2}\cos\frac{3\pi}{7}-\frac{1}{2}\cos\frac{2\pi}{7}\\
&\quad +\cos\frac{2\pi}{7}+\cos\frac{\pi}{7}-\cos\frac{3\pi}{7}-\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}-\cos\frac{\pi}{7}\\
&= \frac{3}{2}+\frac{1}{2}\left(\cos\frac{\pi}{7}-\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}\right)\\
&= \frac{3}{2}+\frac{1}{2}\cdot\frac{1}{2}=\frac{7}{4}.
\end{aligned}$$

We are therefore left with the biquadratic equation of

$$16S^4-32S^2+9=0.$$

Solving gives

$$S=\pm\frac{\sqrt{4\pm\sqrt{7}}}{2}.$$

In selecting the correct root to the biquadratic equation, noting that

$$\cos\frac{5\pi}{28}+\cos\frac{13\pi}{28}-\cos\frac{17\pi}{28}=\cos\frac{5\pi}{28}+\cos\frac{11\pi}{28}+\cos\frac{13\pi}{28}>0,$$

as $\cos x$ is a monotonically decreasing function on the interval $\left(0,\frac{\pi}{2}\right)$, we see that

$$\cos\frac{5\pi}{28}>\cos\frac{\pi}{3}=\frac{1}{2},$$

and

$$-\cos\frac{17\pi}{28}=\cos\frac{11\pi}{28}>\cos\frac{2\pi}{5}=\frac{\sqrt{5}-1}{4}.$$

Thus

$$\cos\frac{5\pi}{28}-\cos\frac{17\pi}{28}>\frac{\sqrt{5}+1}{4}>\frac{\sqrt{4-\sqrt{7}}}{2}.$$

One therefore has

$$\cos\frac{5\pi}{28}+\cos\frac{13\pi}{28}-\cos\frac{17\pi}{28}=\frac{\sqrt{4+\sqrt{7}}}{2},$$

as announced.

Remark

It can in fact be shown that the second positive root of $\dfrac{\sqrt{4-\sqrt{7}}}{2}$ to the biquadratic equation for S corresponds to the value of

$$\cos\frac{\pi}{28}-\cos\frac{3\pi}{28}+\cos\frac{9\pi}{28}.$$

Solution 5 by Julio Cesar Mohnsam, POMAT-IFSul Campus Pelotas-RS, Brazil

Note that: $\cos(\pi-x)=\cos x$, like this:

$$\begin{aligned}\cos\frac{5\pi}{28} &= -\cos\frac{23\pi}{28}=-\cos\left(\frac{\pi}{4}+\frac{4\pi}{7}\right)\\ -\cos\frac{17\pi}{28} &= \cos\frac{11\pi}{28}=\cos\left(\frac{\pi}{4}+\frac{4\pi}{7}\right)\end{aligned}$$

We can also write:

$$\cos\frac{13\pi}{28}=\cos\left(-\frac{13\pi}{28}\right)=\cos\left(\frac{\pi}{4}-\frac{5\pi}{7}\right)$$

like this:

$$\cos\frac{5\pi}{28}+\cos\frac{13\pi}{28}-\cos\frac{137\pi}{28}=-\cos\left(\frac{\pi}{4}+\frac{4\pi}{7}\right)+\cos\left(\frac{\pi}{4}-\frac{5\pi}{7}\right)+\cos\left(\frac{\pi}{4}+\frac{4\pi}{7}\right)$$

$$= \quad -\cos\frac{\pi}{4}\cos\frac{4\pi}{7}+\sin\frac{\pi}{4}\sin\frac{4\pi}{7}+\cos\frac{\pi}{4}\cos\frac{5\pi}{7}+\sin\frac{\pi}{4}\sin\frac{5\pi}{7}+\cos\frac{\pi}{4}\cos\frac{\pi}{7}-\sin\frac{\pi}{4}\sin\frac{\pi}{7}$$

$$\cos\frac{\pi}{4}\left(\cos\frac{\pi}{7}-\cos\frac{4\pi}{7}+\cos\frac{5\pi}{7}\right)+\sin\frac{\pi}{4}\left(-\sin\frac{\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{5\pi}{7}\right).$$

Now note that: $\sin(\pi-x)=\sin x$, $\cos(\pi-x)=-\cos x$ and $\sin(-x)=-\sin x$, we have $-\cos\frac{4x}{7}-=\cos\frac{3\pi}{7}$, $-\sin\frac{\pi}{7}=-\sin\frac{-\pi}{7}=-\sin\frac{8\pi}{7}$ and $\sin\frac{5\pi}{7}=\sin\frac{2\pi}{7}$.
Rewriting we have;

$$\cos\frac{5\pi}{28}+\cos\frac{13\pi}{28}-\cos\frac{17\pi}{28}=\cos\frac{\pi}{4}\underbrace{\left(\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}\right)}_{1/2}+\sin\frac{\pi}{4}\underbrace{\left(\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}\right)}_{\sqrt{7}/2}$$

$$=\frac{\sqrt{2}}{2}\frac{1}{2}+\frac{\sqrt{2}}{2}\frac{\sqrt{7}}{2}=\frac{\sqrt{2}}{4}\left(1+\sqrt{7}\right)$$

Solution 6 by Brian D. Beasley, Presbyterian College, Clinton, SC

We show that the given expression equals $(\sqrt{2}+\sqrt{14})/4$.

Let $a=\cos(5\pi/28)$, $b=\cos(13\pi/28)$, and $c=\cos(17\pi/28)$. Using the triple-angle formula for cosine, we have $b=-\cos(15\pi/28)=-4a^3+3a$ and $c=-\cos(45\pi/28)=4b^3-3b$. Then

$$a+b-c=a(256a^8-576a^6+432a^4-124a^2+13). \qquad (*)$$

Using the multiple-angle formula for $\cos(14\theta)$ with $\theta=5\pi/28$, we obtain $(2a^2-1)f(a)=0$, where

$$f(x)=4096x^{12}-12288x^{10}+13568x^8-6656x^6+1376x^4-96x^2+1.$$

Since $a^2\neq 1/2$, we conclude that $f(a)=0$.

Next, we let $g(x)=16x^4-32x^2+9$ and show that $g(a+b-c)=0$. Using $(*)$ and a considerable amount of algebra, we note that $g(a+b-c)=f(a)h(a)=0$, where $h(x)=\sum_{k=0}^{12}a_{2k}x^{2k}$ with integer coefficients a_{2k} (see the Addendum for the values of these coefficients).

Finally, we observe that $a+b-c>a>\cos(\pi/4)=\sqrt{2}/2$. Since the only zero of $g(x)$ which is greater than $\sqrt{2}/2$ is $(\sqrt{2}+\sqrt{14})/4$, this completes the proof.

Addendum. We calculate the following values for the coefficients of $h(x)$:

$a_{24}=16,777,216$; $a_{22}=-100,663,296$; $a_{20}=265,289,728$; $a_{18}=-404,750,336$;

$a_{16}=397,148,160$; $a_{14}=-263,651,328$; $a_{12}=121,376,768$; $a_{10}=-39,010,304$;

$a_8=8,643,328$; $a_6=-1,266,944$; $a_4=111,536$; $a_2=-4544$; $a_0=9$

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Michel Bataille, Rouen, France; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Daniel Văcaru, Pitesti, Romania, and the proposer.

- **5562:** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Mehedinti, Romania*

Prove: If $a,b,c\geq 1$, then

$$e^{ab}+e^{bc}+e^{ca}>3+\frac{c}{a}+\frac{b}{c}+\frac{a}{b}.$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, NY.

The well-known inequality $e^x>1+x$ for $x\geq 1$ yields

$$e^{ab}+e^{bc}+e^{ca}>(1+ab)+(1+bc)+(1+ca)=3+ab+bc+ca.$$

We also note that since $a,b,c\geq 1$, we have $a\geq 1/c,\ b\geq 1/a$, and $c\geq 1/b$, so that $ab\geq b/c,\ bc\geq c/a$, and $ca\geq a/b$. Thus

$$e^{ab}\ +\ e^{bc}\ +\ e^{ca}\ >\ 3\ +\ ab+bc+ca\ \geq\ 3\ +\ \frac{c}{a}\ +\ \frac{b}{c}\ +\ \frac{a}{b}.$$

Solution 2 by Ed Gray, Highland Beach, FL

1. $e^{ab}>1+ab$
2. $e^{bc}>1+bc$
3. $e^{ca}>1+ca$
4. $e^{ab}+e^{bc}+e^{ca}>3+ab+bc+ca$

5. Claim that $a, b, c \geq 1$ implies that $ab > \frac{a}{b}$ or $ab^2 > a$ because $b \geq 1$; same holds for the others which proves the conjecture.

Solution 3 by Albert Natian, Los Angeles Valley College, Valley Glen, CA

$$e^{ab} + e^{bc} + e^{ca} = \left[1 + ab + \sum_{k=2}^{\infty} \frac{(ab)^k}{k!}\right] + \left[1 + bc + \sum_{k=2}^{\infty} \frac{(bc)^k}{k!}\right] + \left[1 + ca + \sum_{k=2}^{\infty} \frac{(ca)^k}{k!}\right]$$

$$> [1 + ab] + [1 + bc] + [1 + ca]$$

$$= 3 + \left[ab\left(1 - \frac{1}{b^2}\right) + \frac{a}{b}\right] + \left[bc\left(1 - \frac{1}{c^2}\right) + \frac{b}{c}\right] + \left[ca\left(1 - \frac{1}{c^2}\right) + \frac{c}{a}\right]$$

$$\geq 3 + \frac{c}{a} + \frac{b}{c} + \frac{a}{b}.$$

Note: It's interesting that the inequality stands even when $a, b, c \leq -1$.

Solution 4 by Moti Levy, Rehovot, Israel

Since e^x is convex function, then (Jensen's inequality)

$$\frac{e^{ab} + e^{bc} + e^{ca}}{3} \geq e^{\frac{ab+bc+ca}{3}}. \qquad (1)$$

Since $e^x \geq 1 + x$ for $x \geq 0$, then

$$3e^{\frac{ab+bc+ca}{3}} \geq 3\left(1 + \frac{ab + bc + ca}{3}\right) = 3 + ab + bc + ca \qquad (2)$$

Since $a, b, c \geq 1$

$$a^2b^2c + ab^2c^2 + a^2bc^2 \geq a^2c + ab^2 + bc^2$$

or

$$abc\,(ab + bc + ca) \geq a^2c + ab^2 + bc^2.$$

Dividing both sides of the inequality by abc results in

$$ab + bc + ca \geq \frac{c}{a} + \frac{b}{c} + \frac{a}{b}. \qquad (3)$$

The desired inequality follows from (1), (2) and (3).

Solution 5 by Daniel Văcaru, Pitesti, Romania

We know that $e^x \geq x + 1, \forall x \geq 0$, or to be accurate$e^x > x + 1, \forall x \geq 1$. It follows $e^{ab} > 1 + ab \geq 1 + \frac{a}{b} \rightarrow e^{ab} > 1 + \frac{a}{b}$ (1). In the same manner we have $e^{bc} > 1 + \frac{b}{c}$ (2) and $e^{ca} > 1 + \frac{c}{a}$ (3). Summing, we obtain

$$e^{ab} + e^{bc} + e^{ca} > 3 + \frac{c}{a} + \frac{b}{c} + \frac{a}{b},$$

as desired.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Brian Bradie, Christopher Newport News, VA; Michel Bataille, Rouen, France; Michael Brozinsky, Central Islip, NY; Tran Hong, Ben Trey University, Ben Tre, Vietnam; Sanong Huayrerai, Nathom Pathom College, Thailand; Kee-Wai Lau, Hong Kong, China; Ravi Prakash, New Delhi University, New Delhi,India; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Seán M. Stewart, Bomaderry, NSW, Australia; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; and the proposer.

- **5563:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Without the aid of a computer, find the value of

$$\sum_{n=1}^{+\infty} \frac{15}{25n^2+45n-36}.$$

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

Since

$$25n^2+45n-36 = (5n-3)(5n+12),$$

a partial fraction expansion yields

$$\frac{15}{25n^2+45n-36} = \frac{1}{5n-3} - \frac{1}{5n+12}$$

for all $n \geq 1$. Then, for $m \geq 4$, let $i = n-3$ in one of the following sums to obtain

$$\begin{aligned}
\sum_{n=1}^{m} \frac{15}{25n^2+45n-36} &= \sum_{n=1}^{m} \frac{1}{5n-3} - \sum_{n=1}^{m} \frac{1}{5n+12} \\
&= \frac{1}{2} + \frac{1}{7} + \frac{1}{12} + \sum_{n=4}^{m} \frac{1}{5n-3} - \sum_{n=1}^{m} \frac{1}{5n+12} \\
&= \frac{61}{84} + \sum_{i=1}^{m-3} \frac{1}{5i+12} - \sum_{n=1}^{m} \frac{1}{5n+12} \\
&= \frac{61}{84} + \sum_{n=1}^{m-3} \frac{1}{5n+12} - \sum_{n=1}^{m} \frac{1}{5n+12} \\
&= \frac{61}{84} - \frac{1}{5(m-2)+12} - \frac{1}{5(m-1)+12} - \frac{1}{5m+12} \\
&= \frac{61}{84} - \frac{1}{5m+2} - \frac{1}{5m+7} - \frac{1}{5m+12}.
\end{aligned}$$

As a result,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{15}{25n^2+45n-36} &= \lim_{m\to\infty} \sum_{n=1}^{m} \frac{15}{25n^2+45n-36} \\
&= \lim_{m\to\infty} \left[\frac{61}{84} - \frac{1}{5m+2} - \frac{1}{5m+7} - \frac{1}{5m+12}\right] \\
&= \frac{61}{84}.
\end{aligned}$$

Solution 2 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Since $\dfrac{15}{25n^2+45n-36}=\dfrac{1}{5n-3}-\dfrac{1}{5n+12}$, then

$$\begin{aligned}
\sum_{n=1}^{k}\frac{15}{25n^2+45n-36} &= \sum_{n=1}^{k}\left(\frac{1}{5k-3}+\frac{1}{5k+12}\right)\\
&= \frac{1}{2}-\frac{1}{17}+\frac{1}{7}-\frac{1}{22}+\\
&\quad \frac{1}{12}-\frac{1}{27}+\frac{1}{17}-\frac{1}{32}+\\
&\quad \frac{1}{22}-\frac{1}{37}+\frac{1}{27}-\frac{1}{42}+\\
&\quad \frac{1}{32}-\frac{1}{47}+\frac{1}{37}-\frac{1}{52}+\\
&\quad \cdots\\
&\quad \frac{1}{5k-18}-\frac{1}{5k-3}+\frac{1}{5k-13}-\frac{1}{5n+2}+\\
&\quad \frac{1}{5k-8}-\frac{1}{5k+7}+\frac{1}{5k-3}-\frac{1}{5k+12}\\
&= \frac{1}{2}+\frac{1}{7}+\frac{1}{12}-\frac{1}{5n+2}-\frac{1}{5k+7}-\frac{1}{5k+12}\\
&= \frac{61}{84}-\frac{1}{5n+2}-\frac{1}{5k+7}-\frac{1}{5k+12}.
\end{aligned}$$

Therefore, the proposed sum is

$$\sum_{n=1}^{\infty}\frac{15}{25n^2+45n-36}=\lim_{k\to\infty}\left(\frac{61}{84}-\frac{1}{5n+2}-\frac{1}{5k+7}-\frac{1}{5k+12}\right)=\frac{61}{84}.$$

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Michel Bataille, Rouen, France; Brian D. Beasley, Presbyterian College, Clinton, SC; Narendra Bhandari (two solutions), Bajura National College, Nepal, India; Brian Bradie, Christopher Newport New University, Newport News,VA; Bruno Salgueiro Fanego, Viveiro, Spain; Michael C. Faleski, University Center, MI; Michael N. Fried, Ben-Gurion University of the Negev, Beer-Sheva, Israel; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Alexis Llanos, Catolica Colegio, Lima, Peru; David E. Manes, Oneonta, NY; Henry Ricardo, Westchester Area Math Circle NY; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Seán M. Stewart, Bomaderry, NSW, Australia; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Daniel Văcaru, Pitesti, Romania, and the proposer.

- **5564:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a>0$ and let $f:[0,a]\to\Re$ be a Riemann integrable function. Calculate

$$\lim_{n\to\infty}\int_0^a \frac{f(x)}{1+nx^n}\mathrm{d}x.$$

Solution 1 by Brian Bradie, Christopher Newport New University, Newport News,VA

For every positive integer n, let the function $g_n : [0, \infty) \to R$ be defined by

$$g_n(x) = \frac{1}{1 + nx^n}.$$

Then

$$\lim_{n\to\infty} g_n(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases}.$$

On every interval of the form $[0, r]$ with $0 < r < 1$, convergence is uniform. Convergence is also uniform on $[1, \infty)$. We now consider three cases.

– $0 < a < 1$: On the interval $[0, a]$, $g_n \to 1$ uniformly, so

$$\lim_{n\to\infty} \int_0^a \frac{f(x)}{1 + nx^n}\, dx = \int_0^a f(x) \lim_{n\to\infty} g_n(x)\, dx = \int_0^a f(x)\, dx.$$

– $a = 1$: Write

$$\int_0^1 \frac{f(x)}{1 + nx^n}\, dx = \int_0^r \frac{f(x)}{1 + nx^n}\, dx + \int_r^1 \frac{f(x)}{1 + nx^n}\, dx$$

for some $0 < r < 1$. By the previous case,

$$\lim_{n\to\infty} \int_0^r \frac{f(x)}{1 + nx^n}\, dx = \int_0^r f(x)\, dx.$$

For the integral over the interval $[r, 1]$, note $|g_n(x)| \le 1$ for all n and all $x \ge 0$. Moreover, because f is Riemann integrable, it is bounded over the closed interval $[r, 1]$. Let $M = \sup |f(x)|$ over $[r, 1]$. Then, for all n,

$$\left| \int_r^1 \frac{f(x)}{1 + nx^n}\, dx \right| \le (1 - r)M.$$

It then follows that

$$\lim_{r\to 1^-} \int_r^1 \frac{f(x)}{1 + nx^n}\, dx = 0,$$

and

$$\begin{aligned} \lim_{n\to\infty} \int_0^1 \frac{f(x)}{1 + nx^n}\, dx &= \lim_{r\to 1^-} \left(\int_0^r f(x)\, dx + \int_r^1 \frac{f(x)}{1 + nx^n}\, dx \right) \\ &= \int_0^1 f(x)\, dx. \end{aligned}$$

– $a > 1$: Write

$$\int_0^a \frac{f(x)}{1 + nx^n}\, dx = \int_0^1 \frac{f(x)}{1 + nx^n}\, dx + \int_1^a \frac{f(x)}{1 + nx^n}\, dx.$$

By the previous case,

$$\lim_{n\to\infty} \int_0^1 \frac{f(x)}{1 + nx^n}\, dx = \int_0^1 f(x)\, dx.$$

On the interval $[1,a]$, $g_n \to 0$ uniformly, so

$$\lim_{n\to\infty}\int_1^a \frac{f(x)}{1+nx^n}\,dx = \int_1^a f(x)\lim_{n\to\infty} g_n(x)\,dx = 0.$$

Therefore,

$$\lim_{n\to\infty}\int_0^a \frac{f(x)}{1+nx^n}\,dx = \int_0^1 f(x)\,dx.$$

Combining the results from the three cases, we see that

$$\lim_{n\to\infty}\int_0^a \frac{f(x)}{1+nx^n}\,dx = \int_0^{\min(a,1)} f(x)\,dx.$$

Solution 2 by Michel Bataille, Rouen, France

Let $g_n(x) = \frac{f(x)}{1+nx^n}$. The function f, being Riemann integrable, is bounded. We call M a positive real number such that $|f(x)| \le M$ for all $x \in [0,a]$. Note that $|g_n(x)| \le M$ as well (since $1+nx^n \ge 1$).
First, we consider the case $a<1$. When $x \in [0,a]$, we have $\lim\limits_{n\to\infty} nx^n = 0$ (since $0 \le x < 1$) and therefore $\lim\limits_{n\to\infty} g_n(x) = f(x)$. In addition, $|g_n(x)| \le M$ and the constant function $x \mapsto M$ is integrable on $[0,a]$. From Lebesgue's dominated convergence theorem, we deduce

$$\lim_{n\to\infty}\int_0^a \frac{f(x)}{1+nx^n}\,dx = \int_0^a (\lim_{n\to\infty} g_n(x))\,dx = \int_0^a f(x)\,dx.$$

These equalities still hold if $a=1$ since then $\lim\limits_{n\to\infty} g_n(x) = f(x)$ except for $x=1$, that is, almost everywhere on $[0,1]$.
Now suppose that $a>1$. For $1<x\le a$, we have $\lim\limits_{n\to\infty} nx^n = \infty$ and so $\lim\limits_{n\to\infty} g_n(x) = 0$. As above, we obtain $\lim\limits_{n\to\infty}\int_a^1 g_n(x) = 0$ and so

$$\lim_{n\to\infty}\int_0^a \frac{f(x)}{1+nx^n}\,dx = \lim_{n\to\infty}\int_0^1 g_n(x)\,dx + \lim_{n\to\infty}\int_a^1 g_n(x)\,dx = \int_0^1 f(x)\,dx + 0 = \int_0^1 f(x)\,dx.$$

In conclusion,

$$\lim_{n\to\infty}\int_0^a \frac{f(x)}{1+nx^n}\,dx = \int_0^{\min(a,1)} f(x)\,dx.$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that

$$\lim_{n\to\infty}\int_0^a \frac{f(x)}{1+nx^n}dx = \begin{cases} \displaystyle\int_0^a f(x)dx, & 0<a<1 \\ \displaystyle\int_0^1 f(x)dx, & a \ge 1. \end{cases}$$

Since $\lim\limits_{n\to\infty} na^n = 0$ for $0<a<1$ and

$$0 \le \left|\int_0^a \frac{f(x)}{1+nx^n}dx - \int_0^a f(x)dx\right| = n\left|\int_0^a \frac{f(x)x^n}{1+nx^n}dx\right| \le n\int_0^a \frac{|f(x)|\,x^n}{1+nx^n}dx \le na^n\int_0^a |f(x)|\,dx,$$

so $\lim\limits_{n\to\infty}\int_0^a \frac{f(x)}{1+nx^n}dx = \int_0^a f(x)dx$ in this case.

Next , since $\lim_{n\to\infty} n\left(1-\frac{1}{\sqrt{n}}\right)^n = 0, \quad \lim_{n\to\infty}\int_{1-\frac{1}{\sqrt{n}}}^1 |f(x)|\,dx = 0$ and

$$\begin{aligned}
0 \le \left|\int_0^1 \frac{f(x)}{1+nx^n}dx - \int_0^1 f(x)dx\right| &\le n\int_0^1 \frac{|f(x)|x^n}{1+nx^n}dx \\
&\le n\left(1-\frac{1}{\sqrt{n}}\right)^n \int_0^{1-\frac{1}{\sqrt{n}}} |f(x)|\,dx + \int_{1-\frac{1}{\sqrt{n}}}^1 |f(x)|\,dx,
\end{aligned}$$

so $\lim_{n\to\infty}\int_0^1 \frac{f(x)}{1+nx^n}dx = \int_0^1 f(x)dx.$

Finally, for $a>1$, we have $0 \le \left|\int_1^a \frac{f(x)}{1+nx^n}dx\right| \le \frac{1}{n}\int_1^a |f(x)|\,dx$, which tends to zero as n tends to infinity. Hence,

$$\lim_{n\to\infty}\int_0^a \frac{f(x)}{1+nx^n}dx = \lim_{n\to\infty}\int_0^a \frac{f(x)}{1+nx^n}dx + \lim_{n\to\infty}\int_1^a \frac{f(x)}{1+nx^n}dx = \int_0^1 f(x)dx.$$

This completes the proof.

Solution 4 by Albert Natian, Los Angeles Valley College, Valley Glen, CA

First suppose $0 \le a < 1$. Then $\left|\frac{nx^n}{1+nx^n}\right|^2 \le n^2a^{2n} \quad \forall x \in [0,a]$. By Schwarz Inequality

$$\begin{aligned}
0 \le \left|\int_0^a f(x)\ dx - \int_0^a \frac{f(x)}{1+nx^n}\ dx\right|^2 &= \left|\int_0^a f(x)\cdot\frac{nx^n}{1+nx^n}\ dx\right|^2 \\
&\le \int_0^a |f(x)|^2\ dx \cdot \int_0^a \left|\frac{nx^n}{1+nx^n}\right|^2\ dx \\
&\le \int_0^a |f(x)|^2\ dx \cdot \int_0^a n^2a^{2n}\ dx \\
&\le n^2a^{2n+1}\int_0^a |f(x)|^2\ dx.
\end{aligned}$$

Since $\lim_{n\to\infty} n^2a^{2n+1} = 0$, then (by Squeeze Theorem) $\lim_{n\to\infty}\left|\int_0^a f(x)\ dx - \int_0^a \frac{f(x)}{1+nx^n}\ dx\right|^2 = 0$, and so $\lim_{n\to\infty}\int_0^a \frac{f(x)}{1+nx^n}\ dx = \int_0^a f(x)\ dx$.

Now $\forall n \in N,\ \forall a \in (0,1)$: we have $\left|\int_a^1 \frac{f(x)}{1+nx^n}\ dx\right| \le \int_a^1 |f(x)|\ dx$, and so

$$0 \le \left|\int_0^1 \frac{f(x)}{1+nx^n}\ dx - \int_0^1 f(x)\ dx\right| \le \left|\int_0^a \frac{f(x)}{1+nx^n}\ dx - \int_0^a f(x)\ dx\right| + 2\int_a^1 |f(x)|\ dx.$$

So

$$0 \le \lim_{n\to\infty}\left|\int_0^1 \frac{f(x)}{1+nx^n}\ dx - \int_0^1 f(x)\ dx\right| \le 2\int_a^1 |f(x)|\ dx$$

which implies $\lim_{n\to\infty}\int_0^1 \frac{f(x)}{1+nx^n}\ dx = \int_0^1 f(x)\ dx$ since $\int_a^1 |f(x)|\ dx$ can get arbitrarily close to 0 by sending a arbitrarily close to 1.

Now suppose $a > 1$. Since for all x in $[1, a]$: $0 < \frac{1}{1+nx^n} \le \frac{1}{1+n}$, then

$$0 \le \left| \int_1^a \frac{f(x)}{1+nx^n}\ dx \right| \le \int_1^a \left| \frac{f(x)}{1+nx^n} \right|\ dx \le \frac{1}{1+n} \int_1^a |f(x)|\ dx$$

wich implies $\lim_{n\to\infty}\int_1^a \frac{f(x)}{1+nx^n}\ dx = 0$, which in turn implies

$$\lim_{n\to\infty}\int_0^a \frac{f(x)}{1+nx^n}\ dx = \lim_{n\to\infty}\int_0^1 \frac{f(x)}{1+nx^n}\ dx + \lim_{n\to\infty}\int_1^a \frac{f(x)}{1+nx^n}\ dx = \int_0^1 f(x)\ dx.$$

In final conclusion, for $a > 0$:

$$\lim_{n\to\infty}\int_0^a \frac{f(x)}{1+nx^n}\ dx = \int_0^{\min\{1,a\}} f(x)\ dx.$$

Solution 5 by Moti Levy, Rehovot, Israel

Define the sequence of functions

$$f_n(x) := \frac{|f(x)|}{1+nx^n}.$$

For every $1 > \varepsilon > 0$, there exists a number N such that for $n > N$, $\varepsilon > \frac{1}{n+1}$. Hence, for $n > N$ we have

$$x \le \frac{n}{n+1} \quad \mathit{for\ all} \quad x \in [0, 1-\varepsilon),$$

which implies

$$\frac{1}{1+nx^n} \le \frac{1}{1+(n+1)x^{n+1}}.$$

Thus, for $n > N$,

$$\frac{|f(x)|}{1+nx^n} \le \frac{|f(x)|}{1+(n+1)x^{n+1}} \le |f(x)|, \ \mathit{for\ all} \quad x \in [0, 1-\varepsilon).$$

We have shown that the sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ is pointwise non-decreasing sequence of non-negative functions $f_n : [0, 1-\varepsilon] \to [0, +\infty]$, i.e., for every $n \ge N$ and every $x \in [0, 1-\varepsilon)$,

$$0 \le f_n(x) \le f_{n+1}(x) \le \infty.$$

The pointwise limit of the sequence $\{f_n(x)\}$ is $|f(x)|$ for all $x \in [0, 1-\varepsilon)$.
If the function $f(x)$ is Riemann integrable then it is Lebesgue integrable.
If a function is Riemann integrable on a bounded interval then it is Lebesgue measurable, so we can apply the monotone convergence theorem for the Lebesgue integral.
By the monotone convergence theorem

$$\lim_{n\to\infty}\int_{[0,1-\varepsilon)} \frac{|f(x)|}{1+nx^n} dx = \int_{[0,1-\varepsilon)} |f(x)|\, dx. \tag{1}$$

For $x \ge 1$ and for all $n \ge 1$,

$$\frac{|f(x)|}{1+nx^n} \ge \frac{|f(x)|}{1+(n+1)x^{n+1}} \ge 0, \ \mathit{for\ all} \quad x \in [1, +\infty]$$

We have shown that the sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ is pointwise non-increasing sequence of non-negative functions $f_n : [1, +\infty] \to [0, +\infty]$, i.e., for every $n \geq 1$ and every $x \in [1, +\infty]$,

$$\infty \geq f_n(x) \geq f_{n+1}(x) \geq 0.$$

The pointwise limit of the sequence $\{f_n(x)\}$ is 0 for all $x \in [1, +\infty]$.
By the monotone convergence theorem

$$\lim_{n\to\infty} \int_{[1,+A]} \frac{|f(x)|}{1+nx^n} dx = \int_{[1,+A]} 0 dx = 0 \text{ for any finite positive number } A > a. \tag{2}$$

We split the the interval $[0, a]$ into two intervals I_+ and I_- such that $I_+ \cup I_- = [0, a]$ and

$$\begin{aligned} I_+ &: = \{x \in [0,a] | \quad f(x) \geq 0\}, \\ I_- &= \{x \in [0,a] | \quad f(x) < 0\}. \end{aligned}$$

Clearly

$$\begin{aligned} & \int_0^a \frac{f(x)}{1+nx^n} dx \\ = & \int_{I_+ \cap [0,1-\varepsilon)} \frac{|f(x)|}{1+nx^n} dx - \int_{I_- \cap [0,1-\varepsilon)} \frac{|f(x)|}{1+nx^n} dx \\ & + \int_{I_+ \cap [1,A)} \frac{|f(x)|}{1+nx^n} dx - \int_{I_- \cap [1,A)} \frac{|f(x)|}{1+nx^n} dx \\ & + \int_{I_+ \cap [1-\varepsilon,1)} \frac{|f(x)|}{1+nx^n} dx - \int_{I_- \cap [1-\varepsilon,1)} \frac{|f(x)|}{1+nx^n} dx. \end{aligned} \tag{3}$$

By (1) and and (2) it follows from (3) that

$$\begin{aligned} \lim_{n\to\infty} \int_0^a \frac{f(x)}{1+nx^n} dx &= \int_{I_+ \cap [0,1-\varepsilon)} |f(x)| \, dx - \int_{I_- \cap [0,1-\varepsilon)} |f(x)| \, dx \\ & \quad + \int_{I_+ \cap [1-\varepsilon,1)} \frac{|f(x)|}{1+nx^n} dx - \int_{I_- \cap [1-\varepsilon,1)} \frac{|f(x)|}{1+nx^n} dx \\ &= \int_{[0,a] \cap [0,1-\varepsilon)} f(x) \, dx + \int_{[0,a] \cap [1-\varepsilon,1)} f(x) \, dx. \end{aligned}$$

$$\int_{[0,a] \cap [1-\varepsilon,1)} f(x) \, dx \to 0 \quad as \quad \varepsilon \to 0.$$

Since we may take ε to be arbitrarily small, we conclude that

$$\lim_{n\to\infty} \int_0^a \frac{f(x)}{1+nx^n} dx = \int_{[0,a] \cap [0,1)} f(x) \, dx.$$

Solution 6 by Ulrich Abel, Technische Hochschule Mittelhessen, Germany

We show that

$$\int_0^a \frac{f(x)}{1+nx^n} dx = \int_0^{\min\{a,1\}} f(x) \, dx. \tag{*}$$

Case 1: If $0 < a < 1$, we have

$$\int_0^a \frac{f(x)}{1+nx^n} dx = \int_0^a f(x) \, dx + R,$$

where

$$|R| = \left| \int_0^a \frac{-nx^n}{1+nx^n} f(x)\, dx \right| \le na^n \int_0^a |f(x)|\, dx \to 0 \qquad (n \to \infty).$$

Case 2: If $a = 1$, we have, for small $\varepsilon > 0$,

$$\int_0^1 \frac{f(x)}{1+nx^n} dx = \int_0^{1-\varepsilon} \frac{f(x)}{1+nx^n} dx + \int_{1-\varepsilon}^1 f(x)\, dx + \int_{1-\varepsilon}^1 \frac{-nx^n}{1+nx^n} f(x)\, dx,$$

where, by Case 1, the first integral tends to $\int_0^{1-\varepsilon} f(x)\, dx$ and the modulus of the last integral can be estimated by $\varepsilon \int_0^1 |f(x)|\, dx$.

Case 3: If $a > 1$, we have, for small $\varepsilon > 0$,

$$\int_0^a \frac{f(x)}{1+nx^n} dx = \int_0^1 \frac{f(x)}{1+nx^n} dx + \int_1^{1+\varepsilon} \frac{f(x)}{1+nx^n} dx + \int_{1+\varepsilon}^a \frac{f(x)}{1+nx^n} dx.$$

By Case 2, the first integral tends to $\int_0^1 f(x)\, dx$. The modulus of the second integral can be estimated by $\varepsilon \int_0^a |f(x)|\, dx$. Furthermore,

$$\left| \int_{1+\varepsilon}^a \frac{f(x)}{1+nx^n} dx \right| \le \frac{1}{n(1+\varepsilon)^n} \int_0^a |f(x)|\, dx \to 0 \qquad (n \to \infty).$$

This completes the proof of Eq. $(*)$.

Solution 7 by Albert Stadler, Herrliberg, Switzerland

We claim that $\lim\limits_{n\to\infty} \int_0^b \frac{f(x)}{1+nx^n} dx = \int_0^{\min(1,a)} f(x)dx.$

By definition, any Riemann integrable functions is bounded. Therefore there is a positive constant M such that $|f(x)| \le M$ for all x.

Suppose first that $a \le 1$. Then

$$\begin{aligned}\left| \int_0^a \frac{f(x)}{1+nx^n} dx - \int_0^{\min(1,a)} f(x)dx \right| &= \left| \int_0^a \frac{nx^n}{1+nx^n} f(x)dx \right| \le M \int_0^1 \frac{nx^n}{1+nx^n} dx \le \\ &\le \int_0^{1-\frac{1}{\sqrt{n}}} nx^n dx + M \int_{1-\frac{1}{\sqrt{n}}}^1 dx = M\frac{n}{n+1}\left(1-\frac{1}{\sqrt{n}}\right)^{n+1} + \frac{M}{\sqrt{n}} \to 0,\end{aligned}$$

as $n \to \infty$.

Suppose next that $a > 1$. Then

$$\begin{aligned}\left| \int_0^a \frac{fx}{1+nx^n} dx - \int_0^{\min(1,a)} f(x)dx \right| &\le \left| \int_0^1 \frac{f(x)}{1+nx^n} dx - \int_0^1 f(x)dx \right| + \left| \int_0^a \frac{f(x)}{1+nx^n} dx \right| \le \\ &\le o(1) + \frac{M(a-1)}{1+n} \to 0,\end{aligned}$$

as $n \to \infty$.

Also solved by the proposers.