Problems Ted Eisenberg, Section Editor

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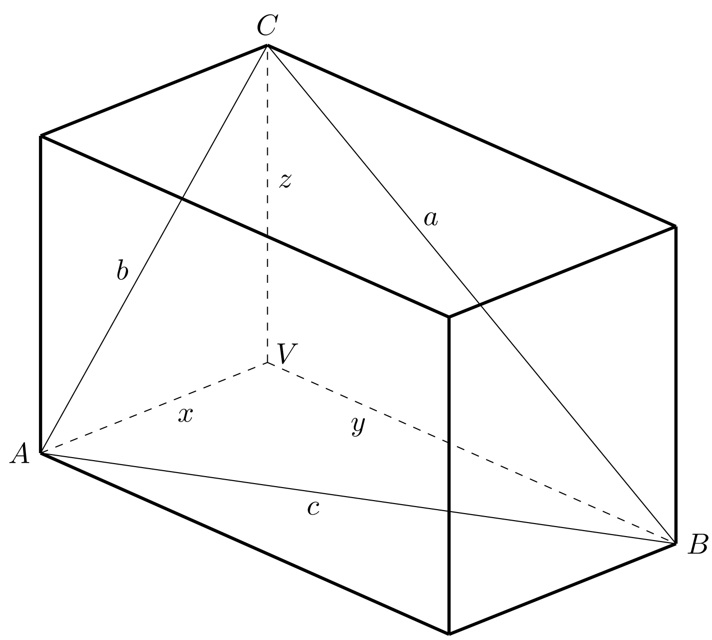
This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before July 15, 2020

- **5589:** *Proposed by Kenneth Korbin, New York, NY*

  Find the dimensions of a triangle with integer length sides if it can be inscribed in a circle with diameter $16\sqrt{7}$.

- **5590:** *Proposed Albert Natian, Los Angeles Valley College, Valley Glen, CA*



  Let V be a vertex of a rectangular box. Let $\overline{VA}$, $\overline{VB}$ and $\overline{VC}$ be the three edges meeting at vertex V. We are given that the perimeter of the triangle $\triangle ABC$ is $(28+\sqrt{106})$, the total surface area of the box is 426, and the length of the main diagonal of the box is $5\sqrt{10}$. Find the area of the triangle $\triangle ABC$.

- **5591:** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Mehedinti, Romania*

  Solve for real numbers:

$$3^{\cos x+\cos y+\cos z} = 3^{\cos^2 x+\cos x} + 3^{\cos^2 y+\cos y} + 3^{\cos^2 z+\cos z}$$

- **5592:** *Proposed by Michel Bataille, Rouen, France*

  Let n be a positive integer. Evaluate $\sum\limits_{k=1}^{n} a_k$ where

$$a_k = \left(\prod_{i=1}^{k}(2i-1)\right)\cdot\left(\prod_{j=k+1}^{n}(k+j)\right).$$

  [The second factor is 1 if $k = n$.]

- **5593:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let A be the set of quadruples of positive integers (i, j, k, l) such that $i + j + k + l = 23$. Compute the following sum

$$\sum_{(i,j,k,l)\in A} ijkl.$$

- **5594:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $k > 1$. Calculate:

  •[(a)] $L = \lim\limits_{n\to\infty}\int_0^1\left(\dfrac{k}{\sqrt[n]{x}+k-1}\right)^n \mathrm{d}x$

  •[(b)] $\lim\limits_{n\to\infty} n\left[L-\int_0^1\left(\dfrac{k}{\sqrt[n]{x}+k-1}\right)^n \mathrm{d}x\right].$

Solutions

- **5571:** *Proposed by Kenneth Korbin, New York, NY*

  Solve the equation: $\sqrt[3]{x^2+x} = \sqrt[3]{x} + \sqrt[3]{x^2-x}$, with $x > 0$.

  Solution 1 by Stanley Rabinowitz, Brooklyn, NY

Clearly, $x = 0$ is a solution, so we may assume that $x \neq 0$. Dividing the given equation $\sqrt[3]{x^2+x} = \sqrt[3]{x} + \sqrt[3]{x^2-x}$, by $\sqrt[3]{x}$ gives us $\sqrt[3]{x+1} = 1 + \sqrt[3]{x-1} + \sqrt[3]{x^2-x}$. Cubing and then adding $-x-1$ to each side gives $0 = 3\left(\sqrt[3]{(x-1)}\right)^2 + 3\sqrt[3]{x-1} - 1$, a quadratic in $\sqrt[3]{x-1}$. Thus, $\sqrt[3]{x-1} = \dfrac{-3 \pm 21}{6}$. Cubing and then adding 1 to each side implies that $x = \pm\dfrac{2}{9}\sqrt{21}$. Since taking odd powers of each side of an equality cannot introduce new real solutions, the real solutions to the given equation are $0, \pm\dfrac{2}{9}\sqrt{21}$. The only positive solution is $\dfrac{2}{9}\sqrt{21}$.

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

To begin, we note that by inspection, $x = 0$ is a solution. If $x \neq 0$, we may divide by $\sqrt[3]{x}$ to obtain

$$\sqrt[3]{x+1} = 1 + \sqrt[3]{x-1}.$$

Then,

$$\begin{aligned} x+1 &= \left(1 + \sqrt[3]{x-1}\right)^3 \\ &= x - 1 + 3\left(\sqrt[3]{x-1}\right)^2 + 3\sqrt[3]{x-1} + 1 \end{aligned}$$

and hence,

$$3\left(\sqrt[3]{x-1}\right)^2 + 3\sqrt[3]{x-1} - 1 = 0.$$

The quadratic formula implies that

$$\sqrt[3]{x-1} = \frac{-3 \pm \sqrt{21}}{6}.$$

If $\sqrt[3]{x-1} = \dfrac{\sqrt{21}-3}{6}$, then

$$\begin{aligned} x-1 &= \frac{\left(\sqrt{21}-3\right)^3}{216} \\ &= \frac{21\sqrt{21} - 9\,(21) + 27\sqrt{21} - 27}{216} \\ &= \frac{48\sqrt{21} - 216}{216} \\ &= \frac{2\sqrt{21}}{9} - 1 \end{aligned}$$

and we have $x = \dfrac{2\sqrt{21}}{9}$.

On the other hand, if $\sqrt[3]{x-1} = \dfrac{-\sqrt{21}-3}{6} = -\dfrac{\sqrt{21}+3}{6}$, then

$$\begin{aligned} x - 1 &= -\left(\frac{\sqrt{21}+3}{6}\right)^3 \\ &= -\frac{21\sqrt{21} + 9\,(21) + 27\sqrt{21} + 27}{216} \\ &= -\frac{48\sqrt{21} + 216}{216} \\ &= -\frac{2\sqrt{21}}{9} - 1 \end{aligned}$$

and it follows that $x = -\dfrac{2\sqrt{21}}{9}$.

Therefore, the solutions to the equation are $x = 0, \pm\dfrac{2\sqrt{21}}{9}$ and the only positive solution is $x = \dfrac{2\sqrt{21}}{9}$.

Solution 3 by David E. Manes, Oneonta, NY

If $x > 0$, then $\sqrt[3]{x^2+x} = \sqrt[3]{x} + \sqrt[3]{x^2-x}$ when $x = \sqrt{\dfrac{28}{27}} = \dfrac{2}{9}\sqrt{21}$.

Consider the equivalent equation: $\sqrt[3]{x} = \sqrt[3]{x^2+x} - \sqrt[3]{x^2-x}$. Cubing both sides of the equation, one obtains

$$x = (x^2+x) - (x^2-x) - 3(x^2+x)^{2/3}(x^2-x)^{1/3} + 3(x^2+x)^{1/3}(x^2-x)^{2/3}.$$

Therefore,

$$x = 2x - 3(x^2+x)^{1/3}(x^2-x)^{1/3}\left[(x^2+x)^{1/3} - (x^2-x)^{1/3}\right].$$

Since $(x^2+x)^{1/3} - (x^2-x)^{1/3} = x^{1/3}$ it follows that

$$\begin{aligned} x &= 3(x^2+x)^{1/3}(x^2-x)^{1/3}x^{1/3} = \left[(x^2+x)(x^2-x)x\right]^{1/3} \\ &= 3\left(x^3(x^2-1)\right)^{1/3}. \end{aligned}$$

Cubing both sides of this equation yields $x^3 = 27x^3(x^2-1)$, whence $1 = 27(x^2-1)$ since $x \neq 0$. Therefore, $27x^2 = 28$ so that $x > 0$ implies $x = \sqrt{\dfrac{28}{27}}$.

Also solved by Arkady Alt, San Jose, CA; Michel Bataille, Rouen, France; Brian D. Beasley, Presbyterian College, Clinton, SC; Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND; Brian Bradie, Christopher Newport University, Newport, News, VA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Ronald Alexandre Martins, Brasília, Brazil; Julio Cesar Mohnsam, IFSul Campus, Pelotas-RS, Brazil, Bruno Oliveira de Almeida, Corsan Rio Grande-RS, Brazil and Luciana de Almedia Mohnsam, IFRS Campus Rio Grande-RS, Brazil; Bin Pan, San Mateo, CA; Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Henry

Ricardo, Westchester Area Math Circle, NY; Albert Stadler, Herrliberg, Switzerland; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Seán M. Stewart, Bomaderry, NSW, Australia; David Stone and John Hawkins, Southern Georgia University, Statesboro GA; Daniel Văcaru, Pitesti, Romania; Titu Zvonaru, Comănesti, Romania, and the proposer.

- **5572:** *Proposed by Titu Zvonaru, Comănesti, Romania*

Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$\frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b} + \frac{64}{a+b+c} \geqslant 34.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We assume (less stringently) that $a, b, c \geqslant 0,\ a+b > 0,\ b+c > 0,\ c+a > 0$, and claim that the minimum is assumed if and only if one variable equals 0 and the other two equal 1.

Put $p = a + b + c$. Then

$$\begin{aligned} a^4b^4 + c^4 &= p^4 - 4p^2(ab+bc+ca) + 2(ab+bc+ca)^2 + 4abcp \\ &= p^4 - 4p^2 + 2 + 4abcp \end{aligned}$$

and

$$\begin{aligned} &\frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b} + \frac{64}{a+b+c} - 34 \\ = \ &\frac{a^4}{ab+ca} + \frac{b^4}{bc+ab} + \frac{c^4}{ca+bc} + \frac{64}{a+b+c} - 34 \\ = \ &\frac{a^4}{1-bc} + \frac{b^4}{1-ca} + \frac{c^4}{1-ab} + \frac{64}{a+b+c} - 34 \\ \geqslant \ &a^4 + b^4 + c^4 + \frac{64}{a+b+c} - 34 \\ \geqslant \ &p^4 - 4p^2 = +2 + \frac{64}{p} - 34 = \frac{(p-2)^2(p^3+4p^2+8p+16)}{p} \geqslant 0. \end{aligned}$$

Equality holds if and only if $a+b+c = 2$ and $ab+bc+ca = 1$ and $abc = 0$, which means that one variable equal 0 and the other two equal 1.

Solution 2 by Moti Levy, Rehovot, Israel

Let

$$\begin{aligned} p &: = a+b+c, \\ q &: = ab+bc+ca, \\ r &: = abc, \end{aligned}$$

and

$$f(a,b,c) := \frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b} + \frac{4^3}{a+b+c}.$$

$$\begin{aligned} & f(a,b,c) \\ = \; & \frac{a^3(c+a)(a+b)+b^3(b+c)(a+b)+c^3(b+c)(c+a)}{(b+c)(c+a)(a+b)}+\frac{4^3}{a+b+c} \\ = \; & \frac{\left(a^5+b^5+c^5\right)+\left(a^3+b^3+c^3\right)(ab+bc+ca)}{(b+c)(c+a)(a+b)}+\frac{4^3}{a+b+c}. \end{aligned}$$

The following identities are known (and useful)

$$\begin{aligned} a^5+b^5+c^5 &= p^5-5p^3q+5rp^2+5pq^2-5rq, \\ a^3+b^3+c^3 &= p^3-3pq+3r, \\ (b+c)(c+a)(a+b) &= pq-r. \end{aligned}$$

So now we can express $f(a,b,c)$ in p,q,r notation:

$$\begin{aligned} f(a,b,c) &= \frac{p^5-5p^3q+5rp^2+5pq^2-5rq+q\left(p^3-3pq+3r\right)}{pq-r} \\ &= \frac{p^5-4p^3q+5rp^2+2pq^2-2rq}{pq-r}+\frac{64}{p}. \end{aligned}$$

Imposing $q=1$, we get

$$f(a,b,c)=\frac{p^5-4p^3+5rp^2+2p-2r}{p-r}+\frac{64}{p}.$$

We observe that the inequality has its equality case when at least one of the variables is on the boundary of their range, i.e., when one of the variables is equal to zero.
Actually the equality holds for the triplet $(a,b,c)=(1,1,0)$ and its cyclic permutations.
When $r=0$, $f(a,b,c)$ becomes $\dfrac{p^5-4p^3+2p}{p}+\dfrac{64}{p}$.

$$\begin{aligned} \frac{p^5-4p^3+2p}{p}+\frac{64}{p}-34 &= \frac{p^5-4p^3-32p+64}{p} \\ &= \frac{\left(16+8p+4p^2+p^3+16\right)(p-2)^2}{p} \geqslant 0 \end{aligned}$$

Hence

$$\frac{p^5-4p^3+2p}{p}+\frac{64}{p}\geqslant 34. \tag{1}$$

We will show that

$$f(a,b,c)\geqslant \frac{p^5-4p^3+2p}{p}+\frac{64}{p}. \tag{2}$$

$$\begin{aligned} & f(a,b,c)-\left(\frac{p^5-4p^3+2p}{p}+\frac{64}{p}\right) \\ = \; & \frac{p^5-4p^3+5rp^2+2p-2r}{p-r}+\frac{64}{p}-\left(\frac{p^5-4p^3+2p}{p}+\frac{64}{p}\right) \\ = \; & p^2\left(p^2+1\right)\frac{r}{p-r}. \end{aligned}$$

So what is left in order to establish (2) is to show that $p > r$.

$$(a+b+c)^2 = p^2 = a^2 + b^2 + c^2 + 2q$$

Since $a^2 + b^2 + c^2 \geqslant ab + bc + ca$ then

$$p^2 \geqslant 3q$$

Imposing $q = 1$,

$$p^2 \geqslant 3. \tag{3}$$

By AM-GM inequality,

$$\frac{q}{3} \geqslant (abc)^{\frac{2}{3}}$$

Hence

$$r \leqslant \left(\frac{q}{3}\right)^{\frac{3}{2}} = \left(\frac{1}{3}\right)^{\frac{3}{2}} = \frac{1}{\sqrt{27}}. \tag{4}$$

Thus, (3) and (4) imply that $p > r$, so that (2) is proved.
Inequalities (1) and (2) imply the inequality of the problem statement.

Solution 3 by Kee-Wai Lau, Hong Kong, China

We suppose that the inequality to be proved is

$$\frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b} + \frac{64}{a+b+c} \geqslant 34. \tag{1}$$

Since $ab + bc + ca = 1$, so

$$\frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b} = \frac{(a+b)(a+c)a^3 + (b+a)(b+c)b^3 + (c+a)(c+b)c^3}{(a+b)(b+c)(c+a)}$$

$$= \frac{(a^3+b^3+c^3) + (a^5+b^5+c^5)}{a+b+c-abc}.$$

Let $p = a+b+c,\ q = ab+bc+ca$ and $r = abc$. It can be checked readily that

$$a^3 + b^3 + c^3 = p^3 - 3pq + 3r = p^3 - 3p + 3r \text{ and}$$

$$a^5 + b^5 + c^5 = p^5 - 5p^3q + 5p^2r + 5pq^2 - 5qr = p^5 - 5p^3 + 5p^2r + 5p - 5r.$$

It follows that the left side (1) equals

$$\frac{p^5 - 4p^3 + 5p^2r + 2p - 2r}{p-r} + \frac{64}{p} = \frac{p^6 - 4p^4 + 5p^3r + 2p^2 - 2pr + 64p - 64r}{p(p-r)}$$

and that (1) is equivalent to

$$p^6 - 4p^4 + 5p^3r - 32p^2 + 32pr + 64p - 64r \geqslant 0 \tag{2}$$

Now the left side of (2) equals $p(p-2)^2(p^3+4p^2+8p+16)+r(5p^3+32p-64)$.

Since $p=\sqrt{3q+\dfrac{(a-b)^2+(b-c)^2+(c-a)^2}{2}}\geqslant\sqrt{3}$, so $5p^3+32p-64\geqslant 0$,

and (2) in fact holds. This completes the solution.

Solution 4 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

$Editor's\ note$: I am taking the liberty of summarizing the author's solution because his solution path involved using partial derivatives that resulted in complicated systems of algebraic equations. These complicated systems were solved with the use of mathematica. The method the author used to solve this problem is valid, but it is also long and involved. Hence, summary.

The conditions in the statement often problem, that a, b and c are positive real numbers with $ab+bc+ca=1$, allowed the author to write c as $c=\dfrac{1-ab}{a+b}$, and to then consider the statement of the problem as a function in two variables x and y with $x=a$ and $y=b$. This gives:

$$f(x,y)=\frac{x^3(x+y)}{y^2+1}+\frac{y^3(x+y)}{x^2+1}+\frac{64(x+y)}{x^2+y^2+xy+1}+\frac{(1-xy)^3}{(x+y)^4}.$$

The problem then becomes one of finding the stationary points of $f(x,y)$, and for this he computed $f_x=\dfrac{\partial f}{\partial x}$ and $f_y=\dfrac{\partial f}{\partial y}$ and set each equal to zero This gives a system of two equations in two unknowns, $\begin{cases} f_x=0 \\ f_y=0 \end{cases}$.

Suppose that (a,b) satisfies the system of equations. (As the author noted there is often more than one solution to the system). The point $f(a,b)$ is now a candidate for being a stationary point; that is, for being a local max or a local min or a saddle point on the graph of $z=f(x,y)$. The usual classification criteria were used to classify the point (a,b).

- If $f_{xx}f_{yy}-f_{xy}^2<0$ at (a,b), then (a,b) is a saddle point.

- If $f_{xx}f_{yy}-f_{xy}^2>0$, at (a,b) then (a,b) is either a local maximum value or a local minimum value.

  We distinguish between them as follows:
- If $f_{xx}<0$ and $f_{yy}<0$ at (a,b), then (a,b) is a local maximum point.
- If $f_{xx}>0$ and $f_{yy}>0$ at (a,b), then (a,b) is a local minimum point.
- But if $f_{xx}f_{yy}=0$, at (a,b), then more advanced methods must be used to classify the solution point to the system of equations.

  With the help of mathematica it was shown that the equation $f(a,b)=34$ is its minimum.

Also solved by the proposer.

- **5573:** *Proposed by D.M.Bătinetu-Giurgiu, National College "Matei Basarab," Bucharest, Anastasios Kotronis, Athens, Greece, and Neulai Stanciu, "George Emil Palade" School,* *Buzău,*

Let $f : \Re_+ \to \Re_+$ be a continuous function such that $\lim_{x\to\infty} \frac{f(x)}{x^2} = a \in \Re_+$. ($\Re_+$ stands for the positive real numbers.) Calculate:

$$\lim_{x\to\infty} \left(\sqrt[n+1]{\prod_{k=1}^{n+1} \frac{f(k)}{k}} - \sqrt[n]{\prod_{k=1}^{n} \frac{f(k)}{k}} \right).$$

Solution 1 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

Let $a_n = \sqrt[n]{\prod_{k=1}^{n} \frac{fk}{k}}$, and let $x_n = \left(\frac{a_n}{n}\right)$. Then, by Stolz-Cesaro:

$$\begin{aligned}
\lim_{n\to+\infty} \ln\left(\frac{a_n}{n}\right) &= \lim_{n\to+\infty} \ln\left(\frac{n\ln a_n - n\ln n}{n}\right) = \lim_{n\to+\infty} \frac{\sum_{k=1}^{n} \ln\frac{f(k)}{k} - n\ln n}{n} \\
&= \lim_{n\to+\infty} \left[\ln\frac{f(n+1)}{(n+1)} - (n+1)\ln(n+1) + n\ln n\right] \\
&= \lim_{n\to+\infty} \left[\ln\frac{f(n+1)}{(n+1)^2} - n\ln\left(1+\frac{1}{n}\right)\right] = \ln\left(\frac{a}{e}\right).
\end{aligned}$$

Hence, $\frac{a_n}{n} = \sqrt[n]{x_n}\frac{a}{e}$, which implies that $\frac{x_{n+1}}{x_n} \to \frac{a}{e}$. Moreover,

$$\frac{a_{n+1}}{a_n} \to 1, \text{ and } \left(\frac{a_{n+1}}{a_n}\right)^n = \frac{(n+1)^n x_{n+1}^{n/n+1}}{n x_n} = \left(1+\frac{1}{n}\right)^n \cdot \frac{x_{n+1}}{x_n} \cdot \frac{1}{\sqrt[n+1]{x_{n+1}}} \longrightarrow e.$$

Therefore,

$$\begin{aligned}
\sqrt[n+1]{\prod_{k=1}^{n+1)} \frac{f(k)}{k}} - \sqrt[n]{\prod_{k=1}^{n} \frac{f(k)}{k}} &= a_{n+1} - a_n \\
&= \frac{a_n}{n} \cdot \frac{\left(\frac{a_{n+1}}{a_n} - 1\right)}{\ln\left[1 + \left(\frac{a_{n+1}}{a_n} - 1\right)\right]} \ln\left[\left(\frac{a_{n+1}}{a_n}\right)^n\right] \longrightarrow \frac{a}{e}.
\end{aligned}$$

Solution 2 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

By the Stolz-Cezaro lemma, the proposed limit, say L, is

$$L = \lim_{n\to\infty} \frac{\sqrt[n+1]{\prod_{k=1}^{n+1} \frac{f(k)}{k}}}{n+1} = \lim_{n\to\infty} \sqrt[n+1]{\prod_{k=1}^{n+1} \frac{f(k)}{k(n+1)}}.$$

Therefore, by the root-quotient criterion,

$$L = \lim_{n\to\infty} \frac{\displaystyle\prod_{k=1}^{n+1} \frac{f(k)}{k(n+1)}}{\displaystyle\prod_{k=1}^{n} \frac{f(k)}{kn}} = \lim_{n\to\infty} \frac{n^n f(n+1)}{(n+1)^{n+1}(n+1)}.$$

Since $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e}$, and $\lim_{n\to\infty} \frac{f(n+1)}{(n+1)^2} = a$, then $L = \frac{a}{e}$.

Solution 3 by Bin Pan, San Mateo, CA

First for any $\epsilon > 0$ there exists an N such that for all $k > N$

$$1 - \epsilon < \frac{f(k)}{ak^2} < 1 + \epsilon.$$

$$\begin{aligned} L &= \lim_{n\to+\infty} \left(\sqrt[n+1]{\prod_{k=1}^{n+1} \frac{f(k)}{k}} - \sqrt[n]{\prod_{k=1}^{n} \frac{f(k)}{k}} \right) \\ &= \lim_{n\to+\infty} \left(a\,\sqrt[n+1]{(n+1!)}\,\sqrt[n+1]{\prod_{k=1}^{n+1} \frac{f(k)}{k^2}} - a\,\sqrt[n]{n!}\,\sqrt[n]{\prod_{k=1}^{n} \frac{f(k)}{ak^2}} \right). \end{aligned}$$

From a known result (e.g., see "Problems in Real Analysis: Advanced Calculus on the Real Axis" by Radulescu T.L., et. al., P1.2.3)

$$\lim_{n\to+\infty} \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \right) = \frac{1}{e}.$$

It is easy to see,

$$a\left(\sqrt[n+1]{M(1-\epsilon)^{a+1-N}}\,\sqrt[n+1]{(n+1)!} - \sqrt[n]{M(1-\epsilon)^{n-N}}\,\sqrt[n]{n!} \right) \leqslant L$$

where

$$M = \prod_{k=1}^{N} \frac{f(k)}{ak^2}.$$

Letting $n \to +\infty$ on both sides, we have $L = \frac{a}{e}$.

Solution 4 by Arkady Alt, San Jose, CA

Let $a_n = \sqrt[n]{\prod_{k=1}^{n} \frac{f(k)}{k}}, n \in \mathbb{N}$ and $b_n := \frac{a_n^n}{n^n} = \frac{1}{n^n n!} \prod_{k=1}^{n} f(k)$.

First we will find $\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \sqrt[n]{b_n}$. Since $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \left(\frac{n^n}{(n+1)^{n+1}} \cdot \frac{f(n+1)}{n+1} \right) =$

$\lim_{n\to\infty}\left(\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{f(n+1)}{(n+1)^2}\right)=\frac{a}{e}$ then by Geometric Mean Limit Theorem (Multiplicative Cesaro's Theorem) $\lim_{n\to\infty}\frac{a_n}{n}=\lim_{n\to\infty}\sqrt[n]{b_n}=\frac{a}{e}$. Coming back to the limit of the problem we obtain $L:=\lim_{n\to\infty}\left(\sqrt[n+1]{\prod_{k=1}^{n+1}\frac{f(k)}{k}}-\sqrt[n]{\prod_{k=1}^{n}\frac{f(k)}{k}}\right)=\lim_{n\to\infty}(a_{n+1}-a_n)=\lim_{n\to\infty}\frac{a_n}{n}\cdot n\left(\frac{a_{n+1}}{a_n}-1\right)=$
$\frac{a}{e}\lim_{n\to\infty}n\left(\frac{a_{n+1}}{a_n}-1\right)$. Since $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{a_{n+1}}{n+1}}{\frac{a_n}{n}}\cdot\frac{n+1}{n}=1$ implies $\lim_{n\to\infty}\ln\left(\frac{a_{n+1}}{a_n}\right)=0$
then $\lim_{n\to\infty}n\left(\frac{a_{n+1}}{a_n}-1\right)=\lim_{n\to\infty}\left(\frac{e^{\ln\frac{a_{n+1}}{a_n}}-1}{\ln\frac{a_{n+1}}{a_n}}\cdot n\ln\frac{a_{n+1}}{a_n}\right)=\lim_{n\to\infty}\left(n\ln\frac{a_{n+1}}{a_n}\right)=\ln\left(\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^n\right)=$
$\ln\left(\lim_{n\to\infty}\left(\frac{a_{n+1}^{n+1}}{a_n^n}\cdot\frac{n+1}{a_{n+1}}\cdot\frac{1}{n+1}\right)\right)=\ln\left(\lim_{n\to\infty}\left(\frac{f(n+1)}{n+1}\cdot\frac{n+1}{a_{n+1}}\cdot\frac{1}{n+1}\right)\right)=$
$\ln\left(\lim_{n\to\infty}\left(\frac{f(n+1)}{(n+1)^2}\cdot\frac{n+1}{a_{n+1}}\right)\right)=\ln\left(\lim_{n\to\infty}\frac{f(n+1)}{(n+1)^2}\cdot\lim_{n\to\infty}\frac{n+1}{a_{n+1}}\right)=\ln\left(a\cdot\frac{e}{a}\right)=1$
and, therefore, $L=\frac{a}{e}$

Also solved by Michel Bataille, Rouen, France; Brian Bradie, Christopher Newport University, Newport News, VA; Bruno Salgueiro Fanego, Viveiro, Spain; Moti Levi, Rehovot, Israel; Kee-Wai Lau, Hong Kong, China; Daniel Văcaru, Pitesti, Romania; Albert Stadler, Herrliberg, Switzerland; Romania;, and the proposers.

- **5574:** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Mehedinti, Romania*

Prove: If $0<a\leqslant b\leqslant c$ then:

$$\frac{1}{1+e^{a-b+c}}+\frac{1}{1+e^b}\leqslant\frac{1}{1+e^a}+\frac{1}{1+e^c}.$$

Solution 1 by Titu Zvonaru, Comănesti, Romania

Clearing denominators, the given inequality is equivalent to:

$$(2+e^{a-b+c}+e^b)(1+e^a)(1+e^c)\geqslant(2+e^a+e^c)(1+e^{a-b+c})(1+e^b)$$

$$2+e^{a-b+c}+e^b+2e^a+e^{2a-b+c}+e^{a+b}+2e^c+e^{a-b+2c}+e^{b+c}+2e^{a+c}+e^{2a-b+2c}+e^{a+b+c}$$

$$\leqslant 2+e^a+e^c+2e^{a-b+c}+e^{2a-b+c}+e^{a-b+2c}+2e^b+e^{a+b}+e^{b+c}+2e^{a+c}+e^{2a+c}+e^{a+2c}$$

$$e^a+e^c+e^{2a-b+2c}+e^{a+b+c}\leqslant e^{a-b+c}+e^b+e^{2a+c}+e^{a+2c}$$

$$(e^b-e^a)(e^{a-b}-1)(e^{a+c}-1)\geqslant 0.$$

The equality holds if and only if $a=b$ or $b=c$

Solution 2 by Brian Bradie, Christopher Newport University, Newport News, VA

If $a = c$, then $a = b = c = a - b + c$ and

$$\frac{1}{1+e^{a-b+c}} + \frac{1}{1+e^b} = \frac{1}{1+e^a} + \frac{1}{1+e^c} = \frac{2}{1+e^a}.$$

Now, suppose $0 < a < c$, and let

$$f(x) = \frac{1}{1+e^x}.$$

Then

$$f''(x) = \frac{e^{2x} - e^x}{(1+e^x)^3} > 0$$

for $x > 0$. Thus, f is convex for $x > 0$. Because

$$0 \leqslant \frac{c-b}{c-a} \leqslant 1 \quad \text{and} \quad a \cdot \frac{c-b}{c-a} + c\left(1 - \frac{c-b}{c-a}\right) = b,$$

it follows that

$$f(b) \leqslant \frac{c-b}{c-a} f(a) + \left(1 - \frac{c-b}{c-a}\right) f(c). \tag{6}$$

Moreover, because

$$0 \leqslant \frac{b-a}{c-a} \leqslant 1 \quad \text{and} \quad a \cdot \frac{b-a}{c-a} + c\left(1 - \frac{b-a}{c-a}\right) = a - b + c,$$

it follows that

$$f(a-b+c) \leqslant \frac{b-a}{c-a} f(a) + \left(1 - \frac{b-a}{c-a}\right) f(c). \tag{7}$$

Adding (6) and (7) yields

$$f(a-b+c) + f(b) \leqslant f(a) + f(c),$$

or

$$\frac{1}{1+e^{a-b+c}} + \frac{1}{1+e^b} \leqslant \frac{1}{1+e^a} + \frac{1}{1+e^c}.$$

Solution 3 by Albert Stadler, Herrliberg, Switzerland

Put $x = e^a, y = e^b, z = e^c$ Then $1 < x \leqslant y \leqslant z$.

$$\frac{1}{1+e^a} + \frac{1}{1+e^c} - \frac{1}{1+e^{a-b+c}} - \frac{1}{1+e^b}$$

$$= \frac{1}{1+x} + \frac{1}{1+z} - \frac{1}{1+\frac{xz}{y}} - \frac{1}{1+y}$$

$$= \frac{(y-x)(z-y)(xz-1)}{(1+x)(1+z)(y+xz)} \geqslant 0.$$

This proof shows that we may even allow $a = 0$.

Solution 4 by Kee-Wai Lau, Hong Kong, China

Let $x = e^a, y = e^b$ and $z = e^c$, so that $1 < x \leqslant y \leqslant z$ and the inequality of the problem is equivalent to

$$\frac{1}{1+x}+\frac{1}{1+z}-\frac{y}{y+xz}-\frac{1}{1+y}\geqslant 0 \tag{1}$$

It is easy to check that the left side of (1) equals $\dfrac{(y-x)(z-y)(xz-1)}{(1+x)(1+y)(1+z)(y+xz)}$

so that (1) and the inequality of the problem indeed hold.

Solution 4 by Daniel Văcaru, Pitesti, Romania

We prove the following fact:
If $f: I\subset\Re\to\Re$ is convex for all $a,b\in I$ with $0<a<b<c$, then

$$f(a-b+c)\leqslant f(a)-f(b)+f(c). \qquad (*)$$

We have $\lambda\in(0,1), \lambda=\dfrac{c-b}{c-a}$ such that $b=\lambda a+(1-\lambda)c$. But f is convex, so

$$f(b)\leqslant \lambda f(a)+(1-\lambda)f(c). \tag{1}$$

We have $a-b+c=(1-\lambda)a+\lambda c$. So again by convexity,

$$f(a-b+c)\leqslant(1-\lambda)f(a)+\lambda f(c). \tag{2}$$

Summing (1) and (2) we obtain $f(a-b+c)+f(b)\leqslant f(a)+f(c)$, proving $(*)$.

Now consider $f:\Re\to\Re,\ f(x)=\dfrac{1}{e^x}=e^{-x}$. We have $f''(x)=e^{-x}$, proving the convexity of f. So using $(*)$, we obtain

$$\frac{1}{1+e^{a-b+c}}+\frac{1}{1+e^{b}}\leqslant\frac{1}{1+e^{a}}+\frac{1}{1+e^{c}},$$

as desired.

Also solved by Arkady Alt, San Jose, CA; Michel Bataille, Rouen, France; Moti Levy, Rehovot, Israel; Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece, and the proposer.

- **5575:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Compute:

$$\int_1^\infty \frac{dt}{[t]^3+9[t]^2+26[t]+24},$$

where $[x]$ represents the integer part of x.

Solution 1 by Henry Ricardo, Westchester Area Math Circle, NY

Let $f(t) = 1/([t]^3 + 9[t]^2 + 26[t] + 24)$. If $n \leqslant t < n+1$, then $[t] = n$ and

$$\begin{aligned}\int_n^{n+1} f(t)\ dt &= \int_n^{n+1} \frac{1}{n^3 + 9n^2 + 26n + 24}\ dt \\ &= \frac{1}{(n+2)(n+3)(n+4)} \int_n^{n+1} 1\ dt \\ &= \frac{1}{(n+2)(n+3)(n+4)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_1^{\infty} \frac{dt}{[t]^3 + 9[t]^2 + 26[t] + 24} &= \sum_{k=1}^{n-1} \int_k^{k+1} f(t)\ dt = \sum_{k=1}^{n-1} \frac{1}{(k+2)(k+3)(k+4)} \\ &= \frac{1}{2} \sum_{k=1}^{n-1} \left(\frac{1}{(k+2)(k+3)} - \frac{1}{(k+3)(k+4)} \right) \\ &= \frac{1}{2} \left(\frac{1}{12} - \frac{1}{(n+2)(n+3)} \right).\end{aligned}$$

Thus

$$\int_1^{\infty} \frac{dt}{[t]^3 + 9[t]^2 + 26[t] + 24} = \lim_{n\to\infty} \int_1^n f(t)\ dt = \lim_{n\to\infty} \left(\frac{1}{24} - \frac{1}{2(n+2)(n+3)} \right) = \frac{1}{24}.$$

Solution 2 by Michael C. Faleski, Delta College, University Center, MI

As an integral returns the area under the curve, we note as $n \leqslant x < n+1$ with integer n, $[x] = n$, the graph of the integrand is a series of steps with unit width. That is, the area under the curve can be recast as

$$\int_1^{\infty} \frac{dt}{[t]^3 + 9[t]^2 + 26[t] + 24} = (width)(height) = (1) \sum_{t=1}^{\infty} \frac{1}{t^3 + 9t^2 + 26t + 24}$$

Noting that $t^3 + 9t^2 + 26t + 24 = (t+2)(t+3)(t+4)$, we can expand as

$$\frac{1}{t^3 + 9t^2 + 26t + 24} = \frac{A}{t+2} + \frac{B}{t+3} + \frac{C}{t+4}$$

Solving for the variables A, B, C, we determine $A = \frac{1}{2}, B = -1$, and $C = \frac{1}{2}$ which leads to

$$\sum_{t=1}^{\infty} \frac{1}{t^3 + 9t^2 + 26t + 24} = \frac{1}{2} \sum_{t=1}^{\infty} \left(\frac{1}{t+2} - \frac{2}{t+3} + \frac{1}{t+4} \right)$$

Now, we rewrite things as

$$\begin{aligned}
\frac{1}{2}\sum_{t=1}^{\infty}\left(\frac{1}{t+2}-\frac{2}{t+3}+\frac{1}{t+4}\right) &= \frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}+\sum_{t=3}^{\infty}\frac{1}{t+2}-\frac{2}{4}+\sum_{t=2}^{\infty}\frac{-2}{t+3}+\sum_{t=1}^{\infty}\frac{1}{t+4}\right)\\
&= \frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}-\frac{1}{2}+\sum_{t=1}^{\infty}\frac{1}{t+4}+\sum_{t=1}^{\infty}\frac{-2}{t+4}+\sum_{t=1}^{\infty}\frac{1}{t+4}\right)\\
&= \frac{1}{2}\left(\frac{1}{12}+\sum_{t=1}^{\infty}\left(\frac{1}{t+4}+\frac{-2}{t+4}+\frac{1}{t+4}\right)\right)\\
&= \frac{1}{24}
\end{aligned}$$

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

$$\int_1^{\infty}\frac{dt}{([t]^3+9[t]^2+26[t]+24} = \int_{\cup_{k=1}^{\infty}[k,k+1)}\frac{dt}{([t]+2)([t]+3)([t]+4)}$$

$$=\sum_{k=1}^{\infty}\int_{[k,k+1)}\frac{dt}{([t]+2)([t]+3)([t]+4)}$$

$$=\sum_{k=1}^{\infty}\int_{[k,k+1)}\frac{dt}{(k+2)(k+3)(k+4)}$$

$$=\sum_{k=1}^{\infty}\left(\lim_{t\to(k+1)^-}\frac{t}{(k+2)(k+3)(k+4)}-\frac{k}{(k+2)(k+3)(k+4)}\right)$$

$$=\sum_{k=1}^{\infty}\frac{k+1-k}{(k+2)(k+3)(k+4)}$$

$$=\sum_{k=1}^{\infty}\frac{1}{(k+2)(k+3)(k+4)}$$

$$=\sum_{k=1}^{\infty}\frac{1}{2}\left(\frac{1}{(k+2)(k+3)}-\frac{1}{(k+3)(k+4)}\right)$$

$$=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(k+2)(k+3)}-\frac{1}{2}\sum_{k=2}^{\infty}\frac{1}{(k+2)(k+3)}=\frac{1}{2}\frac{1}{(1+2)(1+3)}=\frac{1}{24}.$$

Also solved by Arkady Alt, San Jose, CA; Brian Bradie, Christopher Newport University, Newport News, VA; Moti Levy, Rehovot, Israel; Kee-Wai Lau, Hong Kong, China; Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Albert Stadler, Herrliberg, Switzerland; Seán M. Stewart, Bomaderry, NSW, Australia; Ioannis D. Sfikas, National and Kapodistrian University of Athens,

Greece; David Stone and John Hawkins, Southern Georgia University, Statesboro GA; Daniel Văcaru, Pitesti, Romania, and the proposer.

- **5576:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

(a) Calculate

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{2}{n+1}+\frac{2}{n+2}-\frac{2}{n+3}+\cdots\right).$$

(b) Find the domain of convergence and the sum of the power series

$$\sum_{n=1}^{\infty}x^n\left(\frac{1}{n}-\frac{2}{n+1}+\frac{2}{n+2}-\frac{2}{n+3}+\cdots\right).$$

Solution 1 by Ángel Plaza Universidad de Las Palmas de Gran Canaria, Spain

(a) Let $a_n=\dfrac{1}{n}-\dfrac{2}{n+1}+\dfrac{2}{n+2}-\dfrac{2}{n+3}+\cdots$. Note that $\dfrac{1}{n}=\displaystyle\int_0^1 u^{n-1}du$, so

$$\begin{aligned}
a_n &= \int_0^1 u^{n-1}du+2\sum_{k=1}^{\infty}\int_0^1(-1)^k u^{n+k-1}du\\
&= \int_0^1 u^{n-1}du-2\int_0^1 u^n\sum_{k=0}^{\infty}(-1)^k u^k du\\
&= \int_0^1 u^{n-1}du-2\int_0^1\frac{u^n}{1+u}du\\
&= \int_0^1\frac{u^{n-1}-u^n}{1+u}du.
\end{aligned}$$

Therefore

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\int_0^1\frac{u^{n-1}-u^n}{1+u}du=\int_0^1\frac{\sum_{n=1}^{\infty}\left(u^{n-1}-u^n\right)}{1+u}du=\int_0^1\frac{1}{1+u}du=\ln 2.$$

(b) Since $\dfrac{1}{n}-\dfrac{2}{n+1}<|a_n|<\dfrac{1}{n}$, then $R=\left(\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}\right)^{-1}=1$. The series is convergent for $x=1$, since $\displaystyle\sum_{n=1}^{\infty}a_n=\ln 2$. Analogously, it may be seen that $\displaystyle\sum_{n=1}^{\infty}a_n(-1)^n=-1+\ln 2$, so the domain of convergence is the interval $[-1,1]$. Finally,

$$\begin{aligned}
\sum_{n=1}^{\infty}x^n a_n &= \sum_{n=1}^{\infty}x^n\int_0^1\frac{u^{n-1}-u^n}{1+u}du\\
&= \int_0^1\frac{x\sum_{n=0}^{\infty}(xu)^n-\sum_{n=1}^{\infty}(xu)^n}{1+u}\,du\\
&= \int_0^1\frac{x-ux}{(1+u)(1-ux)}\,du\\
&= \frac{x\log(4)+(1-x)\log(1-x)}{x+1}.
\end{aligned}$$

Solution 2 by Seán M. Stewart, Bomaderry, NSW, Australia

•(a) We will show that

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{2}{n+1}+\frac{2}{n+2}-\frac{2}{n+2}+\cdots\right)=\ln 2.$$

Denoting the value of the series to be found by S, we rewrite it as

$$S=\sum_{n=1}^{\infty}\left(2\sum_{k=0}^{\infty}\frac{(-1)^k}{n+k}-\frac{1}{n}\right).$$

Now consider

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{n+k}.$$

After reindexing $k\mapsto k-n$ one has

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-1)^k}{n+k}&=\sum_{k=n}^{\infty}\frac{(-1)^{k-n}}{k}\\&=(-1)^n\sum_{k=n}^{\infty}\frac{(-1)^k}{k}\\&=(-1)^n\left(\sum_{k=1}^{\infty}\frac{(-1)^k}{k}-\sum_{k=1}^{n-1}\frac{(-1)^k}{k}\right)\\&=(-1)^n\left(-\ln 2-\sum_{k=1}^{n-1}\frac{(-1)^k}{k}\right),\end{aligned}\tag{8}$$

where the well-known result for $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$ equalling $-\ln 2$ has been used.

Introducing the so-called nth *skew-harmonic number* $\overline{H}_n$ which is defined by

$$\overline{H}_n=\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k},$$

as

$$\sum_{k=1}^{n-1}\frac{(-1)^k}{k}=-\overline{H}_n-\frac{(-1)^n}{n},$$

in terms of $\overline{H}_n$ the sum appearing in (8) can be rewritten as

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{k+n}=\frac{1}{n}+(-1)^n\left(\overline{H}_n-\ln 2\right).\tag{9}$$

So for the series S, we can write

$$S=\sum_{n=1}^{\infty}\left(\frac{1}{n}+2(-1)^n\left(\overline{H}_n-\ln 2\right)\right).\tag{10}$$

We now show that

$$\overline{H}_n - \ln 2 = (-1)^{n+1} \int_0^1 \frac{x^n}{1+x}\, dx. \tag{11}$$

From (9), starting the summation at $n = 1$ we see that

$$\overline{H}_n - \ln 2 = (-1)^n \sum_{k=1}^{\infty} \frac{(-1)^k}{n+k}.$$

Since

$$\frac{1}{k+n} = \int_0^1 x^{n+k-1}\, dx,$$

one can write

$$\overline{H}_n - \ln 2 = (-1)^n \int_0^1 x^{n-1} \sum_{k=1}^{\infty} (-1)^k x^k\, dx,$$

after the order of the summation with the integration has been changed and is permissible due to Fubini's theorem. Recognising the series as an infinite geometric sum equal to $-x/(1+x)$, the result follows.

Returning to (10), as

$$\frac{1}{n} = \int_0^1 x^{n-1}\, dx,$$

making use of this result together with the result given in (11) we have

$$\begin{aligned}
S &= \sum_{n=1}^{\infty} \left(\int_0^1 x^{n-1}\, dx - 2 \int_0^1 \frac{x^n}{1+x}\, dx \right) \\
&= \sum_{n=1}^{\infty} \int_0^1 \frac{1-x}{x(1+x)} x^n\, dx \\
&= \int_0^1 \frac{1-x}{x(1+x)} \sum_{n=1}^{\infty} x^n\, dx \\
&= \int_0^1 \frac{1-x}{x(1+x)} \cdot \frac{x}{1-x}\, dx \\
&= \int_0^1 \frac{dx}{1+x} = \ln 2,
\end{aligned}$$

as announced. Note here the order of the summation with the integration has been changed and is permissible due to Fubini's theorem while the series, being an infinite geometric sum, is equal to $x/(1-x)$.

•(b) We show that

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{2}{n+1} + \frac{2}{n+2} - \frac{2}{n+3} + \cdots \right) x^n = \frac{2x \ln 2 + (1-x)\ln(1-x)}{1+x},$$

valid for $|x| \leqslant 1$.

Denoting the power series to be found by $S(x)$, from (a) we see in terms of the nth skew-harmonic number $\overline{H}_n$, the power series for $S(x)$ can be expressed as

$$S(x) = \sum_{n=1}^{\infty} \left(2(-1)^n \left(\overline{H}_n - \ln 2 \right) + \frac{1}{n} \right) x^n.$$

As

$$\frac{1}{n} = \int_0^1 t^{n-1}\,dt \quad \text{and} \quad \overline{H}_n - \ln 2 = (-1)^{n+1}\int_0^1 \frac{t^n}{1+t}\,dt,$$

we can express $S(x)$ as

$$\begin{aligned}
S(x) &= \sum_{n=1}^{\infty}\left(\int_0^1 t^{n-1}\,dt - 2\int_0^1 \frac{t^n}{1+t}\,dt\right)x^n \\
&= \int_0^1 \frac{1-t}{t(1+t)}\sum_{n=1}^{\infty}(tx)^n\,dt \\
&= \int_0^1 \frac{1-t}{t(1+t)}\cdot\frac{tx}{1-xt}\,dt, \quad \text{since } |tx|<1 \\
&= x\int_0^1 \frac{1-t}{(1+t)(1-xt)}\,dt \\
&= x\int_0^1 \left(\frac{2}{1+x}\cdot\frac{1}{1+t} - \frac{1-x}{1+x}\cdot\frac{1}{1-tx}\right)dt \\
&= x\left(\frac{2}{1+x}\Big[\ln(1+t)\Big]_0^1 - \frac{1-x}{1+x}\left[\frac{\ln(1-tx)}{-x}\right]_0^1\right) \\
&= \frac{2x\ln 2 + (1-x)\ln(1-x)}{1+x}.
\end{aligned}$$

Note the order of the summation with the integration has been changed and is permissible due to Fubini's theorem while the series $\sum_{n=1}^{\infty}(tx)^n$, being an infinite geometric sum, is equal to $tx/(1-tx)$.

At the end-point $x=1$, from (a) we see the power series converges. Here we have $S(1)=\ln 2$. At the end-point $x=-1$, the power series also converges. This can be established by finding its sum directly. Here

$$\begin{aligned}
S(-1) &= \sum_{n=1}^{\infty}(-1)^n\left(2(-1)^n(\overline{H}_n - \ln 2) + \frac{1}{n}\right) \\
&= 2\sum_{n=1}^{\infty}(\overline{H}_n - \ln 2) + \sum_{n=1}^{\infty}\frac{(-1)^n}{n} \\
&= 2\int_0^1 \frac{1}{1+x}\sum_{n=1}^{\infty}(-1)^{n+1}x^n\,dx - \ln 2 \\
&= 2\int_0^1 \frac{x}{(1+x)^2}\,dx - \ln 2 \\
&= 2\int_1^2\left(\frac{1}{x} - \frac{1}{x^2}\right)dx - \ln 2 \\
&= 2\left(\ln 2 - \frac{1}{2}\right) - \ln 2 \\
&= \ln 2 - 1,
\end{aligned}$$

where in the third line we have made use of (11) followed by a change in the order of the summation with the integration, summed the infinite geometric series that appeared, before enforcing a substitution of $x \mapsto x-1$ and integrating.

Thus

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{2}{n+1}+\frac{2}{n+2}-\frac{2}{n+3}+\cdots\right)x^n=\frac{2x\ln 2+(1-x)\ln(1-x)}{1+x},$$

and is valid for $|x| \leqslant 1$, as announced. Note at $x=-1$ the expression given for the power series needs to be interpreted as a limit as $x \to -1^+$.

Solution 3 by Bruno Salgueiro Fanego,Viveiro, Spain

For any $x \in [-1, 1]$,

$$\sum_{n=1}^{\infty} x^n\left(\frac{1}{n}-\frac{2}{n+1}+\frac{2}{n+2}-\frac{2}{n+3}+\right)$$

$$= \sum_{n=1}^{\infty} x^n\left(\frac{1}{n}+2(-1)^n\left(\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n+2}}{n+2}+\frac{(-1)^{n+3}}{n+3}+\cdots\right)\right)$$

$$= \sum_{n=1}^{\infty} x^n\left(\frac{1}{n}+2(-1)^n\left(\frac{(-1)^1}{1}+\frac{(-1)^2}{2}+\frac{(-1)^1}{3}+\cdots-\left(\frac{(-1)^1}{1}+\frac{(-1)^2}{2}+\frac{(-1)^3}{3}+\cdots+\frac{(-1)^n}{n}\right)\right)\right)$$

$$= \sum_{n=1}^{\infty} x^n\left(\frac{1}{n}+2(-1)^n\left(-\log(1-(-1))-\left(\frac{(1)^1}{1}+\frac{(-1)^2}{2}+\frac{(-1)^3}{3}+\cdots+\frac{(-1)^n}{n}\right)\right)\right)$$

$$= \sum_{n=1}^{\infty} x^n\left(\frac{1}{n}-2(-1)^n\log 2-2(-1)^n\left(\frac{(-1)^1}{1}+\frac{(-1)^2}{2}+\frac{(-1)^3}{3}+\cdots+\frac{(-1)^n}{n}\right)\right)$$

$$= \sum_{n=1}^{\infty}\frac{x^n}{n}-2\log(2)\sum_{n=1}^{\infty}(-x)^n-2\sum_{n=1}^{\infty}(-x)^n\left(\frac{(-1)^1}{1}+\frac{(-1)^2}{2}+\frac{(-1)^3}{3}+\cdots+\frac{(-1)^n}{n}\right)$$

$$= -\log(1-x)-2(\log 2)\frac{(-x)^1}{1-(-x)}-2\Bigg(\frac{(-1)^1}{1}\left(-x+(-x)^2+(-x)^3+\cdots\right)$$

$$+\frac{(-1)^2}{2}\left((-x)^2+(-x)^3+(-x)^4+\cdots\right)+\frac{(-1)^3}{3}\left((-x)^3+(-x)^4+(-x)^5+\cdots\right)+\cdots\Bigg)$$

$$= -\log(1-x)+2(\log 2)\frac{x}{1+x}-2\left(\frac{(-1)^1}{1}\frac{-x}{1-(x)}+\frac{(-1)^2}{2}\frac{(-x)^2}{1-(-x)}+\frac{(-1)^3}{3}\frac{(-x)^3}{1-(-x)}+\cdots\right)$$

$$= -\log(1-x)+2(\log 2)\frac{x}{1+x}-\frac{2}{1+x}\left(\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\right)$$

$$= -\log(1-x)+2(\log 2)\frac{x}{1+x}+\frac{2}{1+x}\log(1-x)=\frac{(1-x)\log(1-x)+2(\log 2)x}{1+x}.$$

Since:

$$= \lim_{x\to(-1)} \frac{(1-x)\log(1-x)+2(\log 2)x}{1+x} = \frac{2\log 2 - 2\log 2}{0} = \left[\frac{0}{0} = \text{Indet}\right]$$

$$= \lim_{x\to(-1)} \frac{-\log(1-x)+(1-x)\frac{-1}{1-x}+2\log}{1} = \log 2 - 1,$$

and

$$\lim_{x\to 1^-}(1-x)\log(1-x)- = [0\cdot\infty = \text{Indet.}] \lim_{x\to 1^-} \frac{\log(1-x)}{\frac{1}{1-x}} = \lim_{x\to 1^-} \frac{\frac{-1}{1-x}}{\frac{1}{(1-x)^2}} = \lim_{x\to 1^-}(-1+x) = 0,$$

then

$$\lim_{x\to 1^-} \frac{(1-x)\log(1-x)+2(\log 2)x}{1+x} = \frac{0+2\log 2}{2} = \log 2,$$

the domain of convergence of the power series at (b) is $[-1, 1]$ and its sum equals

$$\begin{cases} \dfrac{(1-x)\log(1-x)+2(\log 2)x}{1+x}, \text{ if } x \in (-1,1), \\ \\ \log 2 - 1, \text{ if } x = -1 \\ \\ \log 2, \text{ if } x = 1, \end{cases}$$

being the last result asked for in part (a) of the question.

Solution 4 by Ulrich Abel, Technische Hochschule Mittelhessen, Germany

We start with

$$\begin{aligned} a_n := & \quad \frac{1}{n} - \frac{2}{n+1} + \frac{2}{n+2} - \frac{2}{n+3} + \cdots \\ = & \quad \frac{1}{n} - 2\sum_{k=1}^{\infty}(-1)^k \int_0^1 t^{n+k-1}dt = \int_0^1 \left(t^{n-1} - \frac{2t^n}{1+t}\right)dt = \int_0^1 \frac{t^{n-1}-t^n}{1+t}dt, \end{aligned}$$

where the order of summation and integration can be interchanged by Abel's limit theorem. By telescoping we obtain

$$\sum_{n=1}^{N} a_n = \int_0^1 \frac{1-t^{N+1}}{1+t}dt.$$

This implies

$$\sum_{n=1}^{\infty} a_n = \int_0^1 \frac{1}{1+t}dt = \ln 2,$$

which solves part (a). In a similar way, for $|x| < 1$, we have

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} x^n \int_0^1 \frac{t^{n-1} - t^n}{1+t} dt = x \int_0^1 \frac{1}{1+t} \frac{1-t}{1-xt} dt.$$

Using the identity

$$\frac{1}{1+t} \frac{1-t}{1-xt} = \frac{2}{1+x} \frac{1}{1+t} - \frac{1-x}{1+x} \frac{1}{1-xt}$$

we obtain

$$\sum_{n=1}^{\infty} a_n x^n = \frac{2}{1+x} \ln 2 + \frac{1-x}{x(1+x)} \ln(1-x).$$

Considering real numbers x, the sum converges, for $-1 < x \leqslant 1$. In the case $x = 1$, its value is given by the limit, for $x \to 1 - 0$, which recovers part (a).

Also solved by Arkady Alt, San Jose, CA; Michel Bataille, Rouen, France; Brian Bradie, Christopher Newport University, Newport News, VA; Moti Levy, Rehovot, Israel; Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposers.

Mea Culpa

Bruno Salgueiro Fanego of Viveiro, Spain should have been credited for having solved 5564. I inadvertently forgot to credit him with having solved the problem.

An addendum to 5565 by David Stone and John Hawkins, Georgia Southern University, of Statesboro GA.

A number of typographical errors appeared in the featured solution. We offer an organized list of all solutions. In the notion of the featured solution, we exhibit in them with $b < a$ thereby eliminating duplications.

The first four of the solutions have the largest base a, equal to the given diameter 343; That is, they "fill up" the upper semicircle. The other 12 solutions appear in pairs because each pair of bases forms one trapezoid which is properly inscribed in the upper semi-circle and also forms a second trapezoid which straddles the diameter.

$b = upper\ base$	$a = lower\ base$	c	$Perimeter$	
119	343	196	854	
217	343	147	854	
287	343	98	826	
329	343	49	770	
18	294	161	634	
18	294	294	900	
49	143	49	290	MIN
49	143	329	850	
98	262	98	556	
98	262	287	934	
147	333	147	774	
147	333	217	914	
196	332	119	766	
196	332	196	920	
235	245	7	494	
235	245	245	970	MAX