Problems and Solutions

Albert Natian, Section Editor

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This section of the SSMA Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Prof. Albert Natian, Department of Mathematics, Los Angeles Valley College, 5800 Fulton Avenue, Valley Glen, CA, 91401, USA. It's highly preferable that you send your contributions via email.

To propose problems, email them to: **problems4ssma@gmail.com**

To propose solutions, email them to: **solutions4ssma@gmail.com**

Please follow the instructions for submission of problems and solutions provided at the end of this document. Your adherence to all submission requirements is of the greatest help in running this Section successfully. Thank you!

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted before July 15, 2022.

• **5685** *Proposed by D.M. Bătineţu-Giurgiu, Bucharest, Romania and Neculai Stanciu, Buzău, Romania.*

Prove: If $x \in (0, \pi/2)$, then for any triangle $\triangle ABC$ with side lengths a, b, c and area F, the following inequality holds:

$$a^2 + b^2 + c^2 \geqslant 4\sqrt{3}F \sin(2x) + \sum_{cyc} (a \sin x - b \cos x)^2 .$$

• **5686** *Proposed by Albert Stadler, Herrliberg, Switzerland.*

Let n be a natural number. Prove that the following three statements are equivalent:

1. The n-th central trinomial coefficient is divisible by 3.
2. The n-th central binomial coefficient is divisible by 3.
3. The base 3 representation of n has at least one digit "2".

Note: the n-th central trinomial coefficient is the coefficient of x^n in the expansion of $\left(1 + x + x^2\right)^n$, while the n-th central binomial coefficient is the coefficient of x^n in the expansion of $(1 + x)^{2n}$ and equals $\binom{2n}{n}$.

• **5687** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

Find complex numbers u, v such that:

$$\left\{\begin{array}{c}\dfrac{|u|^2}{3}+\dfrac{|v|^2}{4}=\dfrac{|u+v|^2}{7}\\ \\ 8u+v=7+7i\end{array}\right\}.$$

• **5688** *Proposed by Kenneth Korbin, New York, NY.*

Three convex hexagons with integer side lengths are all inscribed in the same circle. The hexagons have perimeters p, $p+1$ and $p+2$. Find the lengths of the sides of each hexagon.

• **5689** *Proposed by Rafael Jakimczuk, Universidad National de Lujá, Buenos Aires, Argentina.*

Let $(F_n)_{n\geqslant 1}$ denote the Fibonacci sequence defined by the recursion $F_n = F_{n-1} + F_{n-2}$ with $F_1 = F_2 = 1$. Find $\lim\limits_{n\to\infty} P_n$ where the sequence $(P_n)_{n\geqslant 1}$ is defined by

$$P_n := \prod_{k=1}^{n}\left(1+\frac{1}{nF_k}\right)^{F_{k+1}}.$$

• **5690** *Proposed by Toyesh Prakash Sharma (Student) St. C.F. Andrews School, Agra, India.*

Find the value of

$$\int_0^{1/\sqrt{2}} \sin^{-1}\left(\cos\left(\sin^{-1}x\right)\right)dx - \int_{\pi/4}^{\pi/2} \sin\left(\cos^{-1}\left(\sin x\right)\right)dx.$$

Solutions

to Formerly Published Problems

• **5667** *Proposed by Albert Stadler, Herrilberg, Switzerland.*

Prove with at most 10 function evaluations that

$$4\cdot 10^{-89} < \prod_{k=1}^{89}\tan^2\left(\frac{k\pi}{360}\right) < 5\cdot 10^{-89}.$$

Solution by the proposer.

We will prove that

$$\prod_{k=1}^{n} \tan\left(\frac{k\pi}{4n}\right) = 2e^{-\frac{4G}{\pi}n}\sqrt{2n}e^{\frac{\pi}{24n}+\theta\frac{\zeta(3)}{32\pi n^2}} \tag{1}$$

where G is Catalan's constant and θ is a real number satisfying $|\theta|\leqslant 1$. We square (1) and set $n = 90$. We get

$$\prod_{k=1}^{89} \tan^2\left(\frac{k\pi}{360}\right) = \prod_{k=1}^{90} \tan^2\left(\frac{k\pi}{360}\right) = 720e^{-\frac{720G}{\pi}+\frac{\pi}{1080}+\theta\frac{\zeta(3)}{129600\pi}}.$$

This approximation shows that $\prod_{k=1}^{89} \tan^2\left(\frac{k\pi}{360}\right)$ lies in the interval

$$\left[4.895808916773337 \times 10^{-89}, 4.89583782529551 \times 10^{-89}\right].$$

Let's turn to the proof of (1). Let n$\geqslant$2 be an integer. Then

$$\prod_{k=1}^{n-1}\left(2\sin\left(x+\frac{k\pi}{n}\right)\right) = \prod_{k=1}^{n-1}\left((-i)\left(e^{ix+\frac{\pi ik}{n}} - e^{-ix-\frac{\pi ik}{n}}\right)\right) =$$

$$= (-i)^{n-1}e^{i(n-1)x+\sum_{k=1}^{n-1}\frac{\pi ik}{n}}\prod_{k=1}^{n-1}\left(1-e^{-2ix-\frac{2\pi ik}{n}}\right) =$$

$$= (-i)^{n-1}e^{\frac{\pi i(n-1)}{2}}\frac{1-e^{-2inx}}{1-e^{-2ix}} = \frac{\sin(nx)}{\sin x}.$$

In particular, letting x tend to 0,

$$\prod_{k=1}^{n-1}\sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}.$$

Clearly, $\sin(x) = \sin(\pi - x)$. So

$$\prod_{k=1}^{n-1}\sin\left(\frac{k\pi}{n}\right) = \prod_{k=1}^{\left[\frac{n-1}{2}\right]}\sin^2\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}$$

which implies

$$\prod_{k=1}^{\left[\frac{n-1}{2}\right]}\sin\left(\frac{k\pi}{n}\right) = \sqrt{\frac{n}{2^{n-1}}}.$$

Hence, using the identity $2\sin(x)\cos(x) = \sin(2x)$,

$$\begin{aligned}\prod_{k=1}^{n}\tan\left(\frac{k\pi}{4n}\right) &= \prod_{k=1}^{n}\sin\left(\frac{k\pi}{4n}\right)\cos\left(\frac{k\pi}{4n}\right)\prod_{k=1}^{n}\frac{1}{\cos^2\left(\frac{k\pi}{4n}\right)}\\ &= \frac{1}{2^n}\prod_{k=1}^{n}\sin\left(\frac{k\pi}{2n}\right)\prod_{k=1}^{n}\frac{1}{\cos^2\left(\frac{k\pi}{4n}\right)}\\ &= \frac{1}{2^n}\sqrt{\frac{2n}{2^{2n-1}}}\prod_{k=1}^{n}\frac{1}{\cos^2\left(\frac{k\pi}{4n}\right)}\\ &= \frac{2\sqrt{n}}{4^n}\prod_{k=1}^{n}\frac{1}{\cos^2\left(\frac{k\pi}{4n}\right)}.\end{aligned}$$

Clearly,

$$\prod_{k=1}^{n}\frac{1}{\cos^2\left(\frac{k\pi}{4n}\right)} = \exp\left(-2\sum_{k=1}^{n}\log\cos\left(\frac{k\pi}{4n}\right)\right).$$

We estimate $\sum_{k=1}^{n}\log\left(\cos\left(\frac{k\pi}{4n}\right)\right)$ by means of the Euler-Maclaurin summation formula. Put

$$f(x) = \log\cos\left(\frac{\pi x}{4n}\right)$$

and note that f(0)=0. Then (e.g., see https://en.wikipedia.org/wiki/Euler%E2%80%93Maclaurin_formula)

$$\sum_{j=0}^{n} f(j) = \int_0^n f(x)\,dx + \frac{1}{2}f(n) + \frac{1}{12}\left(f'(n) - f'(0)\right) + R_3,$$

where

$$|R_3| \leqslant \frac{\zeta(3)}{4\pi^3}\int_0^n \left|f^{(3)}(x)\right| dx$$

and $\zeta(3) = \sum_{n=1}^{\infty}\frac{1}{n^3}$ is Riemann's zeta function evaluated at 3.
We have

$$f'(x) = -\frac{\pi}{4n}\tan\left(\frac{\pi x}{4n}\right),\ f''(x) = -\frac{\pi^2}{16n^2}\cdot\frac{1}{\cos^2\left(\frac{\pi x}{4n}\right)},\ f'''(x) = -\frac{\pi^3}{32n^3}\cdot\frac{1}{\cos^2\left(\frac{\pi x}{4n}\right)}\cdot\tan\left(\frac{\pi x}{4n}\right).$$

Let G be Catalan's constant. It is known (e.g., see https://en.wikipedia.org/wiki/Catalan%27s_constant) that

$$G = -\int_0^{\frac{\pi}{4}}\log(\tan y)dy = -\int_0^{\frac{\pi}{4}}\log(\sin y)dy + \int_0^{\frac{\pi}{4}}\log(\cos y)dy =$$

$$= \underbrace{-\int_0^{\frac{\pi}{2}} \log(\sin y)dy}_{=\frac{\pi}{2}\log 2} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \log(\sin y)dy + \int_0^{\frac{\pi}{4}} \log(\cos y)dy = \frac{\pi}{2}\log 2 \ + 2\int_0^{\frac{\pi}{4}} \log(\cos y)dy.$$

Hence

$$\int_0^n \log\left(\cos\left(\frac{\pi x}{4n}\right)\right)dx \overbrace{=}^{y=\frac{\pi x}{4n}} \frac{4n}{\pi}\int_0^{\frac{\pi}{4}} \log(\cos(y))dy = \frac{2Gn}{\pi} - n\log 2\ .$$

We conclude that there is a constant θ, $|\theta|\leqslant 1$, such that

$$\sum_{j=0}^{n} \log\left(\cos\left(\frac{\pi j}{4n}\right)\right) = \frac{2Gn}{\pi} - n\log 2 \ + \frac{1}{2}\log\left(\cos\left(\frac{\pi n}{4n}\right)\right) + \frac{1}{12}\left(-\frac{\pi}{4n}\tan\left(\frac{\pi n}{4n}\right)\right) + \theta R_3 =$$

$$= \frac{2Gn}{\pi} - n\log 2 \ - \frac{1}{4}\log 2 \ - \frac{\pi}{48n} + \theta\frac{\zeta(3)}{4\pi^3}\left(\frac{\pi^2}{16n^2}\frac{1}{\cos^2\left(\frac{\pi n}{4n}\right)} - \frac{\pi^2}{16n^2}\right) =$$

$$= \frac{2Gn}{\pi} - \left(n + \frac{1}{4}\right)\log 2 \ - \frac{\pi}{48n} + \theta\frac{\zeta(3)}{64\pi n^2}.$$

Hence

$$\prod_{k=1}^{n} \frac{1}{\cos^2\left(\frac{k\pi}{4n}\right)} = \exp\left(-2\sum_{k=1}^{n}\log\left(\cos\left(\frac{k\pi}{4n}\right)\right)\right)$$

$$= \exp\left(-2\left(\frac{2Gn}{\pi} - \left(n + \frac{1}{4}\right)\log 2 \ - \frac{\pi}{48n} + \theta\frac{\zeta(3)}{64\pi n^2}\right)\right)$$

and

$$\prod_{k=1}^{n} \tan\left(\frac{k\pi}{4n}\right) \ = \frac{2\sqrt{n}}{4^n}\prod_{k=1}^{n}\frac{1}{\cos^2\left(\frac{k\pi}{4n}\right)} = 2e^{-\frac{4G}{\pi}n}\sqrt{2n}e^{\frac{\pi}{24n}+\theta\frac{\zeta(3)}{32\pi n^2}}$$

which is (1).

• **5668** *Proposed by Ovidiu-Gabriel Dinu, Technological High School, Petrache Poenaru, Bălceşti, Vâlcea , România.*

Prove that for x and t in $[0,1]$ and for any integer $k \geqslant 2$:

$$\left|e^{-x^{2k}} - \int_0^1 e^{-t^{2k}}dt\right| \leqslant 2k\left(\sqrt[2k]{\frac{2k-1}{2k}}\right)^{2k-1} e^{-\frac{2k-1}{2k}}.$$

Solution 1 by Michel Bataille, Rouen, France.

Let $x \in [0,1]$ and for $t \in [0,1]$, let $f(t) = e^{-t^{2k}}$. First, we have

$$\left| e^{-x^{2k}} - \int_0^1 e^{-t^{2k}}\,dt \right| = \left| \int_0^1 (f(x) - f(t))\,dt \right| \leqslant \int_0^1 \left| f(x) - f(t) \right|\,dt \qquad (1)$$

and, second, from the Mean Value Theorem, $f(x) - f(t) = (x-t)f'(\theta)$ for some θ between x and t so that

$$\left| f(x) - f(t) \right| \leqslant |x-t||f'(\theta)| \leqslant |f'(\theta)|. \qquad (2)$$

A simple calculation gives $|f'(\theta)| = 2kg(\theta)$ where g is the function defined by $g(t) = t^{2k-1}e^{-t^{2k}}$. We readily obtain

$$g'(t) = t^{2k-2}e^{-t^{2k}}(2k-1-2kt^{2k})$$

and deduce that for $t \in [0,1]$,

$$0 \leqslant g(t) \leqslant g\left(\sqrt[2k]{\frac{2k-1}{2k}}\right) = \left(\sqrt[2k]{\frac{2k-1}{2k}}\right)^{2k-1} e^{-\frac{2k-1}{2k}}.$$

Thus

$$|f'(\theta)| \leqslant 2k\left(\sqrt[2k]{\frac{2k-1}{2k}}\right)^{2k-1} e^{-\frac{2k-1}{2k}} \qquad (3)$$

and combining (1), (2) and (3), the required inequality immediately follows.

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

We note that for any (real) k⩾1

$$1 \geqslant \int_0^1 e^{-t^{2k}}dt = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int_0^1 t^{2jk}dt = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(2jk+1)} \geqslant 1 - \frac{1}{2k+1} \geqslant \frac{1}{e}.$$

Clearly, if k⩾1, x∈[0,1], a∈[1/e,1] then

$$\left| e^{-x^{2k}} - a \right| \leqslant \max\left(1-a, a-\frac{1}{e}\right) \leqslant 1 - \frac{1}{e} \approx 0.6321205588285577,$$

since $e^{-x^{2k}}$ assumes only values in the interval [1/e,1]. It is therefore sufficient to prove that

$$2k\left(\sqrt[2k]{\frac{2k-1}{2k}}\right)^{2k-1} e^{-\frac{2k-1}{2k}} \geqslant \sqrt{2}e^{-\frac{1}{2}} \approx 0.8577638849607069 \qquad (1)$$

for k⩾1. Indeed, if

$$f(x) := 2x\left(\sqrt[2x]{\frac{2x-1}{2x}}\right)^{2x-1} e^{-\frac{2x-1}{2x}} = \exp\left(\ln(2x) + \left(1 - \frac{1}{2x}\right)\left(\ln\left(1 - \frac{1}{2x}\right) - 1\right)\right)$$

then

$$f(1) = \sqrt{2}e^{-\frac{1}{2}} \quad \text{and} \quad \frac{f'(x)}{f(x)} = \frac{1}{x} + \frac{1}{2x^2}\ln\left(1 - \frac{1}{2x}\right) \geqslant 0$$

for x⩾1. This proves (1).

Note that the last inequality is equivalent to -y ln(1-y)⩽1 for 0⩽y⩽½ which is true, since

$$-y\ln(1-y) = \sum_{k=1}^{\infty}\frac{y^{k+1}}{k} \leqslant \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k+1} = \frac{1}{2}.$$

Also solved by the proposer.

• **5669** *Proposed by Raluca Maria Caraion, Călăraşi, Romania and Florică Anastase, Lehliu-Gară, Romania.*

Suppose a is a real number. Find:

$$\Omega = \lim_{p\to\infty}\frac{1}{p^a}\cdot\sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2}.$$

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA.

With

$$\sum_{k=1}^{n}\frac{k^2}{2k^2-2kn+n^2} = \sum_{k=0}^{n}\frac{k^2}{k^2+(n-k)^2} = \sum_{k=0}^{n}\frac{(n-k)^2}{k^2+(n-k)^2},$$

it follows that

$$\sum_{k=1}^{n}\frac{k^2}{2k^2-2kn+n^2} = \frac{1}{2}\sum_{k=0}^{n}\frac{k^2+(n-k)^2}{k^2+(n-k)^2} = \frac{n+1}{2}.$$

Then

$$\sum_{n=1}^{m}\sum_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2} = \frac{1}{2}\sum_{n=1}^{m}(n+1) = \frac{1}{2}\left(\frac{m(m+1)}{2}+m\right) = \frac{m(m+3)}{4},$$

and

$$\begin{aligned}\sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2} &= \frac{1}{4}\sum_{m=1}^{p}(m^2+3m)\\ &= \frac{1}{4}\left(\frac{p(p+1)(2p+1)}{6}+\frac{3p(p+1)}{2}\right)\\ &= \frac{p(p+1)(p+5)}{12}.\end{aligned}$$

Finally,

$$\Omega = \lim_{p\to\infty} \frac{1}{p^a} \cdot \sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{k=1}^{n} \frac{k^2}{2k^2-2nk+n^2} = \left\{ \begin{array}{cc} 0 & \text{if } a>3 \\ 1/12 & \text{if } a=3 \\ \infty & \text{if } a<3 \end{array} \right\}.$$

Solution 2 by Moti Levy, Rehovot, Israel.

$$\frac{k^2}{2k^2-2nk+n^2} = \frac{1}{2}\left(1+\frac{2kn-n^2}{2k^2-2kn+n^2}\right)$$

Let

$$a_k := \frac{2kn-n^2}{2k^2-2kn+n^2}.$$

One can check that

$$a_{n-k} = -a_k,$$

hence

$$\sum_{k=1}^{n-1} a_k = 0.$$

$$\sum_{k=1}^{n} a_k = a_n + \sum_{k=1}^{n-1} a_k = 1 + \sum_{k=1}^{n-1} a_k = 1.$$

If follows that

$$\sum_{k=1}^{n} \frac{k^2}{2k^2-2nk+n^2} = \frac{1}{2}\sum_{k=1}^{n}(1+a_k) = \frac{n+1}{2}.$$

$$\begin{aligned} \sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{k=1}^{n} \frac{k^2}{2k^2-2nk+n^2} &= \sum_{m=1}^{p}\sum_{n=1}^{m}\frac{n+1}{2} = \frac{1}{4}\sum_{m=1}^{p} m(m+3) \\ &= \frac{1}{12}p(p+1)(p+5). \end{aligned}$$

$$\Omega = \frac{1}{12}\lim_{p\to\infty}\frac{p(p+1)(p+5)}{p^a} = \left\{ \begin{array}{cc} 0 & \text{if } a>3 \\ 1/12 & \text{if } a=3 \\ \infty & \text{if } a<3 \end{array} \right\}.$$

Solution 3 by Péter Fülöp, Gyömrő, Hungary.

1. Let's start with the inside sum: $S_n = \displaystyle\sum_{k=1}^{n} \frac{k^2}{2k^2-2nk+n^2}$

We can realize that $S_n = 1 + \underbrace{\displaystyle\sum_{k=1}^{n-1} \frac{k^2}{(n-k)^2+k^2}}_{S_{n-1}}$

and that $\sum_{k=n-1}^{1} \frac{(n-k)^2}{(n-k)^2+k^2}$ sum equals to S_{n-1}

So $2S_{n-1} = \sum_{k=n-1}^{1} \frac{(n-k)^2}{(n-k)^2+k^2} + \sum_{k=1}^{n-1} \frac{k^2}{(n-k)^2+k^2} = \sum_{k=1}^{n-1} 1 = n-1$

S_n can be calculated: $S_n = 1 + \frac{n-1}{2} = \frac{n+1}{2}$

2. Sum of the first n integer numbers equals to $\frac{n(n+1)}{2}$. Using this fact we get the followings:

$$\Omega = \lim_{p\to\infty} \frac{1}{p^a}. \sum_{m=1}^{p} \underbrace{\sum_{n=1}^{m} \frac{n+1}{2}}_{\frac{m(m+3)}{4}}$$

3. At the calculation of the last sum, we can use the expression of the sum of the first n integers again. The sum of the squares of the first m integer equals to $\frac{1}{6}p(p+1)(2p+1)$ is also applying.

$$\Omega = \lim_{p\to\infty} \frac{p(p+1)(p+5)}{12p^a}$$

$$\Omega = \lim_{p\to\infty} \frac{1}{12}\left(p^{3-a} + 6p^{2-a} + 5p^{1-a}\right)$$

Taking the limits we have the result for Ω:

$$\Omega = \left\{ \begin{array}{cl} 0 & \text{if } a > 3 \\ 1/12 & \text{if } a = 3 \\ \infty & \text{if } a < 3 \end{array} \right\}.$$

Solution 4 by Michel Bataille, Rouen, France.

Let $S_n = \sum_{k=1}^{n} \frac{k^2}{2k^2-2nk+n^2}$. We observe that $\frac{k^2}{2k^2-2nk+n^2} = \frac{k^2}{k^2+(n-k)^2}$ and, by change of index, that

$$S_n = \sum_{k=0}^{n} \frac{k^2}{k^2+(n-k)^2} = \sum_{k=0}^{n} \frac{(n-k)^2}{(n-k)^2+k^2}.$$

It follows that

$$2S_n = \sum_{k=0}^{n} \frac{k^2+(n-k)^2}{(n-k)^2+k^2} = n+1,$$

that is, $S_n = \dfrac{n+1}{2}$.
As a result, we obtain

$$\sum_{n=1}^{m} S_n = \frac{1}{2}\sum_{n=1}^{m}(n+1) = \frac{1}{2}\left(-1+\sum_{n=1}^{m+1} n\right) = \frac{1}{2}\left(-1+\frac{(m+1)(m+2)}{2}\right) = \frac{m^2+3m}{4}$$

and then

$$\sum_{m=1}^{p}\sum_{n=1}^{m} S_n = \frac{1}{4}\left(\sum_{m=1}^{p} m^2 + 3\sum_{m=1}^{p} m\right) = \frac{p(p+1)(2p+1)}{24} + \frac{3p(p+1)}{8} = \frac{p(p+1)(p+5)}{12}.$$

Thus,

$$\frac{1}{p^a}\sum_{m=1}^{p}\sum_{n=1}^{m} S_n = \frac{p(p+1)(p+5)}{12p^a} \sim \frac{1}{12p^{a-3}} \quad \text{as } p\to\infty.$$

We conclude that the required limit is ∞ if $a<3$, $\dfrac{1}{12}$ if $a=3$ and 0 if $a>3$.

Solution 5 by Albert Stadler, Herrliberg, Switzerland.

We note that

$$\sum_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2} = 1+\sum_{k=1}^{n-1}\frac{k^2}{2k^2-2nk+n^2} =$$

$$= 1+\frac{1}{2}\sum_{k=1}^{n-1}\left(\frac{k^2}{2k^2-2nk+n^2}+\frac{(n-k)^2}{2(n-k)^2-2n\,(n-k)+n^2}\right) = 1+\frac{1}{2}\sum_{k=1}^{n-1}1 = \frac{n+1}{2}.$$

Then

$$\sum_{n=1}^{m}\frac{n+1}{2} = \frac{1}{4}m\,(m+1)+\frac{m}{2} = \frac{m^2+3m}{4},$$

$$\sum_{m=1}^{p}\frac{m^2+3m}{4} = \frac{1}{24}p\,(p+1)\,(2p+1)+\frac{3}{8}p\,(p+1) = \frac{p\,(p+1)\,(p+5)}{12},$$

and finally

$$\Omega = \left\{\begin{array}{cc} 0 & \text{if } a>3 \\ 1/12 & \text{if } a=3 \\ \infty & \text{if } a<3 \end{array}\right\}.$$

Also solved by the proposer.

• **5670** *Proposed by Kenneth Korbin, New York, NY.*

Find a positive real number x such that

$$\sqrt[4]{\pi+x} = 1+\sqrt[4]{\pi-x}\,.$$

Solution 1 by Hossaena Tedla, ADA University, Baku, Azerbaijan.

$$\sqrt[4]{\pi + x} = 1 + \sqrt[4]{\pi - x}$$

Let $u = \sqrt[4]{\pi + x}$ *then* $x = u^4 - \pi$

$$u = 1 + \sqrt[4]{\pi - (u^4 - \pi)}$$

$$(u-1)^4 = 2\pi - u^4$$

$$u^4 - 4u^3 + 6u^2 - 4u + 1 = 2\pi - u^4$$

$$2u^4 - 4u^3 + 6u^2 - 4u - 2\,\pi + 1 = 0$$

Let $y = u - \frac{1}{2}$

$$2\left(y + \frac{1}{2}\right)^4 - 4\left(y + \frac{1}{2}\right)^3 + 6\left(y + \frac{1}{2}\right)^2 - 4\left(y + \frac{1}{2}\right) - 2\,\pi + 1 = 0$$

By simplifying the above equation, we will get

$$2y^4 + 3y^2 - 2\,\pi + \frac{1}{8} = 0$$

Let $y^2 = v$

$$2v^2 + 3v - 2\,\pi + \frac{1}{8} = 0$$

Divide both sides by 2 we will get,

$$\frac{1}{2}\left(-2\,\pi + \frac{1}{8}\right) + \frac{3v}{2} + v^2 = 0$$

Adding $\frac{1}{2}\left(-2\,\pi + \frac{1}{8}\right)$ to both sides we will get

$$v^2 + \frac{3v}{2} = \frac{1}{2}\left(2\,\pi - \frac{1}{8}\right)$$

Adding $\frac{9}{16}$ to both sides,

$v^2 + \frac{3}{2}v + \frac{9}{16} = \frac{1}{2}\left(2\,\pi - \frac{1}{8}\right) + \frac{9}{16}$

$(v + \frac{3}{4})^2 = \frac{1}{2}\left(2\,\pi - \frac{1}{8}\right) + \frac{9}{16}$

$v + \frac{3}{4} = \sqrt{\frac{1}{2}\left(2\,\pi - \frac{1}{8}\right) + \frac{9}{16}}$ or $v + \frac{3}{4} = -\sqrt{\frac{1}{2}\left(2\,\pi - \frac{1}{8}\right) + \frac{9}{16}}$

$v = -\frac{3}{4} + \sqrt{\frac{1}{2}\left(2\,\pi - \frac{1}{8}\right) + \frac{9}{16}}$ or $v = -\frac{3}{4} - \sqrt{\frac{1}{2}\left(2\,\pi - \frac{1}{8}\right) + \frac{9}{16}}$

From above we substitute $y^2 = v$

$y = \sqrt{-\frac{3}{4} + \sqrt{\frac{1}{2}\left(2\,\pi - \frac{1}{8}\right) + \frac{9}{16}}}$ or $y = \sqrt{-\frac{3}{4} - \sqrt{\frac{1}{2}\left(2\,\pi - \frac{1}{8}\right) + \frac{9}{16}}}$

From above we substitute y = u - $\frac{1}{2}$ therefore u = y + $\frac{1}{2}$

$u = \frac{1}{2} + \sqrt{-\frac{3}{4} + \sqrt{\frac{1}{2}\left(2\,\pi - \frac{1}{8}\right) + \frac{9}{16}}}$ or $u = \frac{1}{2} + \sqrt{-\frac{3}{4} - \sqrt{\frac{1}{2}\left(2\,\pi - \frac{1}{8}\right) + \frac{9}{16}}}$

Here the expression $\frac{1}{2}\left(2\,\pi - \frac{1}{8}\right) + \frac{9}{16} = \pi + \frac{1}{2}$,

$u = \frac{1}{2} + \sqrt{-\frac{3}{4} + \sqrt{\pi + \frac{1}{2}}}$ or $u = \frac{1}{2} + \sqrt{-\frac{3}{4} - \sqrt{\pi + \frac{1}{2}}}$

From above we substitute $x = u^4 - \pi$ therefore

$x = \left(\frac{1}{2} + \sqrt{-\frac{3}{4} + \sqrt{\pi + \frac{1}{2}}}\right)^4 - \pi$, x=3.0313

Or

$x = \left(\frac{1}{2} + \sqrt{-\frac{3}{4} - \sqrt{\pi + \frac{1}{2}}}\right)^4 - \pi$ the value of x is complex number. We are asked to find only positive real number.

The answer is x=3.0313.

Solution 2 by Brian D. Beasley, Presbyterian College, Clinton, SC.

We show that the unique positive real solution is

$$x = \frac{1}{16}\left(1 + \sqrt{\sqrt{16\pi + 8} - 3}\right)^4 - \pi \approx 3.03133.$$

For $-\pi \leqslant x \leqslant \pi$, we define $f(x) = \sqrt[4]{\pi + x} - 1 - \sqrt[4]{\pi - x}$. Since $f'(x) = (1/4)(\pi + x)^{-3/4} + (1/4)(\pi - x)^{-3/4} > 0$ for $-\pi < x < \pi$, $f(x)$ is increasing on $[-\pi, \pi]$. Using $f(0) < 0$ and $f(\pi) > 0$, we conclude that $f(x)$ has a unique real zero in $(0, \pi)$.

Next, let $w = x + \pi$, so that the original equation becomes $\sqrt[4]{w} = 1 + \sqrt[4]{2\pi - w}$. Then $2\pi - w = w - 4w^{3/4} + 6w^{1/2} - 4w^{1/4} + 1$. Substituting $v = w^{1/4}$ yields $2v^4 - 4v^3 + 6v^2 - 4v + 1 - 2\pi = 0$, or $(v^2 - v + 1)^2 = \pi + 1/2$. Since $v^2 - v + 1 > 0$ for all real numbers v, we obtain $v^2 - v + 1 = \sqrt{\pi + 1/2}$. Thus

$$v = \frac{1}{2}\left(1 \pm \sqrt{\sqrt{16\pi + 8} - 3}\right),$$

which implies

$$x = v^4 - \pi = \frac{1}{16}\left(1 \pm \sqrt{\sqrt{16\pi + 8} - 3}\right)^4 - \pi \approx \pm 3.03133.$$

The positive choice for x gives $f(x) = 0$ as required, while the negative choice (an extraneous solution) produces $f(x) = -2$ instead.

Solution 3 by Brian Bradie, Christopher Newport University, Newport News, VA.

Let $y = \sqrt[4]{\pi + x}$, so that $x = y^4 - \pi$ and $\pi - x = 2\pi - y^4$. The equation

$$\sqrt[4]{\pi + x} = 1 + \sqrt[4]{\pi - x}$$

then becomes

$$(y-1)^4 = 2\pi - y^4, \quad \text{or} \quad y^4 - 2y^3 + 3y^2 - 2y + \left(\frac{1}{2} - \pi\right) = 0.$$

Now,

$$\begin{aligned} y^4 - 2y^3 + 3y^2 - 2y + \left(\frac{1}{2} - \pi\right) \\ &= \left(y^2 - y + 1 - \sqrt{\pi + \frac{1}{2}}\right)\left(y^2 - y + 1 + \sqrt{\pi + \frac{1}{2}}\right). \end{aligned}$$

The roots of the quadratic

$$y^2 - y + 1 + \sqrt{\pi + \frac{1}{2}}$$

are complex with real part $1/2$ and imaginary not equal to either $\pm 1/2$, so y^4 would be complex. On the other hand, the roots of the quadratic

$$y^2 - y + 1 - \sqrt{\pi + \frac{1}{2}}$$

are

$$y_1 = \frac{1}{2} - \frac{1}{2}\sqrt{\sqrt{16\pi + 8} - 3} \quad \text{and} \quad y_2 = \frac{1}{2} + \frac{1}{2}\sqrt{\sqrt{16\pi + 8} - 3}.$$

Because $7 < \sqrt{16\pi + 8} < 8$, it follows that

$$\frac{1 - \sqrt{5}}{2} < y_1 < -\frac{1}{2}, \quad \text{and} \quad y_1^4 < \frac{7 - 3\sqrt{5}}{2} < \pi,$$

so $y_1^4 - \pi < 0$. However,

$$y_2 > \frac{3}{2}, \quad \text{so} \quad y_2^4 > \frac{81}{16} > \pi.$$

Thus,

$$x = y_2^4 - \pi = \left(\frac{1 + \sqrt{\sqrt{16\pi + 8} - 3}}{2}\right)^4 - \pi$$

is a positive solution of the equation

$$\sqrt[4]{\pi + x} = 1 + \sqrt[4]{\pi - x}.$$

Solution 4 by David A. Huckaby, Angelo State University, San Angelo, TX.

We have

$$\begin{aligned}
\sqrt[4]{\pi + x} - \sqrt[4]{\pi - x} &= 1 \\
\sqrt{\pi + x} - 2\sqrt[4]{\pi^2 - x^2} + \sqrt{\pi - x} &= 1 \\
\sqrt{\pi + x} + \sqrt{\pi - x} &= 1 + 2\sqrt[4]{\pi^2 - x^2} \\
\pi + x + 2\sqrt{\pi^2 - x^2} + \pi - x &= 1 + 4\sqrt[4]{\pi^2 - x^2} + 4\sqrt{\pi^2 - x^2} \\
2\pi - 1 - 2\sqrt{\pi^2 - x^2} &= 4\sqrt[4]{\pi^2 - x^2} \\
4\pi^2 - 4\pi + 1 - 4(2\pi - 1)\sqrt{\pi^2 - x^2} + 4(\pi^2 - x^2) &= 16\sqrt{\pi^2 - x^2} \\
8\pi^2 - 4\pi + 1 - 4x^2 &= (8\pi + 12)\sqrt{\pi^2 - x^2} \\
16x^4 - 64\pi^2 x^2 + 32\pi x^2 - 8x^2 + 64\pi^4 - 64\pi^3 + 32\pi^2 - 8\pi + 1 & \\
= (64\pi^2 + 192\pi + 144)(\pi^2 - x^2) & \\
16x^4 - 64\pi^2 x^2 + 32\pi x^2 - 8x^2 + 64\pi^4 - 64\pi^3 + 32\pi^2 - 8\pi + 1 & \\
= 64\pi^4 + 192\pi^3 + 144\pi^2 - 64\pi^2 x^2 - 192\pi x^2 - 144x^2 & \\
16x^4 + (224\pi + 136)x^2 + (-256\pi^3 - 112\pi^2 - 8\pi + 1) &= 0.
\end{aligned}$$

This is a quadratic equation in x^2 whose possible solutions are

$$\begin{aligned}
x^2 &= \frac{1}{32}\left[-(224\pi + 136) \pm \sqrt{(224\pi + 136)^2 - 4(16)(-256\pi^3 - 112\pi^2 - 8\pi + 1)}\right] \\
&= \frac{1}{32}\left[-(224\pi + 136) \pm \sqrt{64(28\pi + 17)^2 - 64(-256\pi^3 - 112\pi^2 - 8\pi + 1)}\right] \\
&= \frac{1}{32}\left[-(224\pi + 136) \pm 8\sqrt{784\pi^2 + 952\pi + 289 + 256\pi^3 + 112\pi^2 + 8\pi - 1}\right] \\
&= \frac{1}{32}\left[-(224\pi + 136) \pm 8\sqrt{256\pi^3 + 896\pi^2 + 960\pi + 288}\right] \\
&= \frac{1}{32}\left[-(224\pi + 136) \pm 8\sqrt{32(8\pi^3 + 28\pi^2 + 30\pi + 9)}\right] \\
&= \frac{1}{32}\left[-(224\pi + 136) \pm 32\sqrt{2(2\pi + 1)(2\pi + 3)^2}\right] \\
&= -7\pi - \frac{17}{4} \pm (2\pi + 3)\sqrt{2(2\pi + 1)}.
\end{aligned}$$

Since x is real, we reject the solution with the minus sign. Since $-7\pi-\frac{17}{4}+(2\pi+3)\sqrt{2(2\pi+1)} \approx 9.19 > 0$, the positive real solution to the original equation is $x = \sqrt{-7\pi - \frac{17}{4} + (2\pi+3)\sqrt{2(2\pi+1)}}$.

Solution 5 by David E. Manes, Oneonta, NY.

If $\sqrt[4]{\pi + x} = 1 + \sqrt[4]{\pi - x}$ and $0 < x < \pi$, then

$$x = \pi - \left(\frac{1}{16}\right)\left(\sqrt{\sqrt{8+16\pi}-3}-1\right)^4 \approx 3.031\,332\,625\,09.$$

Let $y = \sqrt[4]{\pi - x}$. Then $y^4 = \pi - x$ so that $x = \pi - y^4$. Writing the given equation in terms of y, one obtains $\sqrt[4]{2\pi - y^4} = y + 1$. Eliminating the fractional power, we get

$$2\pi - y^4 = \left(\sqrt[4]{2\pi - y^4}\right)^4 = (y+1)^4 = y^4 + 4y^3 + 6y^2 + 4y + 1$$

or $2y^4 + 4y^3 + 6y^2 + 4y + (1-2\pi) = 0$. Dividing each term by 2 yields the monic polynomial equation

$$P(y) = y^4 + 2y^3 + 3y^2 + 2y + \left(\frac{1-2\pi}{2}\right) = 0.$$

One now writes the polynomial equation $P(y) = 0$ as the difference of two quadratic squares; that is,

$$P(y) = y^4 + 2y^3 + 3y^2 + 2y + \left(\frac{1-2\pi}{2}\right) = (y^2 + hy + k)^2 - (uy+v)^2 = 0,$$

for some real numbers h, k, u, v. To find the values of h, k, u and v, we expand the two squares and equate like coefficients of powers with $P(y)$. In doing so, we get the following simultaneous conditions,

$$2 = 2h,\ \ 3 = h^2 + 2k - u^2,\ \ 2 = 2hk - 2uv,\ \ \frac{1-2\pi}{2} = k^2 - v^2.$$

From the identity $4u^2v^2 = (2uv)^2 = 0$, one finds that that k has to satisfy the cubic equation $2k^3 - 3k^2 + (1+2\pi)k - 2\pi = 0$, an equation that has the immediate and the only real solution, $k = 1$. Then $h = k = 1$, $u = 0$, and $v = \pm\sqrt{(1+2\pi)/2}$. WLOG, we use the positive radical for v. Then

$$P(y) = \left(y^2 + y + 1\right)^2 - \left(\sqrt{\frac{1+2\pi}{2}}\right)^2 = 0.$$

Therefore, either $(y^2 + y + 1 + (\sqrt{(1+2\pi)/2}) = 0$ or $(y^2 + y + 1 - (\sqrt{(1+2\pi)/2}) = 0$. The first quadratic equation has only complex solutions and the second equation has real solutions given by

$$y = \frac{-1 \pm \sqrt{1 - 4\left(1 - \sqrt{\dfrac{1+2\pi}{2}}\right)}}{2}$$
$$= \frac{-1 \pm \sqrt{1 - 4 + 4\left(\dfrac{\sqrt{2+4\pi}}{2}\right)}}{2}$$
$$= \frac{-1 \pm \sqrt{-3 + 2\sqrt{2+4\pi}}}{2}$$
$$= \frac{-1 \pm \sqrt{\sqrt{8+16\pi} - 3}}{2}.$$

The negative radical yields a value of $y^4 > \pi$, thereby giving a value of $x < 0$, a contradiction. Hence,

$$y = \frac{-1 + \sqrt{\sqrt{8+16\pi} - 3}}{2} \approx 0.576\,241\,490\,484$$

so that

$$x = \pi - y^4 = \pi - \left(\frac{1}{16}\right)\left(\sqrt{\sqrt{8+16\pi} - 3} - 1\right)^4 \approx 3.031\,332\,625\,09.$$

This completes the solution.

Solution 6 by the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.

A positive real solution is

$$x = \sqrt{(2\pi + 3)\sqrt{4\pi + 2} - 7\pi - 17/4} \approx 3.03133.$$

We repeatedly manipulate and square both sides.

$$\begin{aligned}
\left(\sqrt[4]{\pi + x} - \sqrt[4]{\pi - x}\right)^2 &= 1^2 \\
\sqrt{\pi + x} - 2\sqrt[4]{\pi^2 - x^2} + \sqrt{\pi - x} &= 1 \\
\left(\sqrt{\pi + x} + \sqrt{\pi - x}\right)^2 &= \left(2\sqrt[4]{\pi^2 - x^2} + 1\right)^2 \\
2\pi - 1 - 2\sqrt{\pi^2 - x^2} &= 4\sqrt[4]{\pi^2 - x^2} \\
(2\pi - 1)^2 - 4(2\pi - 1)\sqrt{\pi^2 - x^2} + 4(\pi^2 - x^2) &= 16\sqrt{\pi^2 - x^2} \\
(2\pi - 1)^2 + 4(\pi^2 - x^2) &= 4(2\pi + 3)\sqrt{\pi^2 - x^2} \\
(2\pi - 1)^4 + 8(2\pi - 1)^2(\pi^2 - x^2) + 16(\pi^2 - x^2)^2 &= 16(2\pi + 3)^2(\pi^2 - x^2) \\
16(\pi^2 - x^2)^2 - 8(4\pi^2 + 28\pi + 17)(\pi^2 - x^2) + (2\pi - 1)^4 &= 0.
\end{aligned}$$

As a quadratic expression of $\pi^2 - x^2$, the determinant is

$$\begin{aligned}
\Delta &= 64(4\pi^2 + 28\pi + 17)^2 - 64(2\pi - 1)^4 \\
&= 64\left[(4\pi^2 + 28\pi + 17)^2 - (2\pi - 1)^4\right] \\
&= 64(4\pi^2 + 28\pi + 17 + 4\pi^2 + 4\pi + 1)(4\pi^2 + 28\pi + 17 - 4\pi^2 + 4\pi - 1) \\
&= 64(8\pi^2 + 24\pi + 18)(32\pi + 16) \\
&= 2^{11}(2\pi + 3)^2(2\pi + 1).
\end{aligned}$$

Using the quadratic formula gives

$$\begin{aligned}
\pi^2 - x^2 &= \frac{8(4\pi^2 + 28\pi + 17) \pm 32(2\pi + 3)\sqrt{4\pi + 2}}{32} \\
-x^2 &= 7\pi + 17/4 \pm (2\pi + 3)\sqrt{4\pi + 2} \\
x^2 &= (2\pi + 3)\sqrt{4\pi + 2} - 7\pi - 17/4 \\
x &= \sqrt{(2\pi + 3)\sqrt{4\pi + 2} - 7\pi - 17/4} \approx 3.03133
\end{aligned}$$

as the only positive real solution.

Solution 7 by G. C. Greubel, Newport News, VA.

Consider the equation

$$\sqrt[4]{a + x} = 1 + \sqrt[4]{a - x}$$

for which $\sqrt[4]{a + x} - \sqrt[4]{a - x} = 1$ and the square of both sides leads to

$$\begin{aligned}
\sqrt[2]{a + x} - 2\sqrt[4]{a^2 - x^2} + \sqrt[2]{a - x} &= 1 \\
\sqrt{a + x} + \sqrt{a - x} &= 1 - 2\sqrt[4]{a^2 - x^2}.
\end{aligned}$$

Square both sides to obtain

$$\sqrt{a^2 - x^2} + 2\sqrt[4]{a^2 - x^2} - \frac{2a - 1}{2} = 0.$$

Let $t = \sqrt[4]{a^2 - x^2}$ to obtain the quadratic equation

$$t^2 + 2t - \frac{2a - 1}{2} = 0.$$

This equation yields the solution

$$t = -1 \pm \frac{\sqrt{4a + 2}}{2}.$$

Another component required is t^4, as will be seen later. In this view then:

$$\begin{aligned} t^4 &= \left(-2\,t + \frac{2a-1}{2}\right)^2 \\ &= 4t^2 - 2(2a-1)\,t + \frac{(2a-1)^2}{4} \\ &= -2(2a+3)\,t + \frac{(2a-1)(2a+7)}{4} \\ &= a^2 + \frac{28a+17}{4} - \pm(2a+3)\,\sqrt{4a+2}. \end{aligned}$$

Now, returning to $t = \sqrt[4]{a^2 - x^2}$ then it is seen that $x^2 = a^2 - t^4$ and yields

$$x^2 = \pm(2a+3)\,\sqrt{4a+2} - \frac{28a+17}{4}.$$

The real solution then can be considered to be

$$x = \pm\frac{1}{2}\,\sqrt{4(2a+3)\,\sqrt{4a+2} - (28a+17)}.$$

For the case of $a = \pi$ then the real solution is:

$$x = \frac{1}{2}\,\sqrt{4(2\pi+3)\,\sqrt{4\pi+2} - (28\pi+17)}.$$

Solution 8 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

We note that the solution must lie in the closed interval $[0, \pi]$. Noting that the sum of the radicands is a constant, 2π, we introduce two new variables: $u = \sqrt[4]{\pi + x} \geqslant 0$ and $v = \sqrt[4]{\pi - x} \geqslant 0$. Thus we have the system of equations

$$u - v = 1 \tag{1}$$

$$u^4 + v^4 = 2\pi. \tag{2}$$

Equation (1) gives us $v = u - 1$ and (2) yields $u^4 + (u-1)^4 = 2\pi$. Making the substitution $u = y + 1/2$, we have $(y+1/2)^4 + (y-1/2)^4 = 2\pi$. Expanding and simplifying, this last equation becomes $16y^4 + 24y^2 + 1 = 16\pi$, and the substitution $z = y^2 \geqslant 0$ gives us $16z^2 + 24z + (1 - 16\pi) = 0$. The quadratic formula provides the solutions

$$z = \frac{-24 \pm \sqrt{512 + 1024\pi}}{32} = \frac{-3 \pm 2\sqrt{2+4\pi}}{4}.$$

We reject the negative solution since $z \geqslant 0$. Therefore we have $z = \left(-3 + 2\sqrt{2+4\pi}\right)/4$. Reversing our substitutions, we find that

$$y = \sqrt{z} = \frac{\sqrt{-3 + 2\sqrt{2+4\pi}}}{2}, \quad u = y + \frac{1}{2} = \frac{\sqrt{-3 + 2\sqrt{2+4\pi}} + 1}{2},$$

$$x = u^4 - \pi = \frac{(\sqrt{2+4\pi} - 1)\sqrt{-3 + 2\sqrt{2+4\pi}}}{2} \approx 3.0313.$$

Solution 9 by Michel Bataille, Rouen, France.

The number x must satisfy $x \leqslant \pi$, hence all amounts to seeking $u \in [0,1)$ such that $x = \pi \cdot \dfrac{1-u^4}{1+u^4}$.
Since $2x = (\sqrt[4]{\pi+x})^4 - (\sqrt[4]{\pi-x})^4$ and $a^4 - b^4 = (a-b)(a^3+a^2b+ab^2+b^3)$, the given equation is equivalent to

$$2x = (\sqrt[4]{\pi+x})^3 + \sqrt{\pi+x}\sqrt[4]{\pi-x} + \sqrt[4]{\pi+x}\sqrt{\pi-x} + (\sqrt[4]{\pi-x})^3.$$

Since $\pi + x = \dfrac{2\pi}{1+u^4}$ and $\sqrt[4]{\dfrac{\pi-x}{\pi+x}} = u$, we obtain the equation for u:

$$\frac{2\pi(1-u^4)}{1+u^4} = \frac{(2\pi)^{3/4}}{(1+u^4)^{3/4}} \cdot (1+u+u^2+u^3).$$

Since $1+u+u^2+u^3 = \dfrac{1-u^4}{1-u}$, the latter easily becomes $2\pi(1-u)^4 = 1+u^4$, which, expanded and arranged, yields

$$u^4(2\pi-1) - 8\pi u^3 + 12\pi u^2 - 8\pi u + (2\pi-1) = 0.$$

We solve this palindromic equation in a classical way: we set $X = u + \dfrac{1}{u}$ and we are led to $(2\pi-1)X^2 - 8\pi X + 8\pi + 2 = 0$. Since $X > 2$, we have $X = X_0$ where $X_0 = \dfrac{4\pi + \sqrt{4\pi+2}}{2\pi-1}$. Then, we get u from $p(u) = 0$ where $p(u) = u^2 - uX_0 + 1$. Since $p(0) > 0$ and $p(1) = 2 - X_0 < 0$, the desired u is the least root, that is, $u = \dfrac{X_0 - \sqrt{X_0^2-4}}{2}$.
Finally,

$$x = \pi \cdot \frac{16 - (X_0 - \sqrt{X_0^2-4})^4}{16 + (X_0 - \sqrt{X_0^2-4})^4}$$

where $X_0 = \dfrac{4\pi + \sqrt{4\pi+2}}{2\pi-1}$.

Solution 10 by Péter Fülöp, Gyömrő, Hungary.

We are looking for the solution among real numbers, so it must be stated that there cannot be a negative number under the root:

So $(\pi - x) \geqslant 0$ and $(\pi + x) \geqslant 0$ that is $|x| \leqslant \pi$

$$\sqrt[4]{\pi+x} - \sqrt[4]{\pi-x} = 1$$

Multiply both sides by $(\sqrt[4]{\pi + x} + \sqrt[4]{\pi - x})$ and add the eqution to the previous one, we get:

$$2\sqrt[4]{\pi + x} = 1 + \sqrt{\pi + x} - \sqrt{\pi + x}$$

Squaring the equation and performing the possible cancellations we get:

$$\frac{2\pi + 1}{2} = \sqrt{\pi + x} + \sqrt{\pi - x} + \sqrt{\pi + x}\sqrt{\pi - x}$$

Let's introduce two new variables:

$$a = \sqrt{\pi + x} \text{ and } b = \sqrt{\pi - x}$$

We got a simple two variables equation system:

I. $a + b + ab = \dfrac{2\pi + 1}{2}$
II. $\sqrt{a} - \sqrt{b} = 1$

Squaring the equation II. we get: $a + b - 2\sqrt{ab} = 1$ then put the value of $(a + b)$ to the equation I., we will have a second order equation in $\sqrt{ab}$

$$\frac{2\pi + 1}{2} = (\sqrt{ab} + 1)^2$$

$$\sqrt{ab} = \sqrt{\pi^2 - x^2}$$

We get the roots:

$$x_i = \pm\sqrt{\pi^2 - \left[-1 \pm \sqrt{\frac{1 + 2\pi}{2}}\right]^4}, \text{ where } i = 1, 2, 3, 4$$

After checking the roots in the original equation, remains only one valid root:

$$x_0 = \sqrt{\pi^2 - \left[-1 + \sqrt{\frac{1 + 2\pi}{2}}\right]^4} \approx 3.0313326251$$

whitch is less then π.

Solution 11 by Seán M. Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia.

We show a positive real number x that satisfies the equation is

$$x = \frac{1}{2}\sqrt{4(3 + 2\pi)\sqrt{4\pi + 2} - 28\pi - 17}.$$

Write the equation as $\sqrt{a} - \sqrt{b} = 1$, where

$$a = \sqrt{\pi + x} > 0 \quad \text{and} \quad b = \sqrt{\pi - x} > 0.$$

So

$$a^2 + b^2 = (\pi + x) + (\pi - x) = 2\pi. \tag{3}$$

And since $\sqrt{a} = 1 + \sqrt{b}$, on squaring and rearranging we have

$$a - b - 1 = 2\sqrt{b}.$$

Squaring and rearranging again we find

$$2a + 2ab + 2b = a^2 + b^2 + 1 = 2\pi + 1, \tag{4}$$

where the result in (3) has been used. Adding (3) and (4) produces

$$(a + b)^2 + 2(a + b) - 4\pi - 1 = 0.$$

On solving this quadratic equation for $a + b$ we find

$$a + b = -1 + \sqrt{4\pi + 2}.$$

Note here the positive case is taken since $a, b > 0$. Substituting this result into (4) yields

$$ab = \frac{2\pi + 3 - 2\sqrt{4\pi + 2}}{2} = \alpha.$$

Since

$$ab = \sqrt{\pi + x}\sqrt{\pi - x} = \sqrt{\pi^2 - x^2} = \alpha,$$

on solving for x we find

$$\begin{aligned} x = \sqrt{\pi^2 - \alpha^2} &= \sqrt{\pi^2 - \frac{1}{4}\left(2\pi + 3 - 2\sqrt{4\pi + 2}\right)^2} \\ &= \frac{1}{2}\sqrt{4(3 + 2\pi)\sqrt{4\pi + 2} - 28\pi - 17}, \end{aligned}$$

as announced. Note the positive value for the square root has been taken since for a solution to the original equation we require $x > 0$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Albert Stadler, Herrliberg, Switzerland and the proposer.

• **5671** *Proposed by Michael Brozinsky, Central Islip, New York.*

Isosceles triangle $\triangle RST$ with $RS = ST$ has the following property:

There are only three points such that the distances from each of these points to the lines $\overleftrightarrow{RT}$, $\overleftrightarrow{RS}$ and $\overleftrightarrow{ST}$ have, respectively, the same ratios as $1:2:3$.

Determine the angles of triangle $\triangle RST$.

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

We may assume that the triangle is given by the following coordinates:
R(-1,0), S(0,h), T(1,0), h>0. Let P(u,v) be a point whose distances from the lines RT, RS, ST have ratios as 1:2:3, respectively. The point of intersection of the two lines $y = \frac{1}{h}(x-u)+v$ and $y = h - hx$ equals $(x, y) = \left(\frac{h^2+u-hv}{1+h^2}, \frac{h(1-u+hv)}{1+h^2}\right)$. The distance from (u,v) to the line ST is equal to the distance from (u,v) to (x,y) and equals

$$\frac{|hu+v-h|}{\sqrt{1+h^2}}.$$

Similarly, the distance from (u,v) to the line RS equals $\frac{|-hu+v-h|}{\sqrt{1+h^2}}$ and the distance from (u,v) to the line RT equals $|v|$. By assumption,

$$|v| : \frac{|-hu+v-h|}{\sqrt{1+h^2}} : \frac{|hu+v-h|}{\sqrt{1+h^2}} = 1:2:3$$

implying the two equations
$4v^2 = \frac{(-hu+v-h)^2}{1+h^2}$ and $9\frac{(-hu+v-h)^2}{1+h^2} = 4\frac{(hu+v-h)^2}{1+h^2}$.
We solve for u and v and find the four solutions (u, v):

$$\left(\frac{\sqrt{1+h^2}}{2-5\sqrt{1+h^2}}, \frac{2h}{2-5\sqrt{1+h^2}}\right), \left(\frac{-\sqrt{1+h^2}}{2+5\sqrt{1+h^2}}, \frac{2h}{2+5\sqrt{1+h^2}}\right),$$

$$\left(\frac{5\sqrt{1+h^2}}{2-\sqrt{1+h^2}}, \frac{2h}{2-\sqrt{1+h^2}}\right), \left(-\frac{5\sqrt{1+h^2}}{2+\sqrt{1+h^2}}, \frac{2h}{2+\sqrt{1+h^2}}\right).$$

These four solutions collapse to three solutions exactly if $h = \sqrt{3}$ which means that the triangle RST is equilateral with three equal angles of $\pi/3$.

Solution 2 by the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.

The angles of ΔRST all have measure $\frac{\pi}{3}$; the triangle is equilateral.

Notice that there are two points X on the line (RT) satisfying $3XR = 2XT$. Let S_1 be the point between R and T satisfying this property (so S_1 is $2/5$ of the way from R to T), and let S_2 be the point on the opposite side of R from T satisfying this property (so R is one third of the way from T to S_2). If P is any point on the line S_1S or S_2S, then draw the line through P parallel to (RT), intersecting (RS) and (ST) at R' and T', respectively. Let Y and Z be the feet of P on (RS) and (ST), respectively. Then the right triangles $PR'Y$ and $PT'Z$ are similar, so

$$\frac{PY}{PZ} = \frac{PR'}{PT'} = \frac{S_iR}{S_iT} = \frac{2}{3},$$

and the distances from P to the lines (RS) and (ST) have the ratio $2 : 3$.

We claim that the points X on the line (ST) whose distances to (RT) and (RS) have the ratio $1 : 2$ are the points that satisfy $XT/XS = \cos\angle TRS$. If X is a point on (ST) with $XT/XS = \cos\angle TRS$, then let X' be the intersection of (RS) with the line through X parallel to (RT), and let Y and Z be the feet of X on (RT) and (RS), respectively. Then

$$\frac{XT}{XS} = \cos\angle TRS = \cos\angle X'XS = \frac{XX'/2}{XS},$$

so $\dfrac{XT}{XX'} = \dfrac{1}{2}$. Since $\angle XX'Z \cong \angle TRS \cong \angle YTX$, then $\Delta XYT \sim \Delta XZX'$ and

$$\frac{XY}{XZ} = \frac{XT}{XX'} = \frac{1}{2}.$$

Let R_1 be the point between S and T satisfying $\dfrac{R_1T}{R_1S} = \cos\angle TRS$, and let R_2 be the point on the opposite side of T from S satisfying $\dfrac{R_2T}{R_2S} = \cos\angle TRS$. As before, if P is any point on the lines (RR_1) or (RR_2), then by a similar argument, the distances from P to the lines (RT) and (RS) have the ratio $1 : 2$. Thus, a point has distances to the lines (RT), (RS), and (ST) in the ratio $1 : 2 : 3$ if and only if

$$P \in \left((SS_1) \cup (SS_2)\right) \cap \left((RR_1) \cup (RR_2)\right).$$

If we orient ΔRST with S above the horizontal line (RT), then (SS_1), (SS_2) and (RR_1) have positive slope, while (RR_2) has negative slope. Thus, (RR_2) always intersects (SS_1) and (SS_2). By construction, (SS_1) and (RR_1) must intersect in the interior of ΔRST. In most cases there would be a fourth point of intersection, but if there are only three such points, then the lines (SS_2) and (RR_1) must be parallel.

In this case $\Delta RR_1T \sim \Delta S_2ST$, so that

$$\frac{R_1T}{ST} = \frac{RT}{S_2T} = \frac{1}{3},$$

and

$$\cos\angle TRS = \frac{R_1T}{R_1S} = \frac{1}{2}.$$

Thus, all three angles of ΔRST have measure $\dfrac{\pi}{3}$.

If the equilateral triangle ΔRST has coordinates $R(-1,0)$, $S(0,\sqrt{3})$, and $T(1,0)$, then $S_1 = (-\frac{1}{5},0)$, $S_2 = (-5,0)$, $R_1 = (\frac{2}{3},\frac{\sqrt{3}}{3})$, $R_2 = (2,-\sqrt{3})$, and the three points with the desired distances are

$$(SS_1) \cap (RR_1) = (-\frac{1}{6},\frac{\sqrt{3}}{6}),$$

$$(SS_1) \cap (RR_2) = (-\frac{1}{4},-\frac{\sqrt{3}}{4}),$$

and

$$(SS_2) \cap (RR_2) = (-\frac{5}{2},\frac{\sqrt{3}}{2}).$$

Also solved by the proposer.

• **5672** *Proposed by Nikos Ntorvas, Athens, Greece.*

Given

$$F(x,y) = (y-x)\Big[y\left(3y^2-28y+3xy-14x+84\right) +x\left(3x^2-28x+3xy-14y+84\right)-96\Big],$$

where $x, y \in \mathbb{R}$, with $0 \leqslant x < y$, find $A = \min F(x,y)$ and the corresponding minimizing values for x and y.

Solution 1 by Péter Fülöp, Gyömrő, Hungary.

A necessary and sufficient condition for the existence of an minimum value are the followings:

Necessary conditions: $\frac{dF}{dx} = 0$ and $\frac{dF}{dy} = 0$. Solution of the equation system results the stacionary points.

Sufficient conditions: $D = \frac{d^2F}{dx^2}\frac{d^2F}{dy^2} - (\frac{d^2F}{dxdy})^2 > 0$ and $\frac{d^2F}{dx^2} > 0$ in the stacioner points.

First order partial derivatives

After performing the derivations:

$$\frac{dF}{dx} = -12(x^3-7x^2+14x-8) = 0$$

and

$$\frac{dF}{dy} = 12(y^3-7y^2+14y-8) = 0$$

From the necessery conditions we get the stacionary points:

$$\begin{aligned} x_1 &= 1, & y_1 &= 1 \\ x_2 &= 2, & y_2 &= 2 \\ x_3 &= 4, & y_3 &= 4 \end{aligned}$$

Second order partial derivatives

$$\frac{d^2F}{dx^2} = -36x^2 + 168x - 168$$
$$\frac{d^2F}{dy^2} = +36y^2 - 168y + 168$$
$$\frac{d^2F}{dxdy} = \frac{d^2F}{dydx} = 0$$

Determination of the minimum location and value

Nine possible pairs can be made from the stationary points:

Extreme value determination				
Stationery points	Determinant	d^2F/dx^2	Type	Value
F(1,1)	-1296	na	na	na
F(1,2)	+864	-36	max	+5
F(1,4)	-2592	na	na	na
F(2,1)	+864	24	min	-5
F(2,2)	-576	na	na	na
F(2,4)	+1728	+24	**min**	**-32**
F(4,1)	-2592	na	na	na
F(4,2)	+1728	-72	max	+32
F(4,4)	-5184	na	na	na

We have two points where F(x,y) has local minimums and one where the $x < y$ condition is completely:

$$F(2,4) = -32$$

Solution 2 by Michel Bataille, Rouen, France.

Let Ω be the open subset of $\mathbb{R}^2$ defined by $\Omega = \{(x,y) : x < y\}$. We show that F has a unique local minimum on Ω, namely at $(2,4)$ with $F(2,4) = -32$, but no absolute minimum on Ω.

Expanding and arranging readily shows that $F(x,y) = p(y) - p(x)$ where p is the polynomial $p(t) = 3t^4 - 28t^3 + 84t^2 - 96t$. Since $p'(t) = 12(t-1)(t-2)(t-4)$, the study of the variations of the function p is easy. Note in particular that $\lim_{t\to-\infty} p(t) = \lim_{t\to\infty} p(t) = \infty,\ p(1) = -37,\ p(2) =$

$-32,\ p(4) = -64$ and that $p(t) \geqslant -64$ for all real t.

Clearly, F is a C^∞ function on Ω and $\frac{\partial F}{\partial x}(x,y) = -p'(x)$, $\frac{\partial F}{\partial y}(x,y) = p'(y)$ so that Ω contains three critical points: $(1,2)$, $(1,4)$, $(2,4)$. We have $\frac{\partial^2 F}{\partial x^2}(x,y) = -p''(x)$, $\frac{\partial^2 F}{\partial y^2}(x,y) = p''(y)$ and $\frac{\partial^2 F}{\partial x \partial y}(x,y) = 0$ with $p''(t) = 12(3t^2 - 14t + 14)$. We readily obtain $p''(1) = 36$, $p''(2) = -24$, $p''(4) = 72$ and we deduce that

- $\frac{\partial^2 F}{\partial x^2}(1,2) = -36$, $\frac{\partial^2 F}{\partial y^2}(1,2) = -24$ and therefore $F(1,2) = 5$ is a local maximum of F on Ω.
- $\frac{\partial^2 F}{\partial x^2}(1,4) = -36$, $\frac{\partial^2 F}{\partial y^2}(1,4) = 72$ and therefore $(1,4)$ is a saddle point of F in Ω.
- $\frac{\partial^2 F}{\partial x^2}(2,4) = 24$, $\frac{\partial^2 F}{\partial y^2}(2,4) = 72$ and therefore $F(2,4) = -32$ is a local minimum of F on Ω.

In addition, $F(0,4) = p(4) = -64 < -32$, hence $F(2,4)$ is not the (absolute) minimum of F on Ω. Thus, F has no absolute minimum on Ω. (This also follows from $\lim_{x\to-\infty} F(x,0) = -\infty$.)

Note. Let $\Omega' = \{(x,y) : \frac{8-2\sqrt{10}}{3} \leqslant x < y\}$. Then, $F(2,4)$ is the absolute minimum of F on Ω'. Since

$$p(t) + 32 = 3(t-2)^2\left(t - \frac{8-2\sqrt{10}}{3}\right)\left(t - \frac{8+2\sqrt{10}}{3}\right),$$

we have

$$p\left(\frac{8-2\sqrt{10}}{3}\right) = p\left(\frac{8+2\sqrt{10}}{3}\right) = -32$$

and from the variations of p we see that $p(t) \leqslant -32$ when $t \in \left[\frac{8-2\sqrt{10}}{3}, \frac{8+2\sqrt{10}}{3}\right]$. Therefore, $F(x,y) = p(y) - p(x) \geqslant -64 + 32 = -32$ when $\frac{8-2\sqrt{10}}{3} \leqslant x \leqslant \frac{8+2\sqrt{10}}{3}$ and $y > x$. In addition $F(x,y) > 0$ if $\frac{8+2\sqrt{10}}{3} < x < y$ since p is increasing on $(4,\infty)$. Thus, $F(x,y) \geqslant -32$ if $(x,y) \in \Omega'$.

Solution 3 by David A. Huckaby, Angelo State University, San Angelo, TX.

From the expanded form $F(x,y) = -3x^4 + 28x^3 - 84x^2 + 96x + 3y^4 - 28y^3 + 84y^2 - 96y$, it is clear that as $x \to -\infty$, $F(x, y_0) \to -\infty$ for any $y_0 \in \mathbb{R}$, and hence the function F has no absolute minimum on the domain $x < y$. We can consider relative minima. Note that $\frac{\partial F}{\partial x} = -12x^3 + 84x^2 - 168x + 96$,

so that $\frac{\partial F}{\partial x} = 0$ for $x = 1$, $x = 2$, and $x = 4$. Similarly, $\frac{\partial F}{\partial y} = 12y^3 - 84y^2 + 168y - 96 = 0$

for $y = 1$, $y = 2$, and $y = 4$. So on the domain $x < y$, the three points $(1,2)$, $(1,4)$, and $(2,4)$ are candidates for yielding relative minima. Now $\dfrac{\partial^2 F}{\partial x^2} = -36x^2 + 168x - 168$, and similarly

$\dfrac{\partial^2 F}{\partial y^2} = 36y^2 - 168y + 168$. Note that $\dfrac{\partial^2 F}{\partial y \partial x} = 0$. To perform a second partial derivative test, consider

$D(x,y) = \dfrac{\partial^2 F(x,y)}{\partial x^2} \cdot \dfrac{\partial^2 F(x,y)}{\partial y^2} - \dfrac{\partial^2 F(x,y)}{\partial y \partial x} = \dfrac{\partial^2 F(x,y)}{\partial x^2} \cdot \dfrac{\partial^2 F(x,y)}{\partial y^2} = (-36x^2 + 168x - 168)(36y^2 - 168y + 168)$. Since $D(1,2) = 864 > 0$, $D(1,4) = -2592 < 0$, and $D(2,4) = 1728 > 0$, the function F on the domain $x < y$ has relative minima for the points $(1,2)$ and $(2,4)$. These minima are $F(1,2) = 5$ and $F(2,4) = -32$.

Solution 4 by Brian Bradie, Christopher Newport University, Newport News, VA.

After simplification, $F(x,y) = g(y) - g(x)$, where $g(x) = 3x^4 - 28x^3 + 84x^2 - 96x$. Note

$$g'(x) \quad = \quad 12x^3 - 84x^2 + 168x - 96 = 12(x-1)(x-2)(x-4);$$

thus, $g'(x) > 0$ and g is increasing for $x > 4$. Let

$$R = \{(x,y) \in \mathbb{R} : 0 \leqslant x < y\}.$$

Along the boundary $x = 0$, $F(0,y) = g(y) - g(0) = g(y)$, so there are critical points at $(0,1)$, $(0,2)$, and $(0,4)$. In the interior of R, critical points are the simultaneous solutions of

$$\frac{\partial F}{\partial x} = -g'(x) = 0 \quad \text{and} \quad \frac{\partial F}{\partial y} = g'(y) = 0;$$

accordingly, there are critical points at $(1,2)$, $(1,4)$ and $(2,4)$. Now,

$$F(0,1) = -37, \quad F(0,2) = -32, \quad \text{and} \quad F(0,4) = -64,$$

and

$$F(1,2) = 5, \quad F(1,4) = -27, \quad \text{and} \quad F(2,4) = -32.$$

For fixed x with $0 \leqslant x \leqslant 4$ and $y > 4$, $F(x,y)$ is increasing, and for fixed x with $x > 4$ and $y > x$, $g(y) > g(x)$ so $F(x,y) > 0$. Moreover, as $x \to y$, $F(x,y) \to 0$. Thus, $A = \min F(x,y) = -64$, and this minimum is achieved for $x = 0$ and $y = 4$.

Solution 5 by Albert Stadler, Herrliberg, Switzerland.

It is easily verified that

$$F(x,y) = 12 \int_x^y (t-1)(t-2)(t-4)\,dt.$$

This representation shows that if x is not bounded from below $F(x,y)$ has no minimum, for

$$12 \int_x^{x+1} (t-1)(t-2)(t-4)\,dt \to -\infty$$

as x tends to $-\infty$. Instead we determine the minimum of $F(x,y)$ subject to the constraint $0 \leqslant x < y$. Clearly, $(t-1)(t-2)(t-4) \leqslant 0$ for $t \in [0,1] \cup [2,4]$. We note that

$$12\int_0^1 (t-1)(t-2)(t-4)\,dt = -37,$$

$$12\int_1^2 (t-1)(t-2)(t-4)\,dt = 5,$$

$$12\int_2^4 (t-1)(t-2)(t-4)\,dt = -32.$$

Therefore $A = -64$ and the minimum is assumed for $x = 0$ and $y = 4$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo and the proposer.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated! . . . And don't worry about making a mistake. All is well!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Recommendations

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to LaTeX.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Problem proposed to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣