Problems and Solutions

Albert Natian, Section Editor

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This section of the SSMA Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Prof. Albert Natian, Department of Mathematics, Los Angeles Valley College, 5800 Fulton Avenue, Valley Glen, CA, 91401, USA. It's highly preferable that you send your contributions via email.

To propose problems, email them to: **problems4ssma@gmail.com**

To propose solutions, email them to: **solutions4ssma@gmail.com**

Please follow the instructions for submission of problems and solutions provided at the end of this document. Your adherence to all submission requirements is of the greatest help in running this Section successfully. Thank you!

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted before August 15, 2022.

• **5691** *Proposed by Mihaly Bencze, Braşov, Romania and Neculai Stanciu, Buzău, Romania.*

Solve for real numbers $x \geqslant 1$:

$$2 + 2^x + 4^x + \log_7 \left(\frac{3^x + 5^x}{2 + 2^x + 4^x} \right) = 3^x + 5^x.$$

• **5692** *Proposed by Shivam Sharma, Delhi University, New Delhi, India.*

Prove that

$$\lim_{n\to\infty} \left(\frac{\sqrt[n+1]{\binom{n+1}{1}\binom{n+1}{2}\cdots\binom{n+1}{n+1}}}{e^{\left(\frac{n+1}{2}\right)}(n+1)^{-\frac{3}{2}}} - \frac{\sqrt[n]{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}}}{e^{\left(\frac{n}{2}\right)}(n)^{-\frac{3}{2}}} \right) = \frac{e}{\sqrt{2\pi}}.$$

• **5693** *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.*

Calculate the integral:

$$\int_0^\infty \frac{\ln x}{x^4 + x^2 + 1} dx.$$

• **5694** *Proposed by Michel Bataille, Rouen, France.*

Given a positive integers m and n, let $S_m(n) = \frac{1}{n^{m+1}} \sum_{k=1}^{n} k^m H_k$ where $H_k = \sum_{i=1}^{k} \frac{1}{i}$. Find real numbers λ_m and μ_m such that

$$\lim_{n\to\infty} (S_m(n) - \lambda_m \ln n - \mu_m) = 0.$$

• **5695** *Proposed by Narendra Bhandari and Yogesh Joshi, Nepal.*

Prove that

$$\int_1^{\sqrt{2}} \frac{xdx}{(1+x)\sqrt{(2-x^2)(x^2-1)}} = \frac{\pi}{2}\left(\frac{\Gamma^4\left(\frac{1}{4}\right)-8\pi^2}{(2\pi)^{3/2}\Gamma^2\left(\frac{1}{4}\right)}\right)$$

where $\Gamma(z) = \displaystyle\int_0^{\infty} x^{z-1}e^{-x}dx$ is Gamma function where the real part $\Re(z)$ of the complex number z is positive.

• **5696** *Proposed by Mohsen Soltanifar, University of Toronto, Toronto, Canada.*

The sequence $(A_n)_{n=1}^{\infty}$ of subsets of the set $\mathbb{R}$ of real numbers is said to be convergent if and only if the two sets

$$B_1 := \bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty} A_m \quad \text{and} \quad B_2 := \bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m$$

are identical. Otherwise, we say the sequence is divergent. For each of the following cases, construct as complicated and fanciful a divergent sequence $(A_n)_{n=1}^{\infty}$ as you can muster while using no more than an aggregate of **42** individual symbols (characters):

1. B_1 is empty and B_2 is non-empty.
2. B_1 is bounded and B_2 is unbounded.
3. B_1 is a singleton set and B_2 is a non-singleton set.
4. B_1 is finite and B_2 is infinite.
5. B_1 is countable and B_2 is uncountable.

Solutions

To Formerly Published Problems

• **5673** *Proposed by Goran Conar, Varaždin, Croatia.*

Let α,β,γ be angles of an arbitrary triangle. Prove the inequality

$$\alpha\cot\alpha+\beta\cot\beta+\gamma\cot\gamma \leqslant \frac{\pi}{\sqrt{3}}.$$

When does equality occur?

Solution 1 by Paolo Perfetti, dipartimento di matematica Università di "Tor Vergata", Roma, Italy.

We need that $x-\tan x<0$ for $0<x<\pi/2$ and $x-\tan x>0$ for $\pi/2<x<\pi$. The first one follows by the definition of tangent while in the second it suffices to observe that $\tan x < 0$ for $\pi/2 < x < \pi$.

$$\left(\frac{x}{\tan x}\right)'' = \frac{2\cos x}{(\sin x)^3}(x - \tan x) \leqslant 0, \qquad 0 < x < \pi$$

because when $0 < x < \pi/2$ the cosine is positive but $x - \tan x < 0$ while the opposite holds for $\pi/2 < x < \pi$.

We conclude that the function $x/\tan x$, $0 < x < \pi$ if concave yielding

$$3\left(\frac{\alpha}{3}\cot\alpha + \frac{\beta}{3}\cot\beta + \frac{\gamma}{3}\cot\gamma\right) \leqslant 3\frac{\alpha+\beta+\gamma}{3}\cot\frac{\alpha+\beta+\gamma}{3} = 3\frac{\pi}{3}\cot\frac{\pi}{3} = \frac{\pi}{\sqrt{3}}$$

and this concludes the proof.

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, Texas.

We begin with a result which will be used at a key point later.

Lemma: If $f(x) = x\cos x - \sin x$, then $f(x) < 0$ on $(0,\pi)$. Proof: We note first that $f(x)$ is continuous on $[0,\pi]$ and $f(0) = 0$. Further,

$$\begin{aligned} f'(x) &= -x\sin x + \cos x - \cos x \\ &= -x\sin x \\ &< 0 \end{aligned}$$

on $(0,\pi)$. Since $f(x)$ is continuous on $[0,\pi]$, $f(0) = 0$, and $f(x)$ is decreasing on $(0,\pi)$, it follows that $f(x) < 0$ on $(0,\pi)$. $\square$

To proceed with our solution, let $g(x) = x\cot x$ on $(0,\pi)$. Then, for all $x \in (0,\pi)$,

$$g'(x) = -x\csc^2 x + \cot x$$

and

$$\begin{aligned} g''(x) &= -x\left[2\csc x\left(-\csc x\cot x\right)\right] - \csc^2 x - \csc^2 x \\ &= 2x\csc^2 x\cot x - 2\csc^2 x \\ &= 2\csc^2 x\left(x\cot x - 1\right) \\ &= 2\frac{x\cos x - \sin x}{\sin^3 x}. \end{aligned}$$

Since the Lemma implies that $g''(x) < 0$ on $(0,\pi)$, it follows that $g(x)$ is concave down on $(0,\pi)$.Then, Jensen's Theorem and the condition $\alpha + \beta + \gamma = \pi$ imply that

$$
\begin{aligned}
\alpha \cot \alpha + \beta \cot \beta + \gamma \cot \gamma &= g(\alpha) + g(\beta) + g(\gamma) \\
&\leqslant 3g\left(\frac{\alpha+\beta+\gamma}{3}\right) \\
&= 3g\left(\frac{\pi}{3}\right) \\
&= 3\left(\frac{\pi}{3}\right)\cot\left(\frac{\pi}{3}\right) \\
&= \pi \cdot \frac{1}{\sqrt{3}} \\
&= \frac{\pi}{\sqrt{3}}. \qquad (1)
\end{aligned}
$$

Further, Jensen's Theorem and the condition $\alpha + \beta + \gamma = \pi$ imply that equality is attained in (1) if and only if $\alpha = \beta = \gamma = \dfrac{\pi}{3}$. ∎

Solution 3 by Albert Stadler, Herrliberg, Switzerland.

We note that the function $x \rightarrow x\cot x$ is concave, since $\dfrac{d^2}{dx^2}x\cot x = -2\left(\dfrac{1 - x\cot x}{\sin^2 x}\right) \leqslant 0$. Hence, by Jensen's inequality,

$$
\alpha\cot\alpha + \beta\cot\beta + \gamma\cot\gamma \leqslant 3\left(\frac{\alpha+\beta+\gamma}{3}\right)\cot\left(\frac{\alpha+\beta+\gamma}{3}\right) = \pi\cot\left(\frac{\pi}{3}\right) = \frac{\pi}{\sqrt{3}},
$$

with equality if and only if $\alpha=\beta=\gamma=\pi/3$.

Solution 4 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Function $f(x) = x\cot x$ is concave for $x \in [0, \pi]$ since $f''(x) = 2(x\cot x - 1) \leqslant 0$, so by Jensen's inequality

$$
\alpha \cot \alpha + \beta \cot \beta + \gamma \cot \gamma \leqslant 3\frac{\alpha+\beta+\gamma}{3}\cot\frac{\alpha+\beta+\gamma}{3} = 3\frac{\pi}{3}\cot\frac{\pi}{3} = \frac{\pi}{\sqrt{3}}
$$

where equality occurs if and only if $\alpha = \beta = \gamma = \dfrac{\pi}{3}$.

Solution 5 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Consider the function $f(x) = x\cot x$ for $x \in (0, \pi)$. Then

$$
f'(x) = \cot x - x\csc^2 x = \frac{\frac{1}{2}\sin 2x - x}{\sin^2 x}
$$

and

$$
f''(x) = 2\csc^2 x(x\cot x - 1).
$$

Because

$$\lim_{x\to 0^+} x\cot x = 1$$

and $\frac{1}{2}\sin 2x - x < 0$ for $x \in (0,\pi)$, it follows that $f'(x) < 0$, $x\cot x < 1$ and $f''(x) < 0$ for $x \in (0,\pi)$. Thus, by Jensen's inequality

$$\alpha\cot\alpha + \beta\cot\beta + \gamma\cot\gamma \leqslant 3\left(\frac{\alpha+\beta+\gamma}{3}\right)\cot\left(\frac{\alpha+\beta+\gamma}{3}\right) = \pi\cot\frac{\pi}{3} = \frac{\pi}{\sqrt{3}}.$$

Equality holds when $\alpha = \beta = \gamma = \frac{\pi}{3}$.

Solution 6 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

The function $f(x) = x\cot x$ is concave on $(0,\pi)$: $f''(x) = 2(\cot^2 x + 1)(x\cot x - 1) \leqslant 0$. Therefore we can apply Jensen's inequality to see that

$$\frac{\sum_{cyclic}\alpha\cot\alpha}{3} \leqslant \left(\frac{\alpha+\beta+\gamma}{3}\right)\cot\left(\frac{\alpha+\beta+\gamma}{3}\right),$$

or

$$\sum_{cyclic}\alpha\cot\alpha \leqslant 3\cdot\frac{\pi}{3}\cdot\cot\left(\frac{\pi}{3}\right) = \frac{\pi}{\sqrt{3}}.$$

Equality holds if and only if $\alpha = \beta = \gamma = \pi/3$.

Solution 7 by Michel Bataille, Rouen, France.

For $x \in (0,\pi)$, let $f(x) = x\cot x = \frac{x\cos x}{\sin x}$. We calculate

$$f'(x) = \frac{\sin x\cos x - x}{\sin^2 x}, \qquad f''(x) = \frac{2g(x)}{\sin^3 x}$$

where $g(x) = x\cos x - \sin x$. We have $g'(x) = -x\sin x < 0$ for $x \in (0,\pi)$, hence the function g is decreasing on $[0,\pi]$ and since $g(0) = 0$, it follows that $g(x) < 0$ for $x \in (0,\pi)$. As a result, $f''(x) < 0$ and f is strictly concave on $(0,\pi)$.
Now, Jensen's inequality gives

$$f(\alpha) + f(\beta) + f(\gamma) \leqslant 3f\left(\frac{\alpha+\beta+\gamma}{3}\right) = 3f(\pi/3),$$

that is,

$$\alpha\cot\alpha + \beta\cot\beta + \gamma\cot\gamma \leqslant 3\cdot\frac{\pi}{3}\cdot\frac{1}{\sqrt{3}} = \frac{\pi}{\sqrt{3}}.$$

Since f is strictly concave, equality holds if and only if $\alpha = \beta = \gamma$, that is, if and only if the triangle is equilateral.

Solution 8 by Péter Fülöp, Gyömrő, Hungary.

Let's see the left hand side of the inequality as a bivariate function, since $\gamma = 180^o - \alpha - \beta$.

Then let's look for the maximum value of this function in the range $0^o < \alpha < 180^o$ and $0^o < \beta < 180^o$.

$$f(\alpha,\beta) = \alpha\Big(\frac{1}{\tan(\alpha)} + \frac{1}{\tan(\alpha+\beta)}\Big) + \beta\Big(\frac{1}{\tan(\beta)} + \frac{1}{\tan(\alpha+\beta)}\Big) - \frac{\pi}{\tan(\alpha+\beta)}$$

A necessary and sufficient condition for the existence of a maximum value are the followings:

1. Necessary conditions: $\frac{df}{d\alpha} = 0$ and $\frac{df}{d\beta} = 0$. Solution of the equation system results the stationary points.

2. Sufficient conditions: $\frac{d^2f}{d\alpha^2}\frac{d^2f}{d\beta^2} - \Big(\frac{d^2f}{d\alpha d\beta}\Big)\Big(\frac{d^2f}{d\beta d\alpha}\Big) > 0$ and $\frac{d^2f}{d\alpha^2} < 0$ in the stationary points.

1. From the necessary conditions we get:

$$\frac{1}{\tan(\alpha)} - \frac{\alpha}{\sin^2(\alpha)} = \frac{1}{\tan(\beta)} - \frac{\beta}{\sin^2(\beta)}$$

The trivial solution arises: $\alpha = \beta$. Let's substitute back to $\frac{df}{d\alpha} = 0$ we get the following equation:

$$\frac{df}{d\alpha} = \frac{1}{\tan(\alpha)} + \frac{1}{\tan(2\alpha)} - \frac{\alpha}{\sin^2(\alpha)} - \frac{2\alpha - \pi}{\sin^2(2\alpha)} = 0$$

Which is true if $\alpha = \frac{\pi}{3}$, and $\beta = \gamma = \frac{\pi}{3}$ as well.

2. From the sufficient conditions we get:

$$\frac{d^2f}{d\alpha^2}_{|\alpha=\beta} = \frac{2}{\sin^2(\alpha)}\Big(\frac{\alpha}{\tan(\alpha)} - 1\Big) - \frac{4}{\sin^2(\pi - 2\alpha)}\Big(\frac{\alpha}{\tan(2\alpha)} + 1\Big)$$

Checking at $\alpha = \frac{\pi}{3}$ point it is equal to $8(\frac{\pi}{3\sqrt{3}} - 1)$, it is less then zero.

$f(\alpha,\beta)$ is symmetrical regarding α, β, so $\frac{d^2f}{d\beta^2}$ provides the same result as we have for $\frac{d^2f}{d\alpha^2}$.

Calculating the second order mixed partial derivatives they equal to zero, the sufficient conditions are met in the $\alpha = \beta = \gamma = \frac{\pi}{3}$ point. $f(\frac{\pi}{3}, \frac{\pi}{3})$ = maximal.

Let's calculate the function value in this point (or put the point into the statement):

$$f(\frac{\pi}{3},\frac{\pi}{3}) = \frac{\pi}{3}\Big(\frac{1}{\tan(\frac{\pi}{3})} + \frac{1}{\tan(\frac{2\pi}{3})}\Big) + \frac{\pi}{3}\Big(\frac{1}{\tan(\frac{\pi}{3}} + \frac{1}{\tan(\frac{2\pi}{3})}\Big) - \frac{\pi}{\tan(\frac{2\pi}{3})}$$

$$f(\frac{\pi}{3},\frac{\pi}{3}) = 2\frac{\pi}{3}\Big(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}\Big) + \frac{\pi}{3}\frac{1}{\sqrt{3}} = \frac{\pi}{\sqrt{3}}$$

The equality occurs when the triangle is equilateral.

Solution 9 by Toyesh Prakash Sharma (Student) Agra College, Agra, India.

Let, $f(x) = x\cot x$ then, $f'(x) = \cot x - x\csc^2 x \Rightarrow f''(x) = 2\csc^2 x\,(x\cot x - 1) < 0$ for $x \in \left(0, \frac{\pi}{2}\right)$ so, $f(x) = x\cot x$ *is convex in nature as a result of which using Jensen's inequality*

$$\frac{\alpha\cot\alpha + \beta\cot\beta + \gamma\cot\gamma}{3} \leqslant \left(\frac{\alpha+\beta+\gamma}{3}\right)\cot\left(\frac{\alpha+\beta+\gamma}{3}\right)$$

$$\alpha\cot\alpha + \beta\cot\beta + \gamma\cot\gamma \leqslant \pi\cot\left(\frac{\pi}{3}\right)$$

$$\alpha\cot\alpha + \beta\cot\beta + \gamma\cot\gamma \leqslant \frac{\pi}{\sqrt{3}}$$

And equality occur when $\alpha = \beta = \gamma = \dfrac{\pi}{3}$

Also solved by Michael Brozinsky, Central Islip, NY and the proposer.

• **5674** *Proposed by Kenneth Korbin, New York, NY.*

Find positive rational numbers x and y such that

$$\left[(x+iy)^7 + (x-iy)^7\right]^2 + \left[(y+ix)^7 + (y-ix)^7\right]^2 = 4,$$

where $i^2 = -1$.

Solution 1 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

Letting $x+iy = z$, we have $y+ix = i(x-iy) = i\,\overline{z}$ and $y - ix = -i(x+iy) = -iz$. Then

$$\begin{aligned}
4 &= \left[(x+iy)^7 + (x-iy)^7\right]^2 + \left[(y+ix)^7 + (y-ix)^7\right]^2 \\
&= \left[z^7 + \overline{z}^7\right]^2 + \left[(i\,\overline{z})^7 + (-i\,z)^7\right]^2 \\
&= \left(z^{14} + 2(z\overline{z})^7 + \overline{z}^{14}\right) + \left(-\overline{z}^{14} + 2(z\overline{z})^7 - z^{14}\right) \\
&= 4(z\overline{z})^7 \iff |z|^{14} = 1 \iff x^2 + y^2 = 1.
\end{aligned}$$

So we are looking for rational points (x, y) on the unit circle. For example, $(x, y) = \left(\frac{3}{5}, \frac{4}{5}\right)$ is a solution. Furthermore, elementary number theory tells us that every Pythagorean triple (a, b, c) corresponds to the rational point $\left(\frac{a}{c}, \frac{b}{c}\right)$. (For a deeper dive into these waters, see "The Group of Rational Points on the Unit Circle" by Lin Tan in the June 1996 issue of *Mathematics Magazine.*")

Solution 2 by David A. Huckaby, Angelo State University, San Angelo, TX.

Write $x + iy = re^{i\theta}$, where $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan\left(\frac{y}{x}\right)$. So

$$\begin{aligned}\left[(x+iy)^7 + (x-iy)^7\right]^2 &= [r^7e^{7i\theta} + r^7e^{-7i\theta}]^2 \\ &= r^{14}e^{14i\theta} + 2r^{14} + r^{14}e^{-14i\theta}.\end{aligned}$$

Since $y + xi$ is the reflection of $x + iy$ about the line $y = x$, $y + xi = re^{i(\frac{\pi}{2}-\theta)}$. So

$$\begin{aligned}\left[(y+ix)^7 + (y-ix)^7\right]^2 &= [r^7e^{7i(\frac{\pi}{2}-\theta)} + r^7e^{-7i(\frac{\pi}{2}-\theta)}]^2 \\ &= r^{14}e^{14i(\frac{\pi}{2}-\theta)} + 2r^{14} + r^{14}e^{-14i(\frac{\pi}{2}-\theta)} \\ &= -r^{14}e^{-14i\theta} + 2r^{14} - r^{14}e^{14i\theta},\end{aligned}$$

where we have used $e^{\pm 14i\frac{\pi}{2}} = (e^{\pm i\frac{\pi}{2}})^{14} = (\pm i)^{14} = -1$. Therefore

$$\begin{aligned}&\left[(x+iy)^7 + (x-iy)^7\right]^2 + \left[(y+ix)^7 + (y-ix)^7\right]^2 \\ &\qquad = r^{14}e^{14i\theta} + 2r^{14} + r^{14}e^{-14i\theta} - r^{14}e^{-14i\theta} + 2r^{14} - r^{14}e^{14i\theta} = 4r^{14}.\end{aligned}$$

So the original equation is $4r^{14} = 4$, or $r^{14} = 1$, that is, $(x^2 + y^2)^7 = 1$. Clearly one solution is $x = y = \frac{\sqrt{2}}{2}$.

Solution 3 by Albert Stadler, Herrliberg, Switzerland.

$$\left[(x+iy)^7 + (x-iy)^7\right]^2 + \left[(y+ix)^7 + (y-ix)^7\right]^2 =$$

$$= (x+iy)^{14} + 2\left(x^2+y^2\right)^7 + (x-iy)^{14} + (y+ix)^{14} + 2\left(x^2+y^2\right)^7 + (y-ix)^{14} =$$

$$= 4\left(x^2+y^2\right)^7,$$

since $(x+iy)^{14} = i^{14}(-ix+y)^{14} = -(-ix+y)^{14}$ and $(x-iy)^{14} = i^{14}(-ix-y)^{14} = -(ix+y)^{14}$. Hence $x^2 + y^2 = 1$, and we need to find the rational points (x,y)$\in\mathbb{Q}^2$ on the unit circle. This is equivalent to finding the integer solutions of $x^2 + y^2 = z^2$. It is well known (see for instance [1], Theorem 225) that the most general solution of the equation $x^2 + y^2 = z^2$ satisfying the conditions x>0, y>0, z>0, x and y coprime and x even is x=2ab, y=a^2-b^2, z=a^2+b^2, where a, b are integers

of opposite parity and a and b are coprime and a>b>0. There is a 1:1 correspondence between different values of a, b and different values of x,y,z.
So

$$\left\{ (x,y) \in \mathbb{Q}^2 \middle| x^2 + y^2 = 1 \right\} =$$

$$= \left\{ \left(\frac{2ab}{a^2+b^2}, \frac{a^2-b^2}{a^2+b^2} \right) \middle| a, b \in \mathbb{Z}, (a,b) \neq (0,0) \right\} \left\{ \left(\frac{a^2-b^2}{a^2+b^2}, \frac{2ab}{a^2+b^2} \right) \middle| a, b \in \mathbb{Z}, (a,b) \neq (0,0) \right\}.$$

References
[1] G.H. Hardy and E.M. Wright, An Introduction to the Theory of Number, 5^{th} edition, Oxford at the Clarendon Press, 1979.

Solution 4 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

If $z = x + iy$, then $\bar{z} = x - iy$, $y + ix = i\bar{z}$, and $y - ix = iz$ so the proposed equation may be written as $(z^7 + \bar{z}^7)^2 + (-i\bar{z}^7 - iz^7)^2 = 4$, which after some algebra becomes $4|z|^4 = 4$, so the solution is $|z| = 1$, that is $x \in (0,1) \cap Q$, and $y = \sqrt{1-x^2}$, with $y \in (0,1) \cap Q$. For example, $x = 3/5$, $y = 4/5$ is a solution to the given equation.

Solution 5 by Seán M. Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia.

Let $z = x + iy$. Then

$$-iz = y - ix, \quad \bar{z} = x - iy, \quad \text{and} \quad i\bar{z} = y + ix.$$

So we can rewrite the given equation as

$$\left(z^7 + \bar{z}^7\right)^2 + \left((i\bar{z})^7 + (-iz)^7\right)^2 = 4.$$

After expanding and simplifying one is left with $|z|^{14} = 1$. Since we are only interested in positive rational solutions for x and y, then $x^2 + y^2 = 1$.

Let (a,b,c) denote the set of all Pythagorean triples where a, b, and c are positive integers such that $0 < a < b < c$. The required solutions to the equation (there are an infinite number of them) are therefore given by

$$x = \frac{a}{c}, y = \frac{b}{c} \quad \text{or} \quad x = \frac{b}{c}, y = \frac{a}{c}.$$

As a simple example of such solutions, since $(3,4,5)$ is a Pythagorean triple two positive rational solutions to the equation will be:

$$x = \frac{3}{5},\ y = \frac{4}{5} \quad \text{or} \quad x = \frac{4}{5},\ y = \frac{3}{5}.$$

Solution 6 by Brian D. Beasley, Presbyterian College, Clinton, SC.

We first note that there are infinitely many pairs of positive rational numbers x and y such that $x^2 + y^2 = 1$; those corresponding to primitive Pythagorean triples have the form

$$x=\frac{2ab}{a^2+b^2}\quad \text{and}\quad y=\frac{a^2-b^2}{a^2+b^2},$$

where a and b are positive integers with $a>b$, $\gcd(a,b)=1$, and $a\not\equiv b \pmod 2$.

Next, we show that all such pairs (x,y) satisfy the given equation. We have

$$(x+iy)^7+(x-iy)^7=2(x^7-21x^5y^2+35x^3y^4-7xy^6);$$

similarly, we calculate

$$(y+ix)^7+(y-ix)^7=2(y^7-21y^5x^2+35y^3x^4-7yx^6).$$

Then substituting $y^2=1-x^2$ yields

$$u=x^7-21x^5y^2+35x^3y^4-7xy^6=64x^7-112x^5+56x^3-7x$$

and

$$v=y^7-21y^5x^2+35y^3x^4-7yx^6=y(-64x^6+80x^4-24x^2+1),$$

so we obtain

$$u^2=4096x^{14}-14336x^{12}+19712x^{10}-13440x^8+4704x^6-784x^4+49x^2$$

and

$$\begin{aligned} v^2 &= (1-x^2)(-64x^6+80x^4-24x^2+1)^2 \\ &= -4096x^{14}+14336x^{12}-19712x^{10}+13440x^8-4704x^6+784x^4-49x^2+1. \end{aligned}$$

Hence $u^2+v^2=1$ as needed.

Solution 7 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Let $x=re^{i\theta}$. Then

$$\begin{aligned} x-iy &= re^{-i\theta}, \\ y+ix &= re^{i(\frac{\pi}{2}-\theta)}=ire^{-i\theta},\ \text{and} \\ y-ix &= -ire^{i\theta}, \end{aligned}$$

so

$$(x+iy)^7+(x-iy)^7=2r^7\cos 7\theta,\quad (y+ix)^7+(y-ix)^7=-2r^7\sin 7\theta,$$

and

$$\Big[(x+iy)^7+(x-iy)^7\Big]^2+\Big[(y+ix)^7+(y-ix)^7\Big]^2=4r^{14}.$$

This will be equal to 4 provided $r=1$. It follows that x and y can be taken as the sine and the cosine values associated with any Pythagorean triple. In particular, we can take

$$x=\frac{3}{5},y=\frac{4}{5}\quad \text{or}\quad x=\frac{5}{13},y=\frac{12}{13}.$$

Solution 8 by the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.

One possible solution is $(x, y) = (3/5, 4/5)$.

Notice that $y + ix = i(x - iy)$ and $y - ix = -i(x + iy)$. Thus,

$$(y + ix)^7 + (y - ix)^7 = i^7(x - iy)^7 + (-i)^7(x + iy)^7 = i\left[(x + iy)^7 - (x - iy)^7\right],$$

and

$$\begin{aligned}\left[(y + ix)^7 + (y - ix)^7\right]^2 &= -\left[(x + iy)^7 - (x - iy)^7\right]^2 \\ &= -(x + iy)^{14} + 2(x + iy)^7(x - iy)^7 - (x - iy)^{14} \\ &= -(x + iy)^{14} + 2(x^2 + y^2)^7 - (x - iy)^{14}.\end{aligned}$$

Meanwhile,

$$\left[(x + iy)^7 + (x - iy)^7\right]^2 = (x + iy)^{14} + 2(x^2 + y^2)^7 + (x + iy)^{14},$$

so that

$$\left[(x + iy)^7 + (x - iy)^7\right]^2 + \left[(y + ix)^7 + (y - ix)^7\right]^2 = 4(x^2 + y^2)^7.$$

Thus, if the sum on the left is equal to 4, then $(x^2 + y^2)^7 = 1$ and $x^2 + y^2 = 1$. So, if (a, b, c) is any Pythagorean triple of positive integers, with $a^2 + b^2 = c^2$, then $x = a/c$ and $y = b/c$ are positive rational numbers that satisfy the given equation. One possible solution is $(x, y) = (3/5, 4/5)$.

Solution 9 by Bataille, Rouen, France.

Let $L = \left[(x + iy)^7 + (x - iy)^7\right]^2 + \left[(y + ix)^7 + (y - ix)^7\right]^2$ and let z be the complex number $x + iy$. Then, we have $x - iy = \overline{z}, y + ix = i\overline{z}$ and $y - ix = -iz$, hence

$$L = (z^7 + \overline{z}^7)^2 + ((i\overline{z})^7 + (-iz)^7)^2 = (z^7 + \overline{z}^7)^2 + (-i\overline{z}^7 + iz^7)^2 = (z^7 + \overline{z}^7)^2 - (z^7 - \overline{z}^7)^2.$$

Thus, $L = 4z^7\overline{z}^7 = 4(x^2 + y^2)^7$ and $L = 4$ if and only if $x^2 + y^2 = 1$.
It follows that $L = 4$ when $x = \frac{3}{5},\ y = \frac{4}{5}$ and more generally when

$$\{x, y\} = \left\{\frac{m^2 - n^2}{m^2 + n^2}, \frac{2mn}{m^2 + n^2}\right\},$$

where m, n are positive integers such that $m > n$.

Solution 10 by Toyesh Prakash Sharma (Student), Agra College, Agra, India.

Let $x = r\cos\theta$ and $y = r\sin\theta$ then,

$$\left[(x+iy)^7 + (x-iy)^7\right]^2$$

$$= \left[r^7(\cos\theta + i\sin\theta)^7 + r^7(\cos\theta - i\sin\theta)^7\right]^2$$

$$= r^{14}\left[(\cos 7\theta + i\sin 7\theta) + (\cos 7\theta - i\sin 7\theta)\right]^2$$

$$= 4r^{14}\cos^2 7\theta$$

While

$$\left[(y+ix)^7 + (y-ix)^7\right]^2$$

$$= \left[-i(x-iy)^7 + i(x+iy)^7\right]^2 = -\left[(x+iy)^7 - (x-iy)^7\right]^2$$

$$= -\left[r^7(\cos\theta + i\sin\theta)^7 - r^7(\cos\theta - i\sin\theta)^7\right]^2$$

$$= -r^{14}\left[(\cos 7\theta + i\sin 7\theta) - (\cos 7\theta - i\sin 7\theta)\right]^2$$

$$= 4r^{14}\sin^2 7\theta$$

Then,

$$\Rightarrow \left[(x+iy)^7 + (x-iy)^7\right]^2 + \left[(y+ix)^7 + (y-ix)^7\right]^2 = 4$$

$$\Rightarrow 4r^{14}\cos^2 7\theta + 4r^{14}\sin^2 7\theta = 4$$

$$\Rightarrow 4r^{14}\left(\cos^2 7\theta + \sin^2 7\theta\right) = 4$$

$$\Rightarrow r^{14} = 1$$

$$\Rightarrow \left(x^2 + y^2\right)^7 = 1 \Rightarrow x = \pm\sqrt{1-y^2}$$

For positive rational numbers x and y we can say that $x = \sqrt{1-y^2}$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo, Spain; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Péter Fülöp, Gyömrő, Hungary; and the proposer.

• **5675** *Proposed by Nikos Ntorvas, Athens, Greece.*

Suppose $a, b, c, n > 0$ and $a + b + c = 1$. Prove:

$$e^n (a+1)^{nb} (b+1)^{nc} (c+1)^{na} < e^4 (na)^{na} (nb)^{nb} (nc)^{nc}.$$

Solution 1 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

By rearranging terms, the inequality becomes

$$e^{1-\frac{4}{n}} < \left(\frac{na}{c+1}\right)^a \left(\frac{nb}{a+1}\right)^b \left(\frac{nc}{b+1}\right)^c.$$

Now, by taking logarithms the inequality reads as

$$1 - \frac{4}{n} < a\ln\left(\frac{na}{c+1}\right) + b\ln\left(\frac{nb}{a+1}\right) + c\ln\left(\frac{nc}{b+1}\right).$$

For $n = 1, 2, 3, 4$ the inequality may be checked easily. Let us consider function

$$f(x) = a\ln\left(\frac{na}{c+1}\right) + b\ln\left(\frac{nb}{a+1}\right) + c\ln\left(\frac{nc}{b+1}\right) - 1 + \frac{4}{n}.$$

Since $f'(x) = \dfrac{a+b+c}{x} - \dfrac{4}{x^2} = \dfrac{x-4}{x^2} \geqslant 0$ for $x \geqslant 4$, so $f(x)$ is increasing for $x > 4$, and so the problem is done.

Solution 2 by Michel Bataille, Rouen, France.

If $n = 4$ and $a = b = c = \dfrac{1}{3}$, the left-hand side equals the right-hand side. So we prove

$$e^n(a+1)^{nb}(b+1)^{nc}(c+1)^{na} \leqslant e^4(na)^{na}(nb)^{nb}(nc)^{nc}.$$

This inequality is successively equivalent to

$$n + nb\ln(a+1) + nc\ln(b+1) + na\ln(c+1) \leqslant 4 + na\ln(na) + nb\ln(nb) + nc\ln(nc)$$

$$1 + b\ln(a+1) + c\ln(b+1) + a\ln(c+1) \leqslant \frac{4}{n} + (a+b+c)\ln(n) + a\ln(a) + b\ln(b) + c\ln(c)$$

$$b\ln\left(\frac{a+1}{b}\right) + c\ln\left(\frac{b+1}{c}\right) + a\ln\left(\frac{c+1}{a}\right) \leqslant \frac{4}{n} + \ln(n) - 1.$$

A quick study of the function f defined by $f(x) = \dfrac{4}{x} + \ln(x) - 1$ shows that $f(x) \geqslant f(4) = \ln(4)$ for all positive real numbers x. It follows that it is sufficient to show that

$$b\ln\left(\frac{a+1}{b}\right) + c\ln\left(\frac{b+1}{c}\right) + a\ln\left(\frac{c+1}{a}\right) \leqslant \ln(4).$$

We are done because ln concave on $(0,\infty)$ and $a+b+c=1$ yield

$$b\ln\left(\frac{a+1}{b}\right)+c\ln\left(\frac{b+1}{c}\right)+a\ln\left(\frac{c+1}{a}\right)\leqslant\ln\left(b\cdot\frac{a+1}{b}+c\cdot\frac{b+1}{c}+a\cdot\frac{c+1}{a}\right),$$

that is,

$$b\ln\left(\frac{a+1}{b}\right)+c\ln\left(\frac{b+1}{c}\right)+a\ln\left(\frac{c+1}{a}\right)\leqslant\ln(4).$$

Solution 3 by Paolo Perfetti, dipartimento di matematica, Università di "Tor Vergata", Roma, Italy.

If $a=b=c=1/3$ we have $e^n4^n<e^4n^n$ and it is false for $n=4$ so I think the inequality should be with a $\leqslant$

We need some facts.

i) for $\lambda_1+\lambda_2+\ldots+\lambda_n=1$

$$\sum_{i=1}^n\lambda_i f(x_i)\geqslant f(\sum_{i=1}^n\lambda_i x_i),\qquad \sum_{i=1}^n\lambda_i g(x_i)\leqslant g(\sum_{i=1}^n\lambda_i x_i)$$

respectively for f convex and g concave.

ii)

$$ab+bc+ca\leqslant\frac{(a+b+c)^2}{3}\iff a^2+b^2+c^2\geqslant ab+bc+ca\iff(a-b)^2+(b-c)^2+(c-a)^2\geqslant 0$$

thus $ab+bc+ca\leqslant 1/3$.

$$a^2+b^2+c^2\geqslant\frac{(a+b+c)^2}{3}\iff a^2+b^2+c^2\geqslant ab+bc+ca$$

which is true and then $a^2+b^2+c^2\geqslant 1/3$

iii)

$$(x\ln x)''=\frac{1}{x}>0,\qquad(\ln(1+x))''=\frac{-1}{(1+x)^2}<0$$

The inequality is

$$4+(a+b+c)n\ln n+n(a\ln a+b\ln b+c\ln c)>n+n(b\ln(1+a)+c\ln(1+b)+a\ln(1+c))$$

Using iii), i) and $a+b+c=1$ we can write

$$4+(a+b+c)n\ln n+n(a\ln a+b\ln b+c\ln c)\geqslant 4+n\ln n+n\ln(a^2+b^2+c^2)\geqslant$$
$$4+n\ln n+n\ln\frac{1}{3}$$

$$n+n\ln(1+ab+bc+ca)\geqslant n+n(b\ln(1+a)+c\ln(1+b)+a\ln(1+c))$$

thus it suffices to show

$$4+n\ln n-n\ln 3\geqslant n+n\ln(1+ab+bc+ca)$$

Let's rewrite it as

$$4+3\frac{n}{3}\ln\frac{n}{3}-\frac{n}{3}(1+\ln(1+ab+bc+ca))3>0, \qquad \frac{n}{3}=x$$

and define the function $f(x)=4+3x\ln x-3x(1+\ln(1+ab+bc+ca))$, $x>0$.

$$\lim_{x\to 0^+} f(x)=4, \quad \lim_{x\to\infty} f(x)=\infty,$$

$$f'(x)=3+3\ln x-3(1+\ln(1+ab+bc+ca))\geqslant 0 \iff x\geqslant(1+ab+bc+ca)$$

namely $x\geqslant(1+ab+bc+ca)\doteq\overline{x}$

$$f(\overline{x})=4+3\overline{x}\ln\overline{x}-3\overline{x}-3\overline{x}\ln\overline{x}=4-3\overline{x}\geqslant 0 \iff ab+bc+ca\leqslant\frac{1}{3}$$

and this follows by ii). The consequence is $f(x)\geqslant 0$ and this concludes the proof.

Also solved by Albert Stadler, Herrliberg, Switzerland and the proposer.

• **5676** *Proposed by Péter Fülöp, Gyömrő, Gyomro, Hungary.*

Without using integral identities of the Catalan's constant G, prove

$$\frac{1}{2}\int_0^{\pi/4}\int_0^{\pi/4}\left[\frac{1}{\cos(u+v)}+\frac{1}{\cos(u-v)}\right]dudv=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

We have

$$\frac{1}{\cos(u+v)}+\frac{1}{\cos(u-v)}=\frac{\cos(u+v)\ +\cos(u-v)}{\cos(u+v)\cos(u-v)}=$$

$$=\frac{2\cos u\cos v}{(\cos u\cos v\ -\sin u\sin v\ )(\cos u\cos v\ +\sin u\sin v\ )}=$$

$$=\frac{2\cos u\cos v}{\cos^2 u\cos^2 v\ -\sin^2 u\sin^2 v}=\frac{2\cos u\cos v}{\left(1-\sin^2 u\right)\left(1-\sin^2 v\right)-\sin^2 u\sin^2 v}=$$

$$=\frac{2\cos u\cos v}{1-\sin^2 u\ -\sin^2 v}.$$

Thus the substitution x = sin u, y = sin v gives

$$I:=\frac{1}{2}\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\left[\frac{1}{\cos(u+v)}+\frac{1}{\cos(u-v)}\right]dudv=\int_0^{\frac{1}{\sqrt{2}}}\int_0^{\frac{1}{\sqrt{2}}}\frac{1}{1-x^2-y^2}dxdy=$$

$$= 2\int_0^{\frac{1}{\sqrt{2}}}\int_0^y \frac{1}{1-x^2-y^2}dxdy.$$

We switch to polar coordinates by applying the substitution x = r cos t, y = r sin t and get

$$I = \int_0^{\frac{\pi}{4}}\int_0^{\frac{1}{\sqrt{2}\cos t}} \frac{2r}{1-r^2}drdt = -\int_0^{\frac{\pi}{4}} \log\left(1-\frac{1}{2\cos^2 t}\right) dt =$$

$$= -\int_0^{\frac{\pi}{4}} \log\left(\frac{\cos(2t)}{2\cos^2 t}\right) dt = -\int_0^{\frac{\pi}{4}} \log\left(\cos(2t)\right) dt + \frac{\pi}{4}\log 2 + 2\int_0^{\frac{\pi}{4}} \log(\cos t)\, dt =$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \log(\cos t)\, dt + \frac{\pi}{4}\log 2 + 2\int_0^{\frac{\pi}{4}} \log(\cos t)\, dt.$$

We claim that

$$\int_0^{\frac{\pi}{2}} \log(2\cos t)\, dt = \int_0^{\frac{\pi}{2}} \log(2\sin t)\, dt = 0. \qquad (1)$$

Indeed,

$$\int_0^{\frac{\pi}{2}} \log(\cos t)\, dt = \int_0^{\frac{\pi}{2}} \log(\sin t)\, dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} \log(\cos t \sin t)\, dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} \log\left(\frac{\sin(2t)}{2}\right) dt =$$

$$= \frac{1}{4}\int_0^{\pi} \log\left(\frac{\sin t}{2}\right) dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} \log\left(\frac{\sin t}{2}\right) dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} \log(\sin t)\, dt - \frac{\pi}{4}\log 2,$$

which implies

$$\int_0^{\frac{\pi}{2}} \log(\cos t)\, dt = \int_0^{\frac{\pi}{2}} \log(\sin t)\, dt = -\frac{\pi}{2}\log 2$$

and (1). Thus, by (1),

$$I = \frac{\pi}{2}\log 2 + 2\int_0^{\frac{\pi}{4}} \log(\cos t)\, dt = 2\int_0^{\frac{\pi}{4}} \log(2\cos t)\, dt =$$

$$= 2\int_0^{\frac{\pi}{2}} \log(2\cos t)\, dt - 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \log(2\cos t)\, dt = -2\int_0^{\frac{\pi}{4}} \log(2\sin t)\, dt =$$

$$= -\int_0^{\frac{\pi}{4}} \log\left(4\sin^2 t\right) dt = -\int_0^{\frac{\pi}{4}} \log\left(\left(1-e^{2it}\right)\left(1-e^{-2it}\right)\right) dt =$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \log\left(1-e^{it}\right) dt - \frac{1}{2}\int_0^{\frac{\pi}{2}} \log\left(1-e^{-it}\right) dt = \sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{2}} \cos(nt)\, dt = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}.$$

Termwise integration is permitted, since the Fourier series $-\log\left(1-e^{it}\right) = \sum_{n=1}^{\infty}\frac{1}{n}e^{int},\ t \neq 0$, is a function in L_1.

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

The change of variable $\theta = u + v$ and $\varphi = u - v$ yields

$$\begin{aligned}
\int_0^{\pi/4}\int_0^{\pi/4} \frac{1}{\cos(u+v)}\,du\,dv &= \frac{1}{2}\int_0^{\pi/4}\int_{-\theta}^{\theta} \frac{1}{\cos\theta}\,d\varphi\,d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2}\int_{-\pi/2+\theta}^{\pi/2-\theta} \frac{1}{\cos\theta}\,d\varphi\,d\theta \\
&= \int_0^{\pi/4} \frac{\theta}{\cos\theta}\,d\theta + \int_{\pi/4}^{\pi/2} \frac{\frac{\pi}{2}-\theta}{\cos\theta}\,d\theta;\ \text{and} \\
\int_0^{\pi/4}\int_0^{\pi/4} \frac{1}{\cos(u-v)}\,du\,dv &= \frac{1}{2}\int_{-\pi/4}^{0}\int_{-\varphi}^{\pi/2+\varphi} \frac{1}{\cos\varphi}\,d\theta\,d\varphi + \frac{1}{2}\int_0^{\pi/4}\int_{\varphi}^{\pi/2-\varphi} \frac{1}{\cos\varphi}\,d\theta\,d\varphi \\
&= \int_{-\pi/4}^{0} \frac{\frac{\pi}{4}+\varphi}{\cos\varphi}\,d\varphi + \int_0^{\pi/4} \frac{\frac{\pi}{4}-\varphi}{\cos\varphi}\,d\varphi \\
&= \int_0^{\pi/4} \frac{\frac{\pi}{2}-2\varphi}{\cos\varphi}\,d\varphi.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}\int_0^{\pi/4}\int_0^{\pi/4}\left[\frac{1}{\cos(u+v)} + \frac{1}{\cos(u-v)}\right]du\,dv \\
&\quad= \frac{1}{2}\left(\int_0^{\pi/4}\frac{\theta}{\cos\theta}\,d\theta + \int_{\pi/4}^{\pi/2}\frac{\frac{\pi}{2}-\theta}{\cos\theta}\,d\theta + \int_0^{\pi/4}\frac{\frac{\pi}{2}-2\theta}{\cos\theta}\,d\theta\right) \\
&\quad= \frac{1}{2}\int_0^{\pi/2}\frac{\frac{\pi}{2}-\theta}{\cos\theta}\,d\theta = \frac{1}{2}\int_0^{\pi/2}\frac{\theta}{\sin\theta}\,d\theta.
\end{aligned}$$

With the substitution $\theta = 2\tan^{-1} x$,

$$\frac{1}{2}\int_0^{\pi/2}\frac{\theta}{\sin\theta}\,d\theta = \int_0^1 \frac{\tan^{-1} x}{x}\,dx;$$

now, with the power series

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1},$$

it follows that

$$\int_0^1 \frac{\tan^{-1} x}{x}\,dx = \sum_{n=0}^{\infty}(-1)^n \int_0^1 \frac{x^{2n}}{2n+1}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}.$$

Finally,

$$\frac{1}{2}\int_0^{\pi/4}\int_0^{\pi/4}\left[\frac{1}{\cos(u+v)} + \frac{1}{\cos(u-v)}\right]du\,dv = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}.$$

Solution 3 by Narendra Bhandari, Bajura district, Nepal.

We notice that $\sec(x \pm y) = \dfrac{\partial}{\partial x}\log\left(\dfrac{1+\sin(x\pm y)}{\cos(x\pm y)}\right)$ and since $\dfrac{1}{\cos(x\pm y)} = \sec(x\pm y)$. Utilizing the facts we noticed, we arrive at

$$g_{\pm}(y) = \int_0^{\frac{\pi}{4}} \frac{d}{dx}\log\left(\frac{1+\sin(x\pm y)}{\cos(x\pm y)}\right)dx = \log\left(\frac{1+\sin(x\pm y)}{\cos(x\pm y)}\right)\Bigg|_0^{\frac{\pi}{4}}$$

and hence

$$\int_0^{\frac{\pi}{4}} \big(g_+(y)+g_-(y)\big)\,dy = \int_0^{\frac{\pi}{4}} \left(\log\left(\frac{1+\sin\left(y+\frac{\pi}{4}\right)}{\cos\left(y+\frac{\pi}{4}\right)}\right) + \log\left(\frac{1+\sin\left(\frac{\pi}{4}-y\right)}{\cos\left(\frac{\pi}{4}-y\right)}\right)\right)dy$$
$$-\int_0^{\frac{\pi}{4}}\left(\log\left(\frac{1+\sin y}{\cos y}\right)+\log\left(\frac{1-\sin y}{\cos y}\right)\right)dy$$

since the latter integral is fairly zero which is easy to justify as $2\log(\cos y) = \log(1-\sin^2 y) = \log(1+\sin y)+\log(1-\sin y)$ and $\log(1/\cos^2 y) = -2\log(\cos y)$. Now enforcing the substitution $\dfrac{\pi}{4}+y \to y$ and $\dfrac{\pi}{4}-y\to y$ in the former integrals, we obtain

$$\int_0^{\frac{\pi}{4}}\big(g_+(y)+g_-(y)\big)\,dy = \left(\int_0^{\frac{\pi}{4}}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right)\log\left(\frac{1+\sin(y)}{\cos y}\right)dy = \int_0^{\frac{\pi}{2}}\log\left(\frac{1+\sin y}{\cos y}\right)dy$$
$$= \int_0^{\frac{\pi}{2}}\log\left(\sqrt{\frac{1+\sin y}{1-\sin y}}\right)dy = \frac{1}{2}\int_0^{\frac{\pi}{2}}\log\left(\frac{1+\sin y}{1-\sin y}\right)dy = \int_0^{\frac{\pi}{2}}\tanh^{-1}(\sin y)dy$$
$$= \int_0^{\frac{\pi}{2}}\tanh^{-1}(\cos y)dy \overset{\text{IBP}}{=} y\tanh^{-1}(\cos y)\Bigg|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\frac{y}{\sin y}dy = 2\int_0^1\frac{\tan^{-1}y}{y}dy$$
$$= 2\int_0^1\frac{1}{y}\left(\sum_{n=0}^{\infty}(-1)^n\frac{y^{2n+1}}{2n+1}\right)dy = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1 y^{2n}dy = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$$

We obtain $\displaystyle\int_0^{\frac{\pi}{2}}\frac{y}{\sin y}dx = 2\int_0^1\frac{\tan^{-1}y}{y}dy$ by using the identity $\sin y = \dfrac{2\tan\left(\frac{y}{2}\right)}{1+\tan^2\left(\frac{y}{2}\right)}$ and further we have used Maclaurin series of $\tan^{-1} y$. On dividing both sides by 2 of the expression above proves the proposed result.

Solution 4 by Seán M. Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia.

Denote the integral to be proved by I. On finding a common denominator in the integrand it may be written as

$$I = \frac{1}{2}\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\frac{\cos(u+v)+\cos(u-v)}{\cos(u+v)\cos(u-v)}\,du\,dv.$$

From elementary trigonometric identities, since

$$\cos(u+v)+\cos(u-v) = 2\cos u\cos v, \text{ and}$$
$$\cos(u+v)\cos(u-v) = \cos^2 u\cos^2 v - \sin^2 u\sin^2 v,$$

we may rewrite the integral as

$$\begin{aligned} I &= \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}} \frac{\cos u\cos v}{\cos^2 u\cos^2 v - \sin^2 u\sin^2 v}\,du\,dv \qquad (2)\\ &= \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}} \frac{\cos u\cos v}{(1-\sin^2 u)\cos^2 v - \sin^2 u(1-\cos^2 v)}\,du\,dv \\ &= \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}} \frac{\cos u\cos v}{\cos^2 v - \sin^2 u}\,du\,dv. \end{aligned}$$

Enforcing a substitution of $t = \sin u$ in the inner u-integral produces

$$\begin{aligned} I &= \int_0^{\frac{\pi}{4}} \cos v\left(\int_0^{\frac{1}{\sqrt{2}}} \frac{dt}{\cos^2 v - t^2}\right)dv = \int_0^{\frac{\pi}{4}} \left[\tanh^{-1}\left(\frac{t}{\cos v}\right)\right]_0^{\frac{1}{\sqrt{2}}} dv \\ &= \int_0^{\frac{\pi}{4}} \tanh^{-1}\left(\frac{\sec v}{\sqrt{2}}\right)dv. \qquad (3) \end{aligned}$$

We now show how the integral appearing in (3) can be evaluated. Let

$$J_S = \int_0^{\frac{\pi}{4}} \log(\sin x)\,dx \quad \text{and} \quad J_C = \int_0^{\frac{\pi}{4}} \log(\cos x)\,dx.$$

Enforcing a substitution of $x \mapsto \frac{\pi}{2} - x$ in J_C gives

$$J_C = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \log(\sin x)\,dx.$$

For the sum of these two integrals we have

$$J_S + J_C = \int_0^{\frac{\pi}{2}} \log(\sin x)\,dx = -\frac{\pi}{2}\log(2),$$

this integral being nothing more than Euler's famous log–sin integral. For the difference between these two integrals we have

$$J_S - J_C = \int_0^{\frac{\pi}{4}} \log(\tan x)\,dx.$$

Enforcing a substitution of $y = \tan x$ yields

$$J_S - J_C = \int_0^1 \frac{\log(y)}{1+y^2}\,dy = \sum_{n=0}^{\infty}(-1)^n \int_0^1 y^{2n}\log(y)\,dy.$$

Integrating by parts produces

$$J_S - J_C = -\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_0^1 y^{2n}\, dy = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = -G. \tag{4}$$

Solving the pair of simultaneous equations for J_S and J_C yields

$$J_S = -\frac{G}{2} - \frac{\pi}{4}\log(2) \quad \text{and} \quad J_C = \frac{G}{2} - \frac{\pi}{4}\log(2). \tag{5}$$

Now let

$$I_p = \int_0^{\frac{\pi}{4}} \log(\sin x + \cos x + 1)\, dx.$$

Enforcing a substitution of $x \mapsto \frac{\pi}{2} - x$ gives

$$I_p = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \log(\sin x + \cos x + 1)\, dx,$$

or

$$2I_p = \int_0^{\frac{\pi}{2}} \log(\sin x + \cos x + 1)\, dx,$$

after adding the above integral to the original integral for I_p. Multiplying both sides of the above integral by a factor of 2 produces

$$\begin{aligned} 4I_p &= \int_0^{\frac{\pi}{2}} \log^2(\sin x + \cos x + 1)\, dx = \int_0^{\frac{\pi}{2}} \log[2(1+\cos x)(1+\sin x)]\, dx \\ &= \frac{\pi}{2}\log(2) + \int_0^{\frac{\pi}{2}} \log(1+\cos x)\, dx + \int_0^{\frac{\pi}{2}} \log(1+\sin x)\, dx. \end{aligned}$$

Enforcing in the rightmost integral a substitution of $x \mapsto \frac{\pi}{2} - x$ produces

$$\begin{aligned} 4I_p &= \frac{\pi}{2}\log(2) + 2\int_0^{\frac{\pi}{2}} \log(1+\cos x)\, dx \\ &= \frac{\pi}{2}\log(2) + 2\int_0^{\frac{\pi}{2}} \log\left(2\cos^2\frac{x}{2}\right) dx \\ &= \frac{3\pi}{2}\log(2) + 4\int_0^{\frac{\pi}{2}} \log\left(\cos\frac{x}{2}\right) dx. \end{aligned}$$

Finally, enforcing a substitution of $x \mapsto 2x$ in the remaining integrals yields

$$4I_p = \frac{3\pi}{2}\log(2) + 8\int_0^{\frac{\pi}{4}} \log(\cos x)\, dx = 4G - \frac{\pi}{2}\log(2),$$

where the result for J_C given in (5) has been used. Upon dividing by a factor of 4 it immediately follows that

$$I_p = G - \frac{\pi}{8}\log(2).$$

Also let

$$I_m = \int_0^{\frac{\pi}{4}} \log(\sin x + \cos x - 1)\, dx.$$

Summing I_p to I_m yields

$$\begin{aligned} I_p + I_m &= \int_0^{\frac{\pi}{2}} \log\left((\sin x + \cos x)^2 - 1\right)\, dx = \int_0^{\frac{\pi}{4}} \log(2 \sin x \cos x)\, dx \\ &= \int_0^{\frac{\pi}{4}} \log(\sin 2x)\, dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \log(\sin x)\, dx = -\frac{\pi}{4} \log(2). \end{aligned}$$

Here a substitution of $x \mapsto \frac{x}{2}$ has been enforced before recognising the remaining integral as Euler's log–sin integral. But since the value for I_p is known, namely $I_p = G - \frac{\pi}{8} \log(2)$, from the above result for the sum of these two integrals we see that $I_m = -G - \frac{\pi}{8} \log(2)$. For the difference, as $I_p - I_m = 2G$ one has

$$I_p - I_m = \int_0^{\frac{\pi}{4}} \log\left(\frac{\sin x + \cos x + 1}{\sin x + \cos x - 1}\right)\, dx = 2G. \tag{6}$$

Recalling

$$\coth^{-1} u = \frac{1}{2} \log\left(\frac{u+1}{u-1}\right), \quad \text{for } |u| > 1,$$

if we set $u = \sin x + \cos x$ in the integral appearing in (6), it can be rewritten as

$$\int_0^{\frac{\pi}{4}} \coth^{-1}(\sin x + \cos x)\, dx = G. \tag{7}$$

Taking advantage of the identity $\sin x + \cos x = \sqrt{2} \cos(\frac{\pi}{4} - x)$ and enforcing a substitution of $x \mapsto \frac{\pi}{4} - x$ in (7), we find

$$\int_0^{\frac{\pi}{4}} \coth^{-1}\left(\sqrt{2} \cos x\right)\, dx = G. \tag{8}$$

Furthermore, the identity $\coth^{-1}(\frac{1}{u}) = \tanh^{-1} u$ for $u \neq 0$ allows one to rewrite integral (8) as

$$\int_0^{\frac{\pi}{4}} \tanh^{-1}\left(\frac{\sec x}{\sqrt{2}}\right)\, dx = G,$$

which is just (3). So in conclusion we have

$$I = \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \left[\frac{1}{\cos(u+v)} + \frac{1}{\cos(u-v)}\right] du\, dv = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = G,$$

as required to prove.

Solution 5 by Paolo Perfetti, dipartimento di matematica, Università di "Tor Vergata", Roma, Italy.

Let's change variables $s = u + v$, $t = u - v$. The integral becomes

$$\frac{1}{2}\iint_R \left[\frac{1}{\cos s} - \frac{1}{\cos t}\right] ds\,dt$$

R is the rombus of vertices $(0,0)$, $(\pi/4, -\pi/4)$, $(\pi/2, 0)$, $(\pi/4, \pi/4)$ and reads as

$$\frac{1}{2}\left[\int_0^{\frac{\pi}{4}} \frac{ds}{\cos s}\int_{-s}^{s} dt + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{ds}{\cos s}\int_{s-\frac{\pi}{2}}^{-s+\frac{\pi}{2}} dt + \int_{\frac{-\pi}{4}}^{0} \frac{dt}{\cos t}\int_{\frac{\pi}{4}}^{t+\frac{\pi}{2}} ds + \int_0^{\frac{\pi}{4}} \frac{dt}{\cos t}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}-t} ds\right] =$$

$$= \frac{1}{2}\left[\int_0^{\frac{\pi}{4}} \frac{2s ds}{\cos s} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{(\pi - 2s)ds}{\cos s} + \int_{\frac{-\pi}{4}}^{0} \frac{(t - \pi/4)dt}{\cos t} + \int_0^{\frac{\pi}{4}} \frac{(\pi/4 - t)dt}{\cos t}\right] =$$

$$+ = \frac{1}{2}\left[\int_0^{\frac{\pi}{4}} \frac{2s ds}{\cos s} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{(\pi - 2s)ds}{\cos s} + \int_0^{\frac{\pi}{4}} \frac{(-t - \pi/4)dt}{\cos t} + \int_0^{\frac{\pi}{4}} \frac{(\pi/4 - t)dt}{\cos t}\right] =$$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{(\pi - 2s)ds}{\cos s} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{u}{\sin u} du \underbrace{=}_{u=2\arctan y} \int_0^1 \frac{\arctan y}{y} dy =$$

$$= \int_0^1 \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{2k+1} dy = \sum_{k=0}^{\infty} (-1)^k \int_0^1 \frac{y^{2k}}{2k+1} dy = G$$

The exchange between the series and the integral is allowed by a Abel's theorem via the uniform convergence of the power series $\sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{2k+1}$ in the interval $[0, 1]$.

Also solved by the proposer.

• **5677** *Proposed by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.*

Solve the differential equation

$$\frac{dy}{dx} = \tan(x + y) - \cot(x - y).$$

Solution 1 by the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.

Notice that

$$\tan(x + y) - \cot(x - y) = \frac{\sin(x + y)\sin(x - y) - \cos(x - y)\cos(x + y)}{\cos(x + y)\sin(x - y)}$$

$$= \frac{-2\cos(2x)}{\sin(2x) - \sin(2y)};$$

hence, the given differential equation is equivalent to

$$2\cos(2x)dx + \big(\sin(2x) - \sin(2y)\big)\,dy = 0.$$

Setting $M(x,y) = 2\cos(2x)$ and $N(x,y) = \sin(2x) - \sin(2y)$, we have

$$\frac{\partial M}{\partial y} = 0 \neq 2\cos(2x) = \frac{\partial N}{\partial x},$$

so the equation is not exact. However, since

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = 1$$

is a function of y, then multiplying by the integration factor $\rho(y) = e^y$ gives the exact equation

$$2e^y\cos(2x)dx + e^y\big(\sin(2x) - \sin(2y)\big)\,dy = 0.$$

A solution to this equation is a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = 2e^y\cos(2x) \text{ and } \frac{\partial F}{\partial y} = e^y\big(\sin(2x) - \sin(2y)\big).$$

Thus, $F(x,y) = e^y\sin(2x) + f(y)$ for some function $f(y)$, and

$$\frac{df}{dy} = -e^y\sin(2y).$$

Using integration by parts, we compute that

$$f(y) = \frac{e^y}{5}\big(2\cos(2y) - \sin(2y)\big),$$

giving the solution to the original differential equation as

$$F(x,y) = e^y\sin(2x) + \frac{e^y}{5}\big(2\cos(2y) - \sin(2y)\big) + C.$$

Solution 2 by Seán M. Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia.

Using elementary trigonometric identities we rewrite the right-hand side of the differential equation as follows

$$\begin{aligned}
\frac{dy}{dx} &= \tan(x+y) - \cot(x-y)\\
&= \frac{\sin(x+y)}{\cos(x+y)} - \frac{\cos(x-y)}{\sin(x-y)}\\
&= \frac{\sin(x+y)\sin(x-y) - \cos(x-y)\cos(x+y)}{\sin(x-y)\cos(x+y)}\\
&= \frac{\frac{1}{2}\big(\cos(2y) - \cos(2x)\big) - \frac{1}{2}\big(\cos(2x) + \cos(2y)\big)}{\frac{1}{2}\big(\sin(2x) - \sin(2y)\big)}\\
&= \frac{-2\cos(2x)}{\sin(2x) - \sin(2y)},
\end{aligned}$$

or after rearranging

$$-2\cos(2x) + \big(\sin(2y) - \sin(2x)\big)\,\frac{dy}{dx} = 0.$$

The differential equation is not exact. Now suppose an integrating factor $R(y)$ can be found that will make the differential equation exact. In this case, writing the differential equation as

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0,$$

where

$$M(x, y) = -2R(y)\cos(2x) \quad \text{and} \quad N(x, y) = R(y)\,\big(\sin(2y) - \sin(2x)\big)\,,$$

it will be exact if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. In this case we see that

$$\frac{dR}{dy} = R(y) \quad \Rightarrow \quad R(y) = e^y.$$

So an integrating factor what will make our differential equation exact is e^y. Multiplying both sides of the differential equation by this integrating factor produces

$$-2e^y\cos(2x) + e^y\,\big(\sin(2y) - \sin(2x)\big)\,\frac{dy}{dx} = 0.$$

Now suppose a function $\Psi(x, y)$ can be found such that $\Psi_x(x, y) = M(x, y)$ and $\Psi_y(x, y) = N(x, y)$. If this can be done then the differential equation can be expressed as

$$\Psi_x(x, y) + \Psi_y(x, y)\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{d}{dx}[\Psi(x, y(x))] = 0.$$

So a solution to the exact differential equation will be $\Psi(x, y) = k$, where k is a constant. In our case

$$\frac{\partial \Psi}{\partial x} = M(x, y) = -2e^y\cos(2x).$$

so

$$\Psi(x, y) = -2e^y\int\cos(2x)\,dx + \eta(y) = -e^y\sin(2x) + \eta(y).$$

Here $\eta(y)$ is an unknown function to be determined. Also

$$\begin{aligned}\frac{\partial \Psi}{\partial y} &= N(x, y)\\ \Rightarrow -e^y\sin(2x) + \eta'(y) &= e^y\,\big(\sin(2y) - \sin(2x)\big)\\ \eta'(y) &= e^y\sin(2y)\\ \Rightarrow \eta(y) &= \int e^y\sin(2y)\,dy + k_1.\end{aligned}$$

Integrating by parts twice produces

$$\eta(y) = \frac{1}{5}e^y\,\big(\sin(2y) - 2\cos(2y)\big) + k_1.$$

So we have found

$$\Psi(x,y) = -e^y \sin(2x) + \frac{1}{5}e^y \left(\sin(2y) - 2\cos(2y)\right) + k_1.$$

The (implicit) solution to the differential equation is

$$e^y \sin(2x) - \frac{1}{5}e^y \left(\sin(2y) - 2\cos(2y)\right) = c,$$

where c is a constant.

Solution 3 by Albert Stadler, Herrliberg, Switzerland.

We note that

$$\cos(x+y)\sin(x-y) = \frac{1}{2}\left(\sin(2x) - \sin(2y)\right),$$
$$\sin(x+y)\sin(x-y) - \cos(x-y)\cos(x+y) = -\cos(2x).$$

Hence

$$\tan(x+y) - \cot(x-y) = \frac{\sin(x+y)\sin(x-y) - \cos(x-y)\cos(x+y)}{\cos(x+y)\sin(x-y)} =$$

$$= -\frac{2\cos(2x)}{\sin(2x) - \sin(2y)}.$$

Hence

$$2\cos(2x)\,dx + \left(\sin(2x) - \sin(2y)\right)dy = 0.$$

We see that $\mu(y) = e^y$ is an integrating factor, since

$$\frac{\partial}{\partial x}\left[\mu(y)\left(\sin(2x) - \sin(2y)\right)\right] = 2\frac{\partial}{\partial y}\left[\mu(y)\cos(2x)\right].$$

We deduce from

$$\frac{\partial}{\partial x}f(x,y) = 2e^y\cos(2x)$$

that $f(x,y) = e^y\sin(2x) + \varphi(y)$.
From

$$\frac{\partial}{\partial y}f(x,y) = e^y\sin(2x) + \varphi'(y) = e^y\left(\sin(2x) - \sin(2y)\right)$$

follows that $\varphi'(y) = -e^y\sin(2y)$ so that $\varphi(y) = -\frac{1}{5}e^y\left(-2\cos(2y) + \sin(2y)\right)$.
Thus the general solution is given by $f(x,y) = e^y\sin(2x) - \frac{1}{5}e^y\left(-2\cos(2y) + \sin(2y)\right) = C$ which is equivalent to

$$\sin(2x) = \frac{1}{5}\left(-2\cos(2y) + \sin(2y)\right) + Ce^{-y}.$$

We verify this solution by differentiation:

$$2\cos(2x) = \frac{2}{5}\left(2\sin(2y) + \cos(2y)\right)y' - Ce^{-y}y'$$

so that

$$y' = \frac{2\cos(2x)}{\frac{2}{5}\left(2\sin(2y) + \cos(2y)\right) - Ce^{-y}} =$$

$$= \frac{2\cos(2x)}{\frac{2}{5}\left(2\sin(2y) + \cos(2y)\right) - \sin(2x) + \frac{1}{5}\left(-2\cos(2y) + \sin(2y)\right)} =$$

$$= \frac{2\cos(2x)}{-\sin(2x) + \sin(2y)} = \tan(x+y) - \cot(x-y).$$

Also solved by the proposer.

• **5678** *Proposed by Seán M. Stewart, Physical Sciences and Engineering Division, King Abdullah University of Science and Technology, Saudi Arabia.*

For positive integers m and n define

$$S_m(n) = \sum_{k=1}^{n} \tan^{2m}\left(\frac{k\pi}{2n+1}\right).$$

Express $S_1(n)$, $S_2(n)$ and $S_3(n)$, each as a polynomial in n.

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

(a) We have for 1⩽k⩽n,

$$\left(\frac{1+i\tan\left(\frac{\pi k}{2n+1}\right)}{1-i\tan\left(\frac{\pi k}{2n+1}\right)}\right)^{2n+1} = \left(\frac{\cos\left(\frac{\pi k}{2n+1}\right) + i\sin\left(\frac{\pi k}{2n+1}\right)}{\cos\left(\frac{\pi k}{2n+1}\right) - i\sin\left(\frac{\pi k}{2n+1}\right)}\right)^{2n+1} = \left(\frac{e^{\frac{\pi i k}{2n+1}}}{e^{-\frac{\pi i k}{2n+1}}}\right)^{2n+1} = 1.$$

So

$$0 = \left(1 + i\tan\left(\frac{\pi k}{2n+1}\right)\right)^{2n+1} - \left(1 - i\tan\left(\frac{\pi k}{2n+1}\right)\right)^{2n+1} =$$

$$= \sum_{m=0}^{2n+1} \binom{2n+1}{m} \tan^{2n+1-m}\left(\frac{\pi k}{2n+1}\right)\left(i^{2n+1-m} - (-i)^{2n+1-m}\right) =$$

$$= 2i(-1)^n \tan\left(\frac{\pi k}{2n+1}\right) \sum_{m=0}^{n} (-1)^m \binom{2n+1}{2m} \tan^{2n-2m}\left(\frac{\pi k}{2n+1}\right).$$

Put $x_k := \tan^2\left(\dfrac{\pi k}{2n+1}\right)$, 1⩽k⩽n. We see from the above equation that the numbers x_k, 1⩽k⩽n, are the roots of

$$\sum_{m=0}^{n}(-1)^m\binom{2n+1}{2m}x^{n-m}=0.$$

Hence

$$\sum_{m=0}^{n}(-1)^m\binom{2n+1}{2m}x^{n-m}=\prod_{m=1}^{n}(x-x_m)=\sum_{m=0}^{n}(-1)^m e_m(x_1,x_2,\ldots,x_n)\,x^{n-m},$$

where $e_m=e_m(x_1,x_2,\ldots,x_n)$ denotes the m*th* elementary symmetric polynomial defined by

$$e_m(x_1,x_2,\ldots,x_n)=\sum_{1\leqslant j_1<j_2<\cdots<j_m\leqslant n}x_{j_1}x_{j_2}\cdots x_{j_m}.$$

We compare coefficients and get

$$e_m(x_1,x_2,\ldots,x_n)=\binom{2n+1}{2m},\ 0\leqslant m\leqslant n$$

Obviously $S_m(n)=\sum_{k=1}^{n}\tan^{2m}\left(\dfrac{k\pi}{2n+1}\right)=\sum_{k=1}^{n}x_k^m$ is a sum of m*th* powers.
According to Newton's identities (see http://en.wikipedia.org/wiki/Newton's_identities),

$$me_m=\sum_{j=1}^{m}(-1)^{j-1}e_{m-j}S_j(n)=(-1)^{m-1}S_m(n)+\sum_{j=1}^{m-1}(-1)^{j-1}e_{m-j}S_j(n),$$

valid for all m⩾1, which readily imply

$$S_m(n)=(-1)^{m-1}me_m-\sum_{j=1}^{m-1}(-1)^j e_j S_{m-j}(n)=$$

$$=(-1)^{m-1}m\binom{2n+1}{2m}-\sum_{j=1}^{m-1}(-1)^j\binom{2n+1}{2j}S_{m-j}(n).$$

This is a recursive formula which allows to calculate $S_m(n)$ one by one. We find

$$S_1(n)=\sum_{k=1}^{n}\tan^2\left(\frac{k\pi}{2n+1}\right)=\sum_{k=1}^{n}x_k=e_1=\binom{2n+1}{2}=n(2n+1)=2n^2+n,$$

$$S_2(n)=-2\binom{2n+1}{4}+\binom{2n+1}{2}S_1(n)=\frac{1}{3}n(2n+1)\left(4n^2+6n-1\right)=$$

$$=\frac{1}{3}\left(8n^4+16n^3+4n^2-n\right),$$

$$S_3(n) = 3\binom{2n+1}{6} + \binom{2n+1}{2} S_2(n) - \binom{2n+1}{4} S_1(n) =$$
$$= \frac{1}{15} n(2n+1)\left(32n^4 + 80n^3 + 40n^2 - 20n + 3\right) =$$
$$= \frac{1}{15}\left(64n^6 + 192n^5 + 160n^4 - 14n^2 + 3n\right).$$

Solution 2 by Henry Ricardo, Westchester Area Math Circle, Purchase, New York.

For any integer k, de Moivre's formula yields

$$\left(\cos\frac{k\pi}{2n+1} + i\sin\frac{k\pi}{2n+1}\right)^{2n+1} = \cos k\pi + i\sin k\pi = (-1)^k.$$

The binomial theorem gives us the equivalent form

$$\sum_{j=0}^{2n+1}\binom{2n+1}{j}\left(\cos\frac{k\pi}{2n+1}\right)^{j}\left(i\sin\frac{k\pi}{2n+1}\right)^{2n+1-j} = (-1)^k,$$

Considering the imaginary part of this last equation, we obtain

$$\sum_{j=0}^{n}\binom{2n+1}{2j}\left(\cos\frac{k\pi}{2n+1}\right)^{2j}\left(i\sin\frac{k\pi}{2n+1}\right)^{2n+1-2j} = 0.$$

Dividing first by $\cos^{2n+1}(k\pi/(2n+1))$ and then by $\tan(k\pi/(2n+1))$, constants with respect to the index of summation, we can write

$$\sum_{j=0}^{n}\binom{2n+1}{2j}\left(i\tan\frac{k\pi}{2n+1}\right)^{2n-2j} = \sum_{j=0}^{n}\binom{2n+1}{2j}\left[-\left(\tan\frac{k\pi}{2n+1}\right)^{2}\right]^{n-j} = 0.$$

This last equation indicates that $(\tan k\pi/(2n+1))^2$, $1 \leqslant k \leqslant n$, are the zeros of the polynomial

$$\sum_{j=0}^{n}\binom{2n+1}{2j}(-x)^{n-j} = x^n - \binom{2n+1}{2}x^{n-1} + \binom{2n+1}{4}x^{n-2} - \binom{2n+1}{6}x^{n-3} + \cdots. \qquad (9)$$

Denoting the sum of the kth power of the zeros of the polynomial $x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n$ by s_k, Newton's identities yield $s_1 = -p_1$, $s_2 = p_1^2 - 2p_2$, $s_3 = -p_1^3 + 3p_1p_2 - 3p_3$.

Applying these to the polynomial in (1), we find that

$$\begin{aligned}
S_1(n) &= \sum_{k=1}^{n} \tan^2\left(\frac{k\pi}{2n+1}\right) = s_1 = -\binom{2n+1}{2} = n(2n+1) = 2n^2+n,\\
S_2(n) &= \sum_{k=1}^{n} \tan^4\left(\frac{k\pi}{2n+1}\right) = s_2 = \binom{2n+1}{2}^2 - 2\binom{2n+1}{4}\\
&= \frac{8}{3}n^4 + \frac{16}{3}n^3 + \frac{4}{3}n^2 - \frac{1}{3}n = \frac{n(2n+1)(4n^2+6n-1)}{3}\\
S_3(n) &= \sum_{k=1}^{n} \tan^6\left(\frac{k\pi}{2n+1}\right) = s_3 = \binom{2n+1}{2}^3 - 3\binom{2n+1}{2}\binom{2n+1}{4} + 3\binom{2n+1}{6}\\
&= \frac{64}{15}n^6 + \frac{64}{5}n^5 + \frac{32}{3}n^4 - \frac{14}{15}n^2 + \frac{1}{5}n\\
&= \frac{n(2n+1)(32n^4+80n^3+40n^2-20n+3)}{15}.
\end{aligned}$$

Viète's formula could also have been used here.

Solution 3 by Michel Bataille, Rouen, France.

In the two featured solutions to problem 11044 of *The American Mathematical Monthly*, Vol. 112, No 7 (Aug.-Sept. 2005) p. 657-9, it is proved that

$$S_m(n) = \frac{2n+1}{2}(-1)^{m-1}\sum_{k=0}^{m-1}\binom{2n}{2m-2k-1}A_k$$

with $A_0 = 1$ and $A_k = \sum (-1)^r r! \prod_{i=1}^{k} \frac{1}{r_i!}\binom{2n+1}{2i}^{r_i}$ where the sum is over all k-tuples of nonnegative integers $(r_1, \ldots, r_k)$ such that $r_1 + 2r_2 + \cdots + kr_k = k$ and where r denotes $r_1 + r_2 + \cdots + r_k$. Applying this general formula, we obtain

$$S_1(n) = \frac{2n+1}{2}\binom{2n}{1}A_0 = n(2n+1).$$

It is easily checked that $A_1 = -n(2n+1)$; it follows that

$$S_2(n) = \frac{2n+1}{2}(-1)\left(\binom{2n}{3}A_0 + \binom{2n}{1}A_1\right) = -\frac{2n+1}{2}\left(\frac{2n(2n-1)(2n-2)}{6} - 2n^2(2n+1)\right),$$

that is,

$$S_2(n) = \frac{n(2n+1)(4n^2+6n-1)}{3}.$$

Similarly, from $A_2 = -\binom{2n+1}{4} + \binom{2n+1}{2}^2 = \frac{n(2n+1)(10n^2+9n-1)}{6}$, long but easy calculations give

$$S_3(n) = \frac{n(2n+1)(32n^4+80n^3+40n^2-20n+3)}{15}.$$

Solution 4 Paolo Perfetti, dipartimento di matematica, Università di "Tor Vergata", Roma, Italy.

The result is

$$S_1(n) = 2n^2 + n, \quad S_2(n) = \frac{n(2n+1)(4n^2+6n-1)}{3},$$
$$S_3(n) = \frac{n(2n+1)(32n^4+80n^3+40n^2-20n+3)}{15}$$

My solution is computer assisted.

Let's start with $m = 1$.

$$C_{k,n} \doteq \cos\frac{k\pi}{2n+1}, \quad S_{k,n} \doteq \sin\frac{k\pi}{2n+1}, \quad \frac{S^2_{k,n}}{C^2_{k,n}} = \frac{1-C^2_{k,n}}{C^2_{k,n}} = \frac{1}{C^2_{k,n}} - 1$$

$$S_1(n) = \sum_{k=1}^{n} \tan^2\frac{k\pi}{2n+1} = \sum_{k=1}^{n}\frac{1}{C^2_{k,n}} - n$$

$$\sum_{k=1}^{2n}\left(\cos\frac{k\pi}{2n+1}\right)^{-2} = \sum_{k=1}^{n}\left(\cos\frac{k\pi}{2n+1}\right)^{-2} + \sum_{k=n+1}^{2n}\left(\cos\frac{k\pi}{2n+1}\right)^{-2} =$$
$$\underbrace{=}_{k\to 2n+1-k} 2\sum_{k=1}^{n}\left(\cos\frac{k\pi}{2n+1}\right)^{-2}$$

$$\sum_{k=1}^{2n}\frac{1}{\left(\cos\frac{k\pi}{2n+1}\right)^2} = -1 + \sum_{k=0}^{2n}\frac{2}{1+\cos\frac{2k\pi}{2n+1}} = -1 + \sum_{k=0}^{2n}\frac{4}{2+e^{\frac{2k\pi i}{2n+1}} + e^{\frac{-2k\pi i}{2n+1}}} =$$
$$\doteq -1 + \sum_{k=0}^{2n}\frac{4}{2+z_k+z_k^{-1}} = -1 + \sum_{k=0}^{2n}\frac{4z_k}{(z_k+1)^2} = \sum_{k=0}^{2n}\frac{4}{z_k+1} - \sum_{k=0}^{2n}\frac{4}{(z_k+1)^2} - 1$$
$$S_1(n) = \frac{1}{2}\left(\sum_{k=0}^{2n}\frac{4}{z_k+1} - \sum_{k=0}^{2n}\frac{4}{(z_k+1)^2} - 1\right) - n \tag{1}$$

$$P(z) \doteq z^{2n+1} - 1 = \prod_{k=0}^{2n}(z - z_k)$$

$$\begin{aligned}
&P(-1) = -2, \quad P'(-1) = 2n+1, \quad P''(-1) = -(2n+1)2n,\\
&P'''(-1) = (2n+1)2n(2n-1), \quad P^{(IV)}(-1) = -(2n+1)2n(2n-1)(2n-2)\\
&P^{(V)}(-1) = (2n+1)(2n)(2n-1)(2n-2)(2n-3)\\
&P^{(VI)}(-1) = -(2n+1)(2n)(2n-1)(2n-2)(2n-3)(2n-4) \qquad (1)
\end{aligned}$$

$$\frac{P'}{P} = \sum_{k=0}^{2n} \frac{1}{z - z_k},$$

$$\sum_{k=0}^{2n} \frac{1}{1+z_k} = -\sum_{k=0}^{2n} \frac{1}{z-z_k}\Big|_{z=-1} = \frac{-P'(-1)}{P(-1)} = n + \frac{1}{2}$$

$$\sum_{k=0}^{2n} \frac{-1}{(z-z_k)^2} = \frac{d}{dz}\frac{P'}{P} = \frac{P''P - (P')^2}{P^2}$$

thus

$$\sum_{k=0}^{2n} \frac{-1}{(1+z_k)^2} = \frac{4n(2n+1) - (2n+1)^2}{4} = \frac{4n^2-1}{4}$$

It follows

$$\sum_{k=1}^{n} \left(\cos\frac{k\pi}{2n+1}\right)^{-2} = \frac{1}{2}\left(4(n+\frac{1}{2}) + 4\frac{4n^2-1}{4} - 1)\right) = 2n^2 + 2n$$

and finally (1)

$$S_1(n) = \frac{1}{2}\left(4(n+\frac{1}{2}) + 4\frac{4n^2-1}{4} - 1)\right) - n = 2n^2 + n$$

$m = 2$

$$\sum_{k=1}^{n} \frac{S^4_{k,n}}{C^4_{k,n}} = \sum_{k=1}^{n}\left(1 - \frac{2}{C^2_{k,n}} + \frac{1}{C^4_{k,n}}\right) \tag{2}$$

$$\sum_{k=1}^{2n} \frac{1}{\left(\cos\frac{k\pi}{2n+1}\right)^4} = -1 + \sum_{k=0}^{2n} \frac{4}{\left(1+\cos\frac{2k\pi}{2n+1}\right)^2} = -1 + \sum_{k=0}^{2n} \frac{16}{\left(2 + e^{\frac{2k\pi i}{2n+1}} + e^{\frac{-2k\pi i}{2n+1}}\right)^2} =$$

$$\doteq -1 + \sum_{k=0}^{2n} \frac{4}{(2+z_k+z_k^{-1})^2} = -1 + \sum_{k=0}^{2n} \frac{16z_k^2}{(z_k+1)^4} =$$

$$= -1 + \sum_{k=0}^{2n}\left(\frac{16}{(1+z_k)^4} - \frac{32}{(1+z_k)^3} + \frac{16}{(1+z_k)^2}\right)$$

and

$$\sum_{k=1}^{n} \frac{1}{\left(\cos\frac{k\pi}{2n+1}\right)^4} = \frac{-1}{2} + \sum_{k=0}^{2n}\left(\frac{8}{(1+z_k)^4} - \frac{16}{(1+z_k)^3} + \frac{8}{(1+z_k)^2}\right) \tag{2.1}$$

We need

$$\frac{d^2}{dz^2}\frac{P'}{P} = \frac{d}{dz}\left[\frac{P''}{P} - \frac{(P')^2}{P^2}\right] = \frac{P'''}{P} - \frac{P''P'}{P^2} - \frac{2P'P''}{P^2} + \frac{2(P')^3}{P^3} = \sum_{k=0}^{2n} \frac{2}{(z-z_k)^3}$$

which evaluated at $z = -1$ gives

$$\sum_{k=0}^{2n} \frac{-2}{(1+z_k)^3} = \frac{(2n-1)2n(2n+1)}{-2} - 3\frac{-2n(2n+1)^2}{4} + 2\frac{(2n+1)^3}{-8} =$$
$$\frac{2n+1}{4}(-4n(2n-1) + 6n(2n+1) - (2n+1)^2) = \frac{12n^2+4n-1}{4}$$

$$\begin{aligned}
\sum_{k=0}^{2n} \frac{-6}{(z-z_k)^4} &= \frac{d^3}{dz^3}\frac{P'}{P} = \frac{d}{dz}\left[\frac{P'''}{P} - \frac{P''P'}{P^2} - \frac{2P'P''}{P^2} + \frac{2(P')^3}{P^3}\right] = \\
&= \frac{P^{(IV)}}{P} - \frac{P'''P'}{P^2} - 3\left[\frac{P'''P' + (P'')^2}{P^2} - \frac{2P''(P')^2}{P^3}\right] + 2\left[\frac{3(P')^2P''}{P^3} - \frac{3(P')^4}{P^4}\right] \\
&= \frac{P^{(IV)}}{P} - 4\frac{P'''P'}{P^2} - 3\frac{(P'')^2}{P^2} + \frac{12P''(P')^2}{P^3} - \frac{6(P')^4}{P^4} \qquad (2.2)
\end{aligned}$$

$$\sum_{k=0}^{2n} \frac{-6}{(1+z_k)^4} = (2n+1)\Big[(2n(2n-1)(n-1) - 2n(2n-1)(2n+1) - 3n^2(2n+1) +$$
$$+3n(2n+1)^2 - \frac{3}{8}(2n+1)^3\Big] = \frac{-1}{8}(2n+1)(2n-3)(4n^2+12n-1)$$

Now we come back to (2.1) and get

$$\sum_{k=1}^{n} \frac{1}{\cos^4 \frac{k\pi}{2n+1}} = 8\frac{1-4n^2}{4} + 8\frac{12n^2+4n-1}{4} + 8\frac{(2n+1)(2n-3)(4n^2+12n-1}{48} - \frac{1}{2} =$$
$$= \frac{8}{3}n(n+1)(n^2+n+1)$$

then (2) is

$$S_2(n) \doteq \sum_{k=1}^{n}\left(\tan\frac{k\pi}{2n+1}\right)^4 =$$
$$= n - 2(2n^2+2n) + \frac{8}{3}n(n+1)(n^2+n+1) = \frac{1}{3}n(2n+1)(4n^2+6n-1)$$

$m = 3$

$$\sum_{k=1}^{n}\left(\tan\frac{k\pi}{2n+1}\right)^6 = \sum_{k=1}^{n}\frac{S_{k,n}^6}{C_{k,n}^6} = \sum_{k=1}^{n}\left(-1+\frac{3}{C_{k,n}^2}-\frac{3}{C_{k,n}^4}+\frac{1}{C_{k,n}^6}\right) \tag{3}$$

We need only the last term.

$$\sum_{k=1}^{2n}\frac{1}{\left(\cos\frac{k\pi}{2n+1}\right)^6} = -1+\sum_{k=0}^{2n}\frac{8}{\left(1+\cos\frac{2k\pi}{2n+1}\right)^3} = -1+\sum_{k=0}^{2n}\frac{64}{\left(2+e^{\frac{2k\pi i}{2n+1}}+e^{\frac{-2k\pi i}{2n+1}}\right)^3} =$$

$$\doteq -1+\sum_{k=0}^{2n}\frac{64}{(2+z_k+z_k^{-1})^3} = -1+\sum_{k=0}^{2n}\frac{64z_k^3}{(z_k+1)^6} =$$

$$= -1+64\sum_{k=0}^{2n}\left(\frac{-1}{(1+z_k)^6}-\frac{3}{(1+z_k)^5}-\frac{3}{(1+z_k)^4}+\frac{1}{(1+z_k)^3}\right)$$

$$\sum_{k=1}^{n}\frac{1}{C_{k,n}^6} = \frac{1}{2}\sum_{k=1}^{2n}\frac{1}{\left(\cos\frac{k\pi}{2n+1}\right)^6} =$$

$$= \frac{-1}{2}+32\sum_{k=0}^{2n}\left(\frac{-1}{(1+z_k)^6}-\frac{3}{(1+z_k)^5}-\frac{3}{(1+z_k)^4}+\frac{1}{(1+z_k)^3}\right) \tag{3.1}$$

We start with (2.2)

$$\sum_{k=0}^{2n}\frac{-6}{(z-z_k)^4} = \frac{P^{(IV)}}{P}-4\frac{P'''P'}{P^2}-3\frac{(P'')^2}{P^2}+\frac{12P''(P')^2}{P^3}-\frac{6(P')^4}{P^4}$$

By differentiating

$$\frac{d}{dz}\sum_{k=0}^{2n}\frac{-6}{(z-z_k)^4} = \sum_{k=0}^{2n}\frac{24}{(z-z_k)^5} =$$

$$= \frac{d}{dz}\left[\frac{P^{(IV)}}{P}-4\frac{P'''P'}{P^2}-3\frac{(P'')^2}{P^2}+\frac{12P''(P')^2}{P^3}-\frac{6(P')^4}{P^4}\right] =$$

$$= \frac{P^{(5)}}{P}-\frac{P^{(4)}P'}{P^2}-\frac{4}{P^2}(P^{(4)}P'+P^{(3)}P'')-6\frac{P''P'''}{P^2}+6\frac{(P'')^2P'}{P^3}+8\frac{P'''(P')^2}{P^3}+$$

$$+12\frac{P'''(P')^2}{P^3}+24\frac{(P'')^2P'}{P^3}-36\frac{(P')^3P''}{P^4}-24\frac{(P')^3P''}{P^4}+24\frac{(P')^5}{P^5} =$$

$$= \frac{P^{(5)}}{P}-5\frac{P^{(4)}P'}{P^2}-10\frac{P'''P''}{P^2}+20\frac{P'''(P')^2}{P^3}+30\frac{(P'')^2P'}{P^3}-60\frac{P''(P')^3}{P^4}+$$

$$+24\frac{(P')^5}{P^5}$$

By using (1) we have

$$\sum_{k=0}^{2n}\frac{1}{(z-z_k)^5}\Big|_{z=-1} = \frac{-(2n+1)(40n^3+60n^2-70n+3)}{96}$$

$$\frac{d}{dz}\sum_{k=0}^{2n}\frac{24}{(z-z_k)^5}=\sum_{k=0}^{2n}\frac{-120}{(z-z_k)^6}$$

$$=\frac{d}{dz}\left[\frac{P^{(5)}}{P}-5\frac{P^{(4)}P'}{P^2}-10\frac{P'''P''}{P^2}+20\frac{P'''(P')^2}{P^3}+30\frac{(P'')^2P'}{P^3}-60\frac{P''(P')^3}{P^4}+\right.$$

$$\left.+24\frac{(P')^5}{P^5}\right]=$$

$$=\frac{P^{(6)}}{P}-6\frac{P^{(5)}P'}{P^2}-15\frac{P^{(4)}P''}{P^2}+30\frac{P^{(4)}(P')^2}{P^3}-120\frac{P'''(P')^3}{P^4}+120\frac{P'P''P'''}{P^3}+$$

$$-10\frac{(P''')^2}{P^2}+30\frac{(P'')^3}{P^3}-270\frac{(P')^2(P'')^2}{P^4}+360\frac{P''(P')^4}{P^5}-120\frac{(P')^6}{P^6}$$

and evaluated for $z=-1$ yields

$$\sum_{k=0}^{2n}\frac{1}{(z-z_k)^6}=\frac{1}{-8\cdot 120}(2n+1)(2n-1)(2n-5)(16n^3+88n^2+114n-3)$$

(3.1) yields

$$\sum_{k=1}^{n}\frac{1}{C_{k,n}^6}=\frac{-1}{2}+\frac{32}{8\cdot 120}((2n+1)(2n-1)(2n-5)(16n^3+88n^2+14n-3)+$$

$$-96\frac{-(2n+1)(40n^3+60n^2-70n+3)}{96}-96\frac{(2n+1)(2n-3)(4n^2+12n-1)}{48}+$$

$$+32\frac{12n^2+4n-1}{-8}=\frac{8}{15}n(n+1)(8n^4+16n^3+19n^2+11n+6)$$

Now we can come back to (3) and write

$$S_3(n)\doteq\sum_{k=1}^{n}\left(\tan\frac{k\pi}{2n+1}\right)^6=$$

$$=n+3(2n^2+2n)-3\frac{8}{3}n(n+1)(n^2+n+1)+$$

$$+\frac{8}{15}n(n+1)(8n^4+16n^3+19n^2+11n+6)=$$

$$=\frac{1}{15}n(2n+1)(32n^4+80n^3+40n^2-20n+3)$$

Also solved by Toyesh Prakash Sharma (Student) Agra College, India and the proposer.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated! . . . And don't worry about making a mistake. All is well!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Recommendations

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to LaTeX.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.

- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Problem proposed to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣