Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
March 15, 2008

- 4996: *Proposed by Kenneth Korbin, New York, NY.*

  Simplify:
  $$\sum_{i=1}^{N}\binom{N}{i}\left(2^{i-1}\right)\left(1+3^{N-i}\right).$$

- 4997: *Proposed by Kenneth Korbin, New York, NY.*

  Three different triangles with integer-length sides all have the same perimeter P and all have the same area K.
  Find the dimensions of these triangles if $K = 420$.

- 4998: *Proposed by Jyoti P. Shiwalkar & M.N. Deshpande, Nagpur, India.*

  Let $A = [a_{i,j}]$, $i = 1, 2, \cdots$ and $j = 1, 2, \cdots, i$ be a triangular array satisfying the following conditions:

  $$\begin{aligned} 1)\ a_{i,1} &= L(i) \text{ for all } i \\ 2)\ a_{i,i} &= i \text{ for all } i \\ 3)\ a_{i,j} &= a_{i-1,j} + a_{i-2,j} + a_{i-1,j-1} - a_{i-2,j-1} \text{ for } 2 \le j \le (i-1). \end{aligned}$$

  If $T(i) = \sum_{j=1}^{i} a_{i,j}$ for all $i \geq 2$, then find a closed form for $T(i)$, where $L(i)$ are the Lucas numbers, $L(1) = 1, L(2) = 3$, and $L(i) = L(i-1) + L(i-2)$ for $i \geq 3$.

- 4999: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Find all real triplets (x, y, z) such that

$$\begin{aligned} x+y+z &= 2 \\ 2^{x+y^2}+2^{y+z^2}+2^{z+x^2} &= 6\sqrt[9]{2} \end{aligned}$$

- 5000: *Proposed by Richard L. Francis, Cape Girardeau, MO.*

  Of all the right triangles inscribed in the unit circle, which has the Morley triangle of greatest area?

- 5001: *Proposed by Ovidiu Furdui, Toledo, OH.*

  Evaluate:

$$\int_0^\infty \ln^2\left(\frac{x^2}{x^2+3x+2}\right)dx.$$

Solutions

- 4978: *Proposed by Kenneth Korbin, New York, NY.*

  Given equilateral triangle ABC with side $\overline{\mathrm{AB}} = 9$ and with cevian $\overline{\mathrm{CD}}$. Find the length of $\overline{\mathrm{AD}}$ if $\triangle$ADC can be inscribed in a circle with diameter equal to 10.

Solution by Dionne Bailey, Elsie Campbell, Charles Diminnie, Karl Havlak, and Paula Koca (jointly), San Angelo, TX.

Let $x = \overline{AD}$ and $y = \overline{CD}$. If A is the area of $\triangle ADC$, then

$$A = \frac{1}{2}(9)\,x\sin 60^o = \frac{9}{4}\sqrt{3}x.$$

Since the circumradius of $\triangle ADC$ is 5, we have

$$5 = \frac{9xy}{4A} = \frac{y}{\sqrt{3}}$$

and hence,

$$y = 5\sqrt{3}.$$

Then, by the Law of Cosines,

$$75 = y^2 = x^2 + 81 - 2(9)\,x\cos 60^o = x^2 - 9x + 81$$

which reduces to

$$x^2 - 9x + 6 = 0.$$

Therefore, there are two possible solutions:

$$\overline{AD} = x = \frac{9 \pm \sqrt{57}}{2}.$$

Also solved by Scott H. Brown, Montgomery, AL; Daniel Copeland, Portland, OR; M.N. Deshpande, Nagpur, India; Paul M. Harms, North

Newton, KS; Jahangeer Kholdi, Portsmouth, VA; Xiezhang Li, David Stone & John Hawkins (jointly), Statesboro, GA; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Charles, McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; David C. Wilson, Winston-Salem, NC, and the proposer.

- 4979: *Proposed by Kenneth Korbin, New York, NY.*

Part I: Find two pairs of positive numbers (x, y) such that

$$\frac{x}{\sqrt{1+y}-\sqrt{1-y}}=\frac{\sqrt{65}}{2},$$

where x is an integer.

Part II: Find four pairs of positive numbers (x, y) such that

$$\frac{x}{\sqrt{1+y}-\sqrt{1-y}}=\frac{65}{2},$$

where x is an integer.

Solution 1 by Brian D. Beasley, Clinton, SC.

(I) We need $0 < y \le 1$, so requiring x to be an integer yields

$$x=\frac{\sqrt{65}}{2}\left(\sqrt{1+y}-\sqrt{1-y}\right)\in\{1,2,3,4,5\}.$$

We solve for y to obtain $y = 2x\sqrt{65-x^2}/65$. Substituting $x \in \{1, 2, 3, 4, 5\}$ yields five solutions for (x, y), with two of these also having y rational, namely

$$(x, y) = (1, 16/65) \quad \text{and} \quad (x, y) = (4, 56/65).$$

(II) We again need $0 < y \le 1$, so requiring x to be an integer yields

$$x=\frac{65}{2}\left(\sqrt{1+y}-\sqrt{1-y}\right)\in\{1,2,\ldots,45\}.$$

We solve for y to obtain $y = 2x\sqrt{4225-x^2}/4225$. Substituting $x \in \{1, 2, \ldots, 45\}$ yields 45 solutions for (x, y), with four of these also having y rational, namely

$$(x, y) = (16, 2016/4225); \quad (x, y) = (25, 120/169);$$
$$(x, y) = (33, 3696/4225); \quad (x, y) = (39, 24/25).$$

Solution 2 by James Colin Hill, Cambridge, MA.

Part I: The given equation yields $4x^2 = 130(1 + \sqrt{1-y^2})$. Let $y = \cos\theta$. Then

$$\sin\theta=\frac{4x^2}{130}-1.$$

For $x \in Z^+$, we find several solutions, including the following (rational) pair:

$$\begin{aligned} x &= 1,\ y = 16/65 \\ x &= 4,\ y = 56/64. \end{aligned}$$

Part II: The given equation yields $\sin\theta = \dfrac{4x^2}{8450} - 1$, where $y = \cos\theta$ as before. For $x \in Z^+$, we find many solutions, including the following (rational) four:

$$x = 16,\ y = 2016/4225$$

$$
\begin{aligned}
x &= 25, \; y = 120/169 \\
x &= 33, \; y = 3696/4225 \\
x &= 39, \; y = 24/25
\end{aligned}
$$

Also solved by John Boncek, Montgomery, AL; Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins (jointly), Statesboro, GA, David C. Wilson, Winston-Salem, NC, and the proposer.

- 4980: *J.P. Shiwalkar and M.N. Deshpande, Nagpur, India.*

An unbiased coin is sequentially tossed until $(r+1)$ heads are obtained. The resulting sequence of heads (H) and tails (T) is observed in a linear array. Let the random variable X denote the number of double heads (HH's, where overlapping is allowed) in the resulting sequence. For example: Let $r = 6$ so the unbiased coin is tossed till 7 heads are obtained and suppose the resulting sequence of H's and T's is as follows:

HHTTT**H**TTTT**HHH**TT**H**

Now in the above sequence, there are three double heads (HH's) at toss number $(1,2), (11,12)$ and $(12,13)$. So the random variable X takes the value 3 for the above observed sequence.

In general, what is the expected value of X?

Solution by N. J. Kuenzi, Oshkosh,WI.

Let $X(r)$ be the number of double heads (HH) in the resulting sequence.
First consider the case $r = 1$. Since the resulting sequence of heads (H) and tails (T) ends in either TH or HH, $P[X(1) = 0] = \dfrac{1}{2}$ and $P[X(1) = 1] = \dfrac{1}{2}$. So $E[X(1)] = \dfrac{1}{2}$.

Next let $r > 1$, an unbiased coin is tossed until $(r+1)$ heads are obtained. If the resulting sequence of $H's$ and $T's$ ends in TH then $X(r) = X(r-1)$. And if the resulting sequence of $H's$ and $T's$ ends in HH then $X(r) = X(r-1)$. So

$$P[X(r) = X(r-1)] = \frac{1}{2} \text{ and } P[X(r) = X(r-1) + 1] = \frac{1}{2}.$$

It follows that

$$E[X(r)] = \frac{1}{2}E[X(r-1)] + \frac{1}{2}E[X(r-1)+1] = E[X(r-1)] + \frac{1}{2}.$$

Finally, the Principle of Mathematical Induction can be used to show that $E[X(r)] = \dfrac{r}{2}$.

Also solved by Kee-Wai Lau, Hong Kong, China; Harry Sedinger, St. Bonvatenture, NY, and the proposers.

- 4981: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

Find all real solutions of the equation

$$5^x + 3^x + 2^x - 28x + 18 = 0.$$

Solution by Paolo Perfetti, Dept. of Mathematics, University of Rome, Italy.

Let $f(x) = 5^x + 3^x + 2^x - 28x + 18$. The values for $x \le 0$ are excluded from being solutions because for these values $f(x) > 0$. It is immediately seen that $f(x) = 0$ for $x = 1, 2$. Moreover, the derivative $f'(x) = 5^x \ln 5 + 3^x \ln 3 + 2^x \ln 2 - 28$ is an increasing continuous function such that:

$$\begin{aligned}
1)\ f'(0) &= \ln 30 - 28 < 0, \quad \lim_{x+\infty} f'(x) = +\infty \\
2)\ f'(1) &= 5\ln 5 + 3\ln 3 + 2\ln 2 - 28 < 10 + 6 + 2 - 28 = -10 \\
3)\ f'(2) &= 25\ln 5 + 9\ln 3 + 4\ln 2 - 28 \ge 34 - 28 > 0.
\end{aligned}$$

By continuity this implies that $f'(x) = 0$ for just one point x_o between 1 and 2, and that the graph of $f(x)$ has a minimum only at $x = x_o$. It follows that there are no values of x other than $x = 1, 2$ satisfying $f(x) = 0$.

Also solved by Brain D. Beasley, Clinton, SC; Pat Costello, Richmond, KY; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; M.N. Deshpande, Nagpur, India; Paul M. Harms, North Newton, KS; Jahangeer Kholdi, Portsmouth, VA; Kee-Wai Lau, Hong Kong, China; Kenneth Korbin, NY, NY; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins, Statesboro, GA, and the proposers.

- 4982: *Proposed by Juan José Egozcue and José Luis Díaz-Barrero, Barcelona, Spain.*

Calculate

$$\lim_{n\to\infty} \frac{1}{n+1}\left(\sum_{1\le i_1\le n+1} \frac{1}{i_1} + \sum_{1\le i_1<i_2\le n+1} \frac{1}{i_1 i_2} + \cdots + \sum_{1\le i_1<\ldots<i_n\le n+1} \frac{1}{i_1 i_2 \ldots i_n}\right).$$

Solution 1 by Paul M. Harms, North Newton, KS.

Let $S(n)$ be the addition of the summations inside the parentheses of the expression in the problem. When $n = 1$. The expression in the problem is

$$\frac{1}{2}\left(\left[\frac{1}{1} + \frac{1}{2}\right] + \left[\frac{1}{1(2)}\right]\right) = (\frac{1}{2})2 = 1, \text{ where } S(1) = 2.$$

When $n = 2$ the expression is

$$\begin{aligned}
&= \frac{1}{3}\left(\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\right] + \left[\frac{1}{1(2)} + \frac{1}{1(3)} + \frac{1}{(2)(3)}\right] + \left[\frac{1}{1(2)(3)}\right]\right) \\
&= \frac{1}{3}\left(S(1) + \frac{1}{3}\left[1 + S(1)\right]\right) \\
&= \frac{1}{3}\left(2 + \frac{1}{3}3\right) = 1, \text{ where } S(2) = 3.
\end{aligned}$$

When $n = 3$ the expression is

$$\frac{1}{4}\left(S(2) + \frac{1}{4}\left[1 + S(2)\right]\right) = \frac{1}{4}\left(3 + \frac{1}{4}\left[1 + 3\right]\right) = 1, \text{ where } S(3) = 4.$$

When $n = k + 1$ the expression becomes

$$\frac{1}{k+2}\left(S(k) + \frac{1}{k+2}\left[1 + S(k)\right]\right) = 1, \text{ where } S(k) = k + 1.$$

The limit in the problem is one.

Solution 2 by David E. Manes,Oneonta, NY.

Let

$$a_n = \frac{1}{n+1}\left(\sum_{1\le i_1\le n+1}\frac{1}{i_1} + \sum_{1\le i_1<i_2\le n+1}\frac{1}{i_1 i_2} + \cdots + \sum_{1\le i_1<\ldots<i_n\le n+1}\frac{1}{i_1 i_2\ldots i_n}\right).$$

Then $a_1 = 3/4, a_2 = 17/18, a_3 = 95/96$, and $a_4 = 599/600$.

We will show that

$$a_n = 1 - \frac{1}{(n+1)(n+1)!}$$

Note that the equation is true for $n = 1$ and assume inductively that it is true for some integer $n \ge 1$. Then

$$\begin{aligned}
a_n &= \frac{1}{n+1}\left(\sum_{1\le i_1\le n+1}\frac{1}{i_1} + \sum_{1\le i_1<i_2\le n+1}\frac{1}{i_1 i_2} + \cdots + \sum_{1\le i_1<\ldots<i_n\le n+1}\frac{1}{i_1 i_2\ldots i_n}\right)\\
&= \frac{1}{n+2}\left[(n+1)a_n + \frac{1}{n+2} + \left(\frac{n+1}{n+2}\right)a_n + \frac{1}{(n+1)!}\right]\\
&= \frac{1}{n+2}\left[(n+1)\left(1 - \frac{1}{(n+1)(n+1)!}\right) + \frac{1}{n+2} + \left(\frac{n+1}{n+2}\right)\left(1 - \frac{1}{(n+1)(n+1)!}\right) + \frac{1}{(n+1)!}\right]\\
&= \frac{1}{n+2}\left[(n+1) - \frac{1}{(n+1)!} + 1 - \frac{1}{(n+2)(n+1)!} + \frac{1}{(n+1)!}\right]\\
&= \frac{1}{n+2}\left[n+2 - \frac{1}{(n+2)!}\right] = 1 - \frac{1}{(n+2)(n+2)!}.
\end{aligned}$$

Therefore, the result is true for $n+1$. By induction $a_n = 1 - \dfrac{1}{(n+1)(n+1)!}$ is valid for all integers $n \ge 1$. Hence $\lim\limits_{n\to\infty} a_n = 1$.

Also solved by Carl Libis, Kingston, RI; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- 4983: *Proposed by Ovidiu Furdui, Kalamazoo, MI.*

Let k be a positive integer. Evaluate

$$\int_0^1 \left\{\frac{k}{x}\right\} dx,$$

where $\{a\}$ is the *fractional part* of a.

Solution by Kee-Wai Lau, Hong Kong, China.

We show that

$$\int_0^1 \left\{\frac{k}{x}\right\} dx = k\left(\sum_{n=1}^{k}\frac{1}{n} - \ln k - \gamma\right),$$

where γ is Euler's constant. By substituting $x = ky$, we obtain

$$\int_0^1 \left\{\frac{k}{x}\right\} dx = k\int_0^{1/k} \left\{\frac{1}{y}\right\} dy.$$ For any integer $M > k$, we have

$$\begin{aligned}
\int_{1/M}^{1/k} \left\{\frac{1}{y}\right\} dy &= \sum_{n=k}^{M-1} \int_{1/(n+1)}^{1/n} \left\{\frac{1}{y}\right\} dy \\
&= \sum_{n=k}^{M-1} \int_{1/(n+1)}^{1/n} \left\{\frac{1}{y} - n\right\} dy \\
&= \sum_{n=k}^{M-1} \left(\ln(\frac{n+1}{n}) - \frac{1}{n+1}\right) \\
&= \ln\left(\frac{M}{k}\right) - \sum_{n=k+1}^{M} \frac{1}{n} \\
&= \sum_{n=1}^{k} \frac{1}{n} - \ln k - \left(\sum_{n=1}^{M} \frac{1}{n} - \ln M\right).
\end{aligned}$$

Since $\lim\limits_{M\to\infty} \left(\sum_{n=1}^{M} \frac{1}{n} - \ln M\right) = \gamma$, we obtain our result.

Also solved by Brian D. Beasley, Clinton, SC; Jahangeer Kholdi, Portsmouth, VA; David E. Manes, Oneonta, NY; Paolo Perfetti, Dept. of Mathematics, University of Rome, Italy; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Acknowledgments

The name of **Dr. Peter E. Liley of Lafayette, IN** should have been listed as having solved problems 4966, 4973 and 4974. His name was inadvertently omitted from the listing; mea culpa.

Problem 4952 was posted in the January 07 issue of this column. It was proposed by **Michael Brozinsky of Central Islip, NY & Robert Holt of Scotch Plains, NJ.** I received one solution to this problem; it was from **Paul M. Harms of North Newton, KS.** His solution, which was different from the one presented by proposers, made a lot of sense to me and it was published in the October 07 issue of this column. Michael then wrote to me stating that he thinks Paul misinterpreted the problem. For the sake of completeness, here is the proposers' solution to their problem.

- 4952: An archeological expedition discovered all dwellings in an ancient civilization had $1, 2$, or 3 of each of k independent features. Each plot of land contained three of these houses such that the k sums of the number of each of these features were all divisible by 3. Furthermore, no plot contained two houses with identical configurations of features and no two plots had the same configurations of three houses. Find **a)** the maximum number of plots that a house with a given configuration might be located on, and **b)** the maximum number of distinct possible plots.

Solution by the proposers: a) Clearly these maximum numbers will be attained using the 3^k possible configurations for a house.

Note: For any two houses on a plot:
1) if they have the same number of any given feature then the third house will necessarily have this same number of that feature since the sum must be divisible by three, and
2) if they have a different number of a given feature then the third house will have a different number of that feature than the first two houses since the sum must be divisible by three.

It follows then that any fixed house can be adjoined with $\dfrac{3^k-1}{2}$ possible pairs of houses to be placed on a plot since the second house can be any of the remaining 3^k-1 house configurations but the third configuration is uniquely determined by the above note and the fact that no two houses on a plot can be identically configured. These 3^k-1 permutations of the second and third house thus must have arisen from the $\dfrac{3^k-1}{2}$ possible pairs claimed above. The answer is thus $\dfrac{3^k-1}{2}$.
b) The above note shows that for any two differently configured houses only one of the remaining 3^k-2 configurations will form a plot with these two. Hence, the probability that 3 configurations chosen randomly from the 3^k configurations are suitable for a plot is $\dfrac{1}{3^k-2}$. Since there are $\dbinom{3^k}{3}$ subsets of size three that can be formed from the 3^k configurations, it follows that the maximum number of distinct possible plots is
$$\frac{\dbinom{3^k}{3}}{3^k-2}=\frac{3^{k-1}(3^k-1)}{2}.$$