# Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

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*Solutions to the problems stated in this issue should be posted before*
*May 15, 2016*

- **5391:** *Proposed by Kenneth Korbin, New York, NY*

  A triangle with integer length sides $(49, b, b+1)$ has integer area. Find two possible values of $b$.

- **5392:** *Proposed by Titu Zvonaru, Comănesti, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

  Prove that if $x, y, z > 0$, then
$$\frac{4\left(x^2+y^2+z^2\right)}{27\left(xy+yz+zx\right)}+\frac{x}{7x+y+z}+\frac{y}{x+7y+z}+\frac{z}{x+y+7z}\geq\frac{13}{27}.$$

- **5393:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Through the midpoint of the diagonal $BD$ in the convex quadrilateral $ABCD$ we draw a straight line parallel to the diagonal $AC$. This line intersects the side $AD$ at the point $E$. Show that
$$\frac{1}{[ABC]}+\frac{1}{[AEC]}\geq\frac{4}{[CED]}.$$
  Here $[XYZ]$ represents the are of $\triangle XYZ$.

**5394:** *Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain*

Let $a$, $b$ and $c$ be positive real numbers such that $ab+bc+ca=3$ and $n>1$. Prove that
$$\sqrt[n]{a+\frac{1}{abc}}+\sqrt[n]{b+\frac{1}{abc}}+\sqrt[n]{c+\frac{1}{abc}}\geq 3\sqrt[n]{2}.$$

- **5395:** *Proposed by Mohsen Soltanifar (Ph.D. student), Biostatistics Division, Dalla Lana School of Public Health, University of Toronto, Canada.*

  Given the sequence $\{\sigma_n^2\}_{n=1}^{\infty}$ of positive numbers and $X_1 \sim N(\mu, \sigma_1^2)$ . Define recursively a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ via
$$X_{n+1}|X_n \sim N\left(X_n, \sigma_{n+1}^2\right) \qquad n=1,2,3,\ldots$$

Calculate the limit distribution $X$ of $\{X_n\}_{n=1}^{\infty}$.

Reference: Rosenthal, J.S. (2007). A First Look at Rigorous Probability (2nd edition), World Scientific, p. 139.

*Proposer's note concerning the problem:*

This is a Bayesian Hierarchical Model of Human Heights from Adam & Eve to the end of time.

Consider a family with its children. We know that height has a normal distribution. We also know that height of children is due to genetic factors which are dependent on the height of their parents, but usually this distribution has the same mean as the mean height of their parents but may vary (some children are taller, some shorter, some are average- versus their parents). So, the height of children may be modeled as the normal distribution conditioned to the height of their parents with same mean but potentially different variance.

The first term in the sequence is the distribution of height of Adam & Eve. The second term is the conditional distribution of their children's height. This goes till the end of time consecutively when, according to some beliefs, the Messiah returns. Accordingly, the Messiah will return and a generation of humans will observe this return . But we do not know when this will occur. So, we may assume the Messiah will return as time approaches infinity, and that the distribution of the height of generations of humans that observe the return is "X". We are interested in knowing certain features of this distribution.

This problem is a mathematical modeling of the above belief.

- **5396:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Find all continuous functions $f : \Re \to \Re$ such that

$$f(-x) = x + \int_0^x e^{-t} f(x-t)dt, \quad \forall x \in \Re.$$

*Solutions*

- **5373:** *Proposed by Kenneth Korbin, New York, NY*

  Given the equation $\dfrac{2\sqrt{2}}{\sqrt{343-147\sqrt{5}}-\sqrt{315-135\sqrt{5}}} = \sqrt{x+y\sqrt{5}}.$

  Find positive integers $x$ and $y$.

  **Solution 1 by Albert Stadler, Herrliberg, Switzerland**

  We note that

$$\frac{2\sqrt{2}}{\sqrt{343-147\sqrt{5}}-\sqrt{315-135\sqrt{5}}} = \frac{2\sqrt{2}}{\left(7-3\sqrt{5}\right)\sqrt{7-3\sqrt{5}}}.$$

$$\frac{2\sqrt{2}}{\left(7-3\sqrt{5}\right)\sqrt{76-3\sqrt{5}}} = \sqrt{x+y\sqrt{5}} \text{ implies}$$

$$x+y\sqrt{5}=\frac{8}{\left(7-3\sqrt{5}\right)^3}=\frac{8\cdot\left(7+3\sqrt{5}\right)^3}{4^3}=161+72\sqrt{5}.$$

So $(x,y)=(161,72)$.

**Solution 2 by Neculai Stanciu "George Emil Palade" School, Buzău and Titu Zvonaru, Comănesti, Romania**

We have

$$\sqrt{343-147\sqrt{5}} = \sqrt{\frac{441}{2}}-\sqrt{\frac{245}{2}}=\frac{21\sqrt{2}}{2}-\frac{7\sqrt{10}}{2}.$$

$$\sqrt{315-135\sqrt{5}} = \sqrt{\frac{405}{2}}-\sqrt{\frac{225}{2}}=\frac{9\sqrt{10}}{2}-\frac{15\sqrt{2}}{2}.$$

$$\sqrt{343-147\sqrt{5}}-\sqrt{315-135\sqrt{5}} = \frac{21\sqrt{2}}{2}-\frac{7\sqrt{10}}{2}-\frac{9\sqrt{10}}{2}+\frac{15\sqrt{2}}{2}=18\sqrt{2}-8\sqrt{10},\ \text{and so}$$

$$\frac{2\sqrt{2}}{\sqrt{343-147\sqrt{5}}-\sqrt{315-135\sqrt{5}}} = \frac{1}{9-4\sqrt{5}}=9+4\sqrt{5}.$$

Solving the equation $\sqrt{x+y\sqrt{5}}=9+4\sqrt{5}$ gives $x=161, y=72$.

**Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain**

Since $343-147\sqrt{5}=49\left(7-3\sqrt{5}\right)$ and $315-135\sqrt{5}=45\left(7-3\sqrt{5}\right)$, we obtain

$$\begin{aligned}\sqrt{343-147\sqrt{5}}-\sqrt{315-135\sqrt{5}} &= 7\sqrt{7-3\sqrt{5}}-\sqrt{45}\sqrt{7-3\sqrt{5}}\\ &= \left(7-3\sqrt{5}\right)\sqrt{7-3\sqrt{5}}\\ &= \sqrt{\left(7-3\sqrt{5}\right)^3}=\sqrt{8\left(161-72\sqrt{5}\right)}=2\sqrt{2}\sqrt{161-72\sqrt{5}}.\end{aligned}$$

Hence,

$$\frac{2\sqrt{2}}{\sqrt{343-147\sqrt{5}}-\sqrt{315-135\sqrt{5}}}=\sqrt{\frac{1}{161-72\sqrt{5}}}=\sqrt{161+72\sqrt{5}}.$$

So, $x+y\sqrt{5}=161+72\sqrt{5}$, that is $x-161=(72-y)\sqrt{5}$. If $72-y\neq 0$, then $\sqrt{5}=\dfrac{x-161}{72-y}$, which is impossible because, since $x-161$ and $72-y$ are integers, the left hand side is irrational and the right hand side is rational. Thus, $72-y=0$ and henceforth, $x-161=0\sqrt{5}$. That is $(x,y)=(161,72)$, is valid and the only solution, because of the stipulation in the problem that $x$ and $y$ be positive integers.

**Solution 4 by Kee-Wai Lau, Hong Kong, China**

Squaring both sides of the given equation, we obtain

$$\begin{aligned} x+y\sqrt{5} &= \frac{8}{658-282\sqrt{5}-2\sqrt{207270-92610\sqrt{5}}} \\ &= \frac{8}{658-282\sqrt{5}-2\left(147\sqrt{5}-315\right)} \\ &= \frac{1}{161-72\sqrt{5}} \\ &= 161+72\sqrt{5}. \end{aligned}$$

Hence, $x=161$ and $y=72$.

**Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State Unversity, San Angelo TX; Brian D. Beasley, Presbyterian College, Clinton, SC; Ed Gray, Highland Beach, FL; G.C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Toshihiro Shimizu, Kawasaki, Japan; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania, and the author.**

- **5374:** *Proposed by Roger Izard, Dallas TX*

In a certain triangle, three circles are tangent to the incircle, and all of these circles are tangent to two sides of the triangle. Derive a formula which gives the radius of the incircle in terms of the radii of these three circles.

**Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain**

Let $ABC$ be the triangle, $a=\overline{BC}, b=\overline{CA}, c=\overline{AB}, A=\angle BAC, B=\angle CBA$, $C=\angle ACB, I$ the center of the incircle $ABC$, $r$ its radius, $O_a$ and $r_a$ the respective center and radius of the circle tangent to that incircle and to the side $AB$ and $AC, O_b$ and $r_b$ the center and radius of the circle tangent to that incircle and to the sides $BA$ and $BC$, respectively, $O_c$ and $r_c$ the respective radius of the circle tangent to that incircle and to the sides $CA$ and $CB, T_a$ the point of tangency of the incircle of $ABC$ and the circle with center $O_a$ and radius $r_a, O_a^{'}$ the point of tangency of that circle with $AB$ and $I^{'}$ the point of tangency of the incircle of $ABC$ with $AB$. We shall prove that

$$r=\sqrt{r_ar_b}+\sqrt{r_br_c}+\sqrt{r_cr_a}.$$

From triangles $AO_a'O_a$ and $AI'I$, since $\angle O_a'AO_a=\angle BAC/2=\angle I'AI$, we deduce that $\dfrac{r_a}{AO_a}=\sin\left(\dfrac{A}{2}\right)=\dfrac{r}{AI}$.

Since $AI = AO_a + O_aT_a + T_aI$, $r\csc\left(\frac{A}{2}\right) = r_a\csc\left(\frac{A}{2}\right) + r_a + r$ and , so,

$$\frac{r_a}{r} = \frac{1-\sin\left(\frac{A}{2}\right)}{1+\sin\left(\frac{A}{2}\right)} = \frac{1-2\sin\left(\frac{A}{4}\right)\cos\left(\frac{A}{2}\right)}{1+2\sin\left(\frac{A}{4}\right)\cos\left(\frac{A}{2}\right)} = \frac{1-2\dfrac{\tan\left(\frac{A}{4}\right)}{1+\tan^2\left(\frac{A}{4}\right)}}{1+2\dfrac{\tan\left(\frac{A}{4}\right)}{1+\tan^2\left(\frac{A}{4}\right)}} = \left(\frac{1-\tan\left(\frac{A}{4}\right)}{1-\tan\left(\frac{A}{4}\right)}\right)^2.$$

Analogously,

$$\frac{r_b}{r} = \left(\frac{1-\tan\left(\frac{B}{4}\right)}{1-\tan\left(\frac{B}{4}\right)}\right)^2 \text{ and } \frac{r_c}{r} = \left(\frac{1-\tan\left(\frac{C}{4}\right)}{1-\tan\left(\frac{C}{4}\right)}\right)^2.$$

Let us dnote $t_a = \tan\left(\frac{A}{4}\right)$, $t_b = \tan\left(\frac{B}{4}\right)$ and $t_c = \tan\left(\frac{C}{4}\right)$. Since $0 < \frac{A}{4} < \frac{\pi}{4}$, $0 < t_a < 1$, so since $1 - t_a > 0$, analogously we have $1 - t_b > 0$ and $1 - t_c > 0$. The equality to prove, $\sqrt{\frac{r_a}{r}}\sqrt{\frac{r_b}{r}} + \sqrt{\frac{r_b}{r}}\sqrt{\frac{r_c}{r}} + \sqrt{\frac{r_c}{r}}\sqrt{\frac{r_a}{r}} = 1$ is successively equivalent to showing that

$$\frac{1-t_a}{1+t_a}\cdot\frac{1-t_b}{1+t_b} + \frac{1-t_b}{1+t_b}\cdot\frac{1-t_c}{1+t_c} + \frac{1-t_c}{1+t_c}\cdot\frac{1-t_a}{1+t_a} = 1.$$

And this is equivalent to showing that

$$(1-t_a)(1-t_b)(1+t_c) + (1-t_b)(1-t_c)(1+t_a) + (1-t_a)(1-t_c)(1+t_b) = (1+t_a)(1+t_b)(1+t_c).$$

Expanding and simplifying we obtain:

$$t_at_b + t_bt_c + t_ct_a + t_a + t_b + t_c = t_at_bt_c + 1.$$

But this is true because

$$1 = \tan\left(\frac{\pi}{4}\right) = \tan\left(\frac{A+B+C}{4}\right) = \tan\left(\frac{A}{4}+\frac{B}{4}+\frac{C}{4}\right) = \frac{t_a+t_b+t_c-t_at_bt_c}{1-t_at_b-t_bt_c+t_ct_a}.$$

So, the formula $r = \sqrt{r_ar_b} + \sqrt{r_br_c} + \sqrt{r_cr_a}$ holds.

**Solution 2 by Toshihiro Shimizu, Kawasaki, Japan**

Let the triangle be $A_1A_2A_3$, the incenter and radius of the triangle be $I$,$r$, the centers of the three circles be $I_1, I_2, I_3$ and radius of them be $r_1, r_2, r_3$, respectively. Let the foot of perpendicular from $I$ to $A_2A_3, A_3A_1, A_1A_2$ be $H_1, H_2, H_3$, respectively. Let $\alpha_i = \angle A_iIA_{i+1} = \angle A_iIA_{i+2}$ where the indices are considered to be the same $\pmod 3$. Now, we calculate $\tan\alpha_1$. Let the foot of perpendicular from $I_1$ to $IH_2$ be $K$. Then, $IK = r - r_1$, $II_1 = r + r_1$. Thus, $KI_1 = \sqrt{II_1^2 - IK^2} = 2\sqrt{rr_1}$. Therefore,

$$\tan\alpha_1 = \frac{2\sqrt{rr_1}}{r-r_1}.$$

Similarly,

$$\tan\alpha_2 = \frac{2\sqrt{rr_2}}{r-r_2}$$
$$\tan\alpha_3 = \frac{2\sqrt{rr_3}}{r-r_3}.$$

On the other hand, since $\alpha_1+\alpha_2+\alpha_3=\pi$, it follows that

$$\begin{aligned}\tan\alpha_1+\tan\alpha_2+\tan\alpha_3 &= \tan\alpha_1+\tan\alpha_2-\tan(\alpha_1+\alpha_2)\\ &= \tan\alpha_1+\tan\alpha_2-\frac{\tan\alpha_1+\tan\alpha_2}{1-\tan\alpha_1\tan\alpha_2}\\ &= \frac{-\tan\alpha_1\tan\alpha_2(\tan\alpha_1+\tan\alpha_2)}{1-\tan\alpha_1\tan\alpha_2}\\ &= -\tan\alpha_1\tan\alpha_2\tan(\alpha_1+\alpha_2)\\ &= \tan\alpha_1\tan\alpha_2\tan\alpha_3.\end{aligned}$$

Thus,

$$\frac{2\sqrt{rr_1}}{r-r_1}+\frac{2\sqrt{rr_2}}{r-r_2}+\frac{2\sqrt{rr_3}}{r-r_3} = \frac{2\sqrt{rr_1}}{r-r_1}\cdot\frac{2\sqrt{rr_2}}{r-r_2}\cdot\frac{2\sqrt{rr_3}}{r-r_3}$$
$$\sum_{cyc}\sqrt{r_1}(r-r_2)(r-r_3) = 4r\sqrt{r_1r_2r_3}$$
$$r^2\sum_{cyc}\sqrt{r_1} - r\left(\sum_{sym}\sqrt{r_1}r_2+4\sqrt{r_1r_2r_3}\right)+\sqrt{r_1r_2r_3}\sum_{cyc}\sqrt{r_2r_3}=0$$

We see it as a quadratic equation of $r$. Then the discriminant is

$$\begin{aligned}D &= \left(\sum_{sym}\sqrt{r_1}r_2+4\sqrt{r_1r_2r_3}\right)^2 - 4\sum_{cyc}\sqrt{r_1}\cdot\sqrt{r_1r_2r_3}\sum_{cyc}\sqrt{r_2r_3}\\ &= \left(\sum_{sym}\sqrt{r_1}r_2\right)^2+8\sqrt{r_1r_2r_3}\sum_{sym}\sqrt{r_1}r_2+16r_1r_2r_3-4\sqrt{r_1r_2r_3}\left(3\sqrt{r_1r_2r_3}+\sum_{sym}\sqrt{r_1}r_2\right)\\ &= \left(\sum_{sym}\sqrt{r_1}r_2\right)^2+4\sqrt{r_1r_2r_3}\sum_{sym}\sqrt{r_1}r_2+4r_1r_2r_3\\ &= \left(\sum_{sym}\sqrt{r_1}r_2+2\sqrt{r_1r_2r_3}\right)^2.\end{aligned}$$

Thus,

$$r = \frac{\sum_{sym}\sqrt{r_1}r_2+4\sqrt{r_1r_2r_3}\pm\left(\sum_{sym}\sqrt{r_1}r_2+2\sqrt{r_1r_2r_3}\right)}{2\sum_{cyc}\sqrt{r_1}}.$$

We first consider the minus sign of the case. In this case,

$$\begin{aligned} r &= \frac{\sqrt{r_1r_2r_3}}{\sum_{cyc}\sqrt{r_1}} \\ &\le \frac{\sqrt{r_1r_2r_3}}{3\sqrt[3]{\sqrt{r_1r_2r_3}}} \\ &= \frac{1}{3}(r_1r_2r_3)^{1/3} \\ &\le \frac{1}{3}\max\{r_1,r_2,r_3\} \end{aligned}$$

It contradicts with the fact that $r$ is larger any of $r_1, r_2, r_3$.
Thus, the plus sign must be ocurred. Then,

$$\begin{aligned} r &= \frac{\sum_{sym}\sqrt{r_1}r_2 + 3\sqrt{r_1r_2r_3}}{\sum_{cyc}\sqrt{r_1}} \\ &= \frac{(\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3})(\sqrt{r_1r_2}+\sqrt{r_2r_3}+\sqrt{r_3r_1})}{\sum_{cyc}\sqrt{r_1}} \\ &= \sqrt{r_1r_2}+\sqrt{r_2r_3}+\sqrt{r_3r_1} \end{aligned}$$

**Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Albert Stadler, Herrliberg, Switzerland, and the proposer**

- **5375*:** *Proposed by Kenneth Korbin, New York, NY*

Prove or disprove the following conjecture. Let $k$ be the product of $N$ different prime numbers each congruent to $1(mod\ 4)$. Let $a$ be any positive integer.

Conjecture: The total number of different rectangles and trapezoids with integer length sides that can be inscribed in a circle with diameter $k$ is exactly $\dfrac{5^N - 3^N}{2}$.

*Editor′s comment*: The number for this problem carries with it an asterisk. The asterisk signifies that neither the proposer nor the editor are aware of a proof of this conjecture. **Toshihiro Shimizu of Kawasaki, Japan** considered the case $k = 5 \cdot 17$, and stated:
"There are four rectangles satisfying the conditions of the problem:
$(a, b) = (13, 84), (36, 77), (40, 75), (51, 68)$, where $a$ and $b$ are the lengths of the sides of the rectangle."
and
"There are six trapezoids satisfying the conditions of the problem:

$$(a, b, c) = (13, 77, 40), (13, 77, 68), (36, 84, 40), (36, 84, 51), (43, 83, 34), (43, 83, 50),$$

where $a$, and $b$ are the lengths of the two parallel sides of the trapezoid and $c$ is the length of the other two sides." He went on to state that he came to these conclusions with the aid of a computer.

*Editor′s update* : No analytic solutions to the conjecture were received, so the problem will remain open. Ken Korbin, the author of 5375, sent a comment that we should also note that
$\dfrac{5^N - 3^N}{2} = \displaystyle\sum_{j=1}^{N} \binom{N}{j} \left(2^{j-1}\right)\left(3^{N-j}\right).$

When a complete solution is received, it will be published.

- **5376:** *Proposed by Arkady Alt , San Jose ,CA*

Let $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$ be positive real numbers such that
$b_1 < a_1 < b_2 < a_2 < ... < a_{n-1} < b_n < a_n$.
Let
$$F(x) = \frac{(x-b_1)(x-b_2)...(x-b_n)}{(x-a_1)(x-a_2)...(x-a_n)}.$$
Prove that $F'(x) < 0$ for any $x \in Dom(F)$.

**Solution 1 by Albert Stadler, Herrliberg, Switzerland**

We note that $F(x) = \prod_{m=1}^{n} \frac{x-b_m}{x-a_m}$ is a rational function with simple poles at $x = a_m$, $1 \leq m \leq n$.
The residue of $F(x)$ at $x = a_\mu$ equals $(a_\mu - b_\mu) \prod_{m \neq \mu} \frac{a_\mu - b_m}{a_\mu - a_m} > 0$, since $a_\mu > b_\mu$ and

$\frac{a_\mu - b_m}{a_\mu - a_m} > 0$, for $m \neq \mu$.

So $f(x) = F(x) - \sum_{\mu=1}^{n} \frac{a_\mu - b_\mu}{x - a_\mu} \prod_{m \neq \mu} \frac{a_\mu - b_m}{a_\mu - a_m}$ is a bounded entire function which implies that $f(x)$ is a constant. We conclude $f'(x) = 0$ which implies
$F'(x) = -\sum_{\mu=1}^{n} \frac{a_\mu - b_\mu}{(x-a_\mu)^2} \prod_{m \neq \mu} \frac{a_\mu - b_m}{a_\mu - a_m} < 0$ for any $x \in$Dom (F).

**Solution 2 by Ethan Gegner (student), Taylor University, Upland, IN**

For all $x \in Dom(F)$, we have
$$F'(x) = \frac{\left(\prod_{i=1}^n (x-a_i)\right)\left(\prod_{i=1}^n (x-b_i)\right)' - \left(\prod_{i=1}^n (x-b_i)\right)\left(\prod_{i=1}^n (x-a_i)\right)'}{\left(\prod_{i=1}^n (x-a_i)\right)^2} \qquad (1)$$
Suppose $x = b_j$ for some $1 \leq j \leq n$. Then
$$F'(x) = \frac{\prod_{i \neq j}(x-b_i)}{\prod_{i=1}^n (x-a_i)} = \frac{1}{(x-a_j)} \prod_{i \neq j} \frac{x-b_i}{x-a_i} < 0$$
since $x = b_j < a_j$ and $\frac{x-b_i}{x-a_i} > 0$ for all $i \neq j$.

Now suppose $x \notin \{b_1, \ldots, b_n\}$. Then $F(x) \neq 0$, so by equation (1) we have
$$\frac{F'(x)}{F(x)} = \frac{\left(\prod_{i=1}^n (x-b_i)\right)'}{\prod_{i=1}^n (x-b_i)} - \frac{\left(\prod_{i=1}^n (x-a_i)\right)'}{\prod_{i=1}^n (x-a_i)} = \sum_{i=1}^n \left(\frac{1}{x-b_i} - \frac{1}{x-a_i}\right)$$
$$= \sum_{i=1}^n \frac{b_i - a_i}{(x-b_i)(x-a_i)} \qquad (2)$$
If $x < b_1$ or $x > a_n$, then $F(x) > 0$, and $\frac{b_i - a_i}{(x-b_i)(x-a_i)} < 0$ for all $1 \leq i \leq n$, whence $F'(x) < 0$. Suppose there exists some $1 \leq j \leq n-1$ such that $a_j < x < b_{j+1}$. Then for every

$1 \le i \le n$, $x - b_i$ and $x - a_i$ have the same sign, whence $\dfrac{b_i - a_i}{(x-b_i)(x-a_i)} < 0$ and $F(x) = \prod_{i=1}^{n} \dfrac{x-b_i}{x-a_i} > 0$. Thus, equation (2) implies $F'(x) < 0$.

Finally, suppose that $b_j < x < a_j$ for some $1 \le j \le n$. Then

$$\frac{F'(x)}{F(x)} = \sum_{i=1}^{n}\left(\frac{1}{x-b_i} - \frac{1}{x-a_i}\right) = \frac{1}{x-b_1} - \frac{1}{x-a_n} + \sum_{i=1}^{n-1}\left(\frac{1}{x-b_{i+1}} - \frac{1}{x-a_i}\right) > 0$$

since every term on the right hand side is positive. Moreover, $F(x) = \dfrac{x-b_j}{x-a_j}\prod_{i\neq j}\dfrac{x-b_i}{x-a_i} < 0$, so again $F'(x) < 0$.

**Solution 3 by the proposer**

**Lemma.**

$F(x)$ can be represented in form

$$F(x) = 1 + \sum_{k=1}^{n} \frac{c_k}{x-a_k},$$

where $c_k, k = 1, 2, ..., n$ are some positive real numbers.

**Proof.**

Let $F_k(x) := \dfrac{(x-b_1)(x-b_2)\dots(x-b_k)}{(x-a_1)(x-a_3)\dots(x-a_k)}$, $k \le n$.

We will prove by Math Induction that for any $k \le n$ there are positive numbers $c_k(i), i = 1, ..., k$ such that $F_k(x) = 1 + \sum\limits_{i=1}^{k} \dfrac{c_k(i)}{x-a_i}$.

Let $d_k := a_k - b_k > 0, k = 1, 2, ..., n$.

Note that $F_1(x) = \dfrac{x-b_1}{x-a_1} = \dfrac{x-a_1+a_1-b_1}{x-a_1} = 1 + \dfrac{d_1}{x-a_1}$.

Since $\dfrac{x-b_{k+1}}{x-a_{k+1}} = 1 + \dfrac{d_{k+1}}{x-a_{k+1}}$ then in supposition $F_k(x) = 1 + \sum\limits_{i=1}^{k}\dfrac{c_k(i)}{x-a_i}$ , where $c_k(i) > 0, i = 1, ..., k < n$ we obtain

$$\begin{aligned}
F_{k+1}(x) &= F_k(x)\cdot\frac{x-b_{k+1}}{x-a_{k+1}} = \left(1 + \sum_{i=1}^{k}\frac{c_k(i)}{x-a_i}\right)\left(1 + \frac{d_{k+1}}{x-a_{k+1}}\right) \\
&= 1 + \frac{d_{k+1}}{x-a_{k+1}} + \sum_{i=1}^{k}\frac{c_k(i)}{x-a_i} + \sum_{i=1}^{k}\frac{d_{k+1}c_k(i)}{(x-a_i)(x-a_{k+1})} \\
&= 1 + \frac{d_{k+1}}{x-a_{k+1}} + \sum_{i=1}^{k}\frac{c_k(i)}{x-a_i} - \sum_{i=1}^{k}\frac{d_{k+1}c_k(i)}{a_{k+1}-a_i}\left(\frac{1}{x-a_i} - \frac{1}{x-a_{k+1}}\right) \\
&= 1 + \frac{d_{k+1}}{x-a_{k+1}}\left(1 + \sum_{i=1}^{k}\frac{c_k(i)}{a_{k+1}-a_i}\right) + \sum_{i=1}^{k}\frac{c_k(i)}{x-a_i}\left(1 - \frac{d_{k+1}}{a_{k+1}-a_i}\right)
\end{aligned}$$

$$= \quad 1 + \frac{d_{k+1} F_k\left(a_{k+1}\right)}{x - a_{k+1}} + \sum_{i=1}^{k} \frac{c_k\left(i\right)}{x - a_i} \cdot \frac{b_{k+1} - a_i}{a_{k+1} - a_i}.$$

Since $F_k\left(a_{k+1}\right) > 0$ and $b_{k+1} - a_i = \left(b_{k+1} - a_k\right) + \left(a_k - a_i\right) > 0$ then

$c_{k+1}\left(k+1\right) = d_{k+1} F_k\left(a_{k+1}\right) > 0,\ c_{k+1}\left(i\right) := \dfrac{\left(b_{k+1} - a_i\right) c_k\left(i\right)}{a_{k+1} - a_i} > 0, i = 1, 2, ..., k$

and $F_{k+1}\left(x\right) = 1 + \sum\limits_{i=1}^{k+1} \dfrac{c_{k+1}\left(i\right)}{x - a_i}$.

Therefore, since $F\left(x\right) = 1 + \sum\limits_{k=1}^{n} \dfrac{c_k}{x - a_k}$ and $c_k > 0, k = 1, 2, ..., n$ then

$F'\left(x\right) = -\sum\limits_{k=1}^{n} \dfrac{c_k}{\left(x - a_k\right)^2} < 0$ for any $x \in Dom\left(F\right) = \left\{a_1, a_2, ..., a_n\right\}$.

**Solution 4 by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC**

We find $F'(x)$,

$$\begin{aligned} F'(x) \quad = \quad & \frac{b_1 - a_1}{(x - a_1)^2} \cdot \frac{x - b_2}{x - a_2} \cdot \frac{x - b_3}{x - a_3} \cdots \frac{x - b_n}{x - a_n} + \frac{x - b_1}{x - a_1} \cdot \frac{b_2 - a_2}{(x - a_2)^2} \cdot \frac{x - b_3}{x - a_3} \cdots \frac{x - b_n}{x - a_n} + \ldots + \\ & \frac{x - b_1}{x - a_1} \cdot \frac{x - b_2}{x - a_2} \cdots \frac{x - b_{j-1}}{x - a_{j-1}} \cdot \frac{b_j - a_j}{(x - a_j)^2} \cdot \frac{x - b_{j+1}}{x - a_{j+1}} \cdots \frac{x - b_n}{x - a_n} + \ldots + \\ & \frac{x - b_1}{x - a_1} \cdot \frac{x - b_2}{x - a_2} \cdot \frac{x - b_3}{x - a_3} \cdots \frac{b_n - a_n}{(x - a_n)^2}. \qquad (1) \end{aligned}$$

We set

$$\begin{aligned} D_1(x) &= \frac{b_1 - a_1}{(x - a_1)^2} \cdot \frac{x - b_2}{(x - a_2)} \cdot \frac{x - b_3}{(x - a_3)} \cdots \frac{x - b_n}{x - a_n} \\ D_2(x) &= \frac{x - b_1}{x - a_1} \cdot \frac{b_2 - a_2}{(x - a_2)^2} \cdot \frac{x - b_3}{(x - a_3)} \cdots \frac{x - b_n}{x - a_n} \\ &\vdots \\ D_j(x) &= \frac{x - b_1}{x - a_1} \cdot \frac{x - b_2}{x - a_2} \cdots \frac{x - b_{j-1}}{x - a_{j-1}} \cdot \frac{b_j - a_j}{(x - a_j)^2} \cdot \frac{x - b_{j+1}}{x - a_{j+1}} \cdots \frac{x - b_n}{x - a_n} \\ &\vdots \\ D_n(x) &= \frac{x - b_1}{x - a_1} \cdot \frac{x - b_2}{x - a_2} \cdot \frac{x - b_3}{x - a_3} \cdots \frac{b_n - a_n}{(x - a_n)^2}. \text{ Then} \end{aligned}$$

$$F'(x) = \sum_{k=1}^{n} D_k(x). \qquad (2)$$

We note that because

$$0 < b_1 < a_1 < b_2 < a_2 < \cdots < a_{n-1} < b_n < a_n \qquad (3)$$

we have

$$\frac{b_j - a_j}{(x-a_j)^2} < 0, \text{for all } j \text{ with } 1 \le j \le n. \quad (4)$$

Let $x \in Dom(F)$, then we consider the following cases:

**Case 1.** Let $x = b_{j_0}$, for some $j_0 \in \{1, 2, \cdots, n\}$, then $D_j(b_{j_0}) = 0$, for all $j \neq j_0$, and because of (3), $\dfrac{b_{j_0} - b_j}{b_{j0} - a_j} > 0$, for all $j \neq j_0$ and with (4) we conclude that $F'(b_{j_0}) < 0$.

**Case 2.** Let $x < b_1$, then for all $j$ with $1 \le j \le n$, and by using (3), we conclude that and that $\dfrac{x - b_j}{x - a_j} > 0. \quad (5)$

And then by (4) and (5) we get equation (6) that $D_j(x < b_1) < 0$, for all $j$ with $1 \le j \le n$, and this implies that $F'(x < b_1) < 0$.

**Case 3.** Let $x \in (b_{j_0}, a_{j_0})$ for some $j_0 \in \{1, 2, \cdots, n\}$, we will show that $F(x)$ is decreasing on $(b_{j_0}, a_{j_0})$. We know that by (4) and (3),, each function $f_j(x) = \dfrac{x - b_j}{x - a_j}$ is decreasing and positive on $(b_{j_0}, a_{j_0})$, when $j \neq j_0$, then for all $s, t \in (b_{j_0}, a_{j_0})$ with $s < t$ we have

$$f_j(t) > f_j(s), \quad (7)$$

also $f_{j_0}(x) = \dfrac{x - b_{j_0}}{x - a_{j_0}}$ is decreasing but negative on $(b_{j_0}, a_{j_0})$ and

$$f_{j_0}(t) > f_{j_0}(s). \quad (8)$$

Now using (7) and (8), we have $\prod_{j=1}^{n} f_j(t) > \prod_{j=1}^{n} f_j(s)$, that is $F(t) > F(s)$, whenever $s, t \in (b_{j_0}, a_{j_0})$ with $s < t$, the means $F(x)$ is decreasing on $(b_{j_0}, a_{j_0})$ or $F(x) < 0$ on $(b_{j_0}, a_{j0})$.

**Case 4.** Let $x \in \left(a_{j_0}, b_{j_{0+1}}\right)$, for some $j_0 \in \{1, 2, \cdots, n-1\}$, then $f_j(x) = \dfrac{x - b_j}{x - a_j} > 0$, on $\left(a_{j_0}, b_{j_{0+1}}\right)$, for $j \in \{1, 2, \cdots, n\}$, and by (4) and (2), we conclude that $F'(x) < 0$, on $\left(a_n, b_{j_{0+1}}\right)$.

**Case 5.** Let $x \in (b_n, \infty)$, then $f_j(x) = \dfrac{x - b_j}{x - a_j} > 0$, on $(b_n, \infty)$ for all $j \in \{1, 2, \cdots, n\}$, and by (4) and (2), we conclude that $F'(x) < 0$, on $(b_n, \infty)$.

Combining the results of Cases 1-5, we conclude that $F'(x) < 0$ for any $x \in Dom(F)$.

**Also solved by Ed Gray, Highland Beach, FL; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy, and Toshihiro Shimizu, Kawasaki, Japan.**

- **5377:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Show that if $A, B, C$ are the measures of the angles of any triangle $ABC$ and $a, b, c$ the measures of the length of its sides, then holds

$$\prod_{cyclic} \sin^{1/3}(|A - B|) \le \sum_{cyclic} \frac{a^2 + b^2}{3ab} \sin(|A - B|).$$

**Solution 1 by Andrea Fanchini Cantú, Italy**

We know that

$$\sin A = \frac{2K}{bc}, \quad \sin B = \frac{2K}{ac}, \quad \sin C = \frac{2K}{ab}$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \quad \cos B = \frac{c^2+a^2-b^2}{2ca}, \quad \cos C = \frac{a^2+b^2-c^2}{2ab}$$

where $K$ is the area of the triangle.

So we have that

$$\sin(A-B) = \sin A \cos B - \cos A \sin B = \frac{2K(a^2-b^2)}{abc^2},$$

and cyclically,

$$\sin(B-C) = \frac{2K(b^2-c^2)}{a^2bc}, \quad \sin(C-A) = \frac{2K(c^2-a^2)}{ab^2c},$$

so we have

$$\prod_{cyc} \sin^{1/3}(A-B) = \frac{2K}{abc}\sqrt[3]{\frac{(a^2-b^2)(b^2-c^2)(c^2-a^2)}{abc}}, \text{ and}$$

$$\sum_{cyc} \frac{a^2+b^2}{3ab}\sin(A-B) = \frac{2K}{3a^2b^2c^2}\left[(a^2+b^2)(a^2-b^2)+(b^2+c^2)(b^2-c^2)+(c^2+a^2)(c^2-a^2)\right].$$

Now if we assume $C > B > A$ then

$$\prod_{cyc} \sin^{1/3}|A-B| = \frac{2K}{abc}\sqrt[3]{\frac{(a+b)(b-a)(b+c)(c-b)(c+a)(c-a)}{abc}}, \text{ and}$$

$$\sum_{cyc} \frac{a^2+b^2}{3ab}\sin|A-B| = \frac{2K}{3a^2b^2c^2}\left[(a^2+b^2)(a+b)(b-a)+(b^2+c^2)(b+c)(c-b)+(c^2+a^2)(c+a)(c-a)\right].$$

Therefore we need to prove

$$\begin{aligned} & 3\sqrt[3]{a^2b^2c^2(a+b)(b-a)(b+c)(c-b)(c+a)(c-a)} \\ \le\ & (a^2+b^2)(a+b)(b-a)+(b^2+c^2)(b+c)(c-b)+(c^2+a^2)(c+a)(c-a). \end{aligned}$$

But the AM-GM inequality gives us

$$\begin{aligned} & 3\sqrt[3]{a^2b^2c^2(a+b)(b-a)(b+c)(c-b)(c+a)(c-a)} \\ \le\ & 3\sqrt[3]{(a^2+b^2)(a+b)(b-a)(b^2+c^2)(b+c)(c-b)(c^2+a^2)(c+a)(c-a)}. \end{aligned}$$

So it remains to show that

$$a^2b^2c^2 \le (a^2+b^2)(b^2+c^2)(c^2+a^2).$$

But this follows immediately because this inequality is equivalent to $0 \le a^4(b^2+c^2)+b^4(a^2+c^2)+c^4(a^2+b^2)+a^2b^2c^2$ which is immediately evident. Therefore, the statement of the problem is true, with equality holding if and only if the triangle is isosceles.

**Solution 2 Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX**

We will prove the slightly stronger inequality

$$\prod_{cyclic} \sin^{\frac{1}{3}}(|A-B|) \le \sum_{cyclic} \frac{a^2+b^2}{6ab} \sin(|A-B|).$$

Since $0 < A, B, C < \pi$, it follows that

$$-\pi < A-B, B-C, C-A < \pi$$

and hence,

$$0 \le |A-B|, |B-C|, |C-A| < \pi.$$

Then,

$$\sin(|A-B|), \sin(|B-C|), \sin(|C-A|) \ge 0$$

and the Arithmetic - Geometric Mean Inequality implies that

$$\begin{aligned} \prod_{cyclic} \sin^{\frac{1}{3}}(|A-B|) &= \sqrt[3]{\prod_{cyclic} \sin(|A-B|)} \\ &\le \frac{1}{3} \sum_{cyclic} \sin(|A-B|). \end{aligned} \tag{1}$$

Further, The Arithmetic - Geometric Mean Inequality also yields

$$a^2+b^2 \ge 2ab, \quad \text{i.e.,} \quad \frac{a^2+b^2}{2ab} \ge 1.$$

Similar results hold for $\dfrac{b^2+c^2}{2bc}$ and $\dfrac{c^2+a^2}{2ca}$. If we combine these facts with condition (1), we get

$$\begin{aligned} \prod_{cyclic} \sin^{\frac{1}{3}}(|A-B|) &\le \frac{1}{3} \sum_{cyclic} \sin(|A-B|) \\ &\le \sum_{cyclic} \frac{a^2+b^2}{6ab} \sin(|A-B|). \end{aligned}$$

**Solution 3 by Moti Levy, Rehovot, Israel**

By AM-GM inequality and $x + \dfrac{1}{x} \ge 2$ for $x > 0$,

$$\begin{aligned} \prod_{cyclic} \sin^{\frac{1}{3}}|A-B| &\le \frac{1}{3} \sum_{cyclic} \sin|A-B| \\ &\le \frac{1}{3} \sum_{cyclic} 2\sin|A-B| \le \frac{1}{3} \sum_{cyclic} \left(\frac{a}{b} + \frac{b}{a}\right) \sin|A-B| \\ &= \sum_{cyclic} \frac{a^2+b^2}{3ab} \sin|A-B|. \end{aligned}$$

**Also solved by Bruno Salgueiro Fanego, Viveiro, Spain, Albert Stadler, Herrliberg, Switzerland; Henry Ricardo, New York Math Circle, NY; Neculai Stanciu "George Emil Palade" General School, Buzău and Titu Zvonaru, Comănesti, Romania Toshihiro Shimizu, Kawasaki, Japan, and the proposer**

- **5378:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $k \geq 1$ be an integer. Calculate

$$\int_0^\infty \ln^k \left(\frac{e^x+1}{e^x-1}\right) dx.$$

**Solution 1 by Toshihiro Shimizu, Kawasaki, Japan**

Let $y = \ln\left(\frac{e^x+1}{e^x-1}\right)$. Then $e^x = \frac{e^y+1}{e^y-1}$ or $x = \ln\left(\frac{e^y+1}{e^y-1}\right)$ and

$$\begin{aligned}
\frac{dx}{dy} &= \frac{d}{dy}\left(\ln(e^y+1) - \ln(e^y-1)\right) \\
&= \frac{e^y}{e^y+1} - \frac{e^y}{e^y-1} \\
&= \frac{-1}{e^y+1} + \frac{-1}{e^y-1}
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^\infty \ln^k\left(\frac{e^x+1}{e^x-1}\right) dx &= \int_\infty^0 y^k \left(\frac{-1}{e^y+1} + \frac{-1}{e^y-1}\right) dy \\
&= \int_0^\infty \frac{y^k}{e^y+1} dy + \int_0^\infty \frac{y^k}{e^y-1} dy \\
&= \Gamma(k+1)\eta(k+1) + \Gamma(k+1)\zeta(k+1) \\
&= k!(1-2^{-k})\zeta(k+1) + k!\zeta(k+1) \\
&= k!(2-2^{-k})\zeta(k+1)
\end{aligned}$$

**Solution 2 by Ulrich Abel, Technische Hochschule Mittelhessen, Germany**

We calculate, for $k \geq 1$,

$$I(k) = \int_0^\infty \left(\log \frac{e^x+1}{e^x-1}\right)^k dx.$$

The change of variable

$$\begin{aligned}
t &= \log \frac{e^x+1}{e^x-1}, \text{ or equivalently, } x = \log \frac{e^t+1}{e^t-1}, \\
dx &= \left(\frac{e^t+1}{e^t-1}\right)^{-1} \frac{-2e^t}{(e^t-1)^2} dt = \frac{-2e^t}{e^{2t}-1} dt = \frac{-2e^{-t}}{1-e^{-2t}} dt
\end{aligned}$$

yields

$$I(k) = \int_0^\infty t^k \frac{2e^{-t}}{1-e^{-2t}} dt.$$

Rewriting as a geometric series we have

$$I(k) = 2\sum_{j=0}^\infty \int_0^\infty t^k e^{-(2j+1)t} dt = 2\Gamma(k+1) \sum_{j=0}^\infty \frac{1}{(2j+1)^{k+1}},$$

since $\int_0^\infty t^k e^{-t} dt = \Gamma(k+1)$ and the interchange of integration and summation is justified by the Monotone Convergence Theorem. It is well known (and easy to verify) that

$$\sum_{j=0}^{\infty} \frac{1}{(2j+1)^{k+1}} = \left(1 - 2^{-(k+1)}\right) \zeta(k+1).$$

Hence,

$$I(k) = \left(2 - 2^{-k}\right) \Gamma(k+1) \zeta(k+1)$$

Remark: The above formula is valid even for complex $k$ with $Re(k) > 0$.

**Solution 3 by Moti Levy, Rehovot, Israel**

Let $I_k := \int_0^\infty \ln^k \left(\frac{e^x+1}{e^x-1}\right) dx$. By change of variable $v = \ln\left(\frac{e^x+1}{e^x-1}\right)$,

$$I_k = 2\int_0^\infty v^k \frac{e^v}{e^{2v}-1} dv.$$

$$\frac{e^v}{e^{2v}-1} = \frac{1}{e^v-1} - \frac{1}{e^{2v}-1},$$

$$\begin{aligned} I_k &= 2\int_0^\infty \frac{v^k}{e^v-1} dv - 2\int_0^\infty \frac{v^k}{e^{2v}-1} dv \\ &= 2\int_0^\infty \frac{v^k}{e^v-1} dv - \left(\frac{1}{2}\right)^k \int_0^\infty \frac{v^k}{e^v-1} dv \\ &= \left(2 - \left(\frac{1}{2}\right)^k\right) \int_0^\infty \frac{v^k}{e^v-1} dv. \end{aligned}$$

An integral representation of the Zeta function is

$$\Gamma(s)\zeta(s) = \int_0^\infty v^{s-1} \frac{1}{e^v-1} dv, \quad \mathrm{Re}(s) > 1.$$

$$\int_0^\infty \ln^k \left(\frac{e^x+1}{e^x-1}\right) dx = \left(2 - \left(\frac{1}{2}\right)^k\right) k! \zeta(k+1).$$

**Also solved by Ed Gray, Highland Beach, FL; G.C. Greubel, Newport News, VA; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland, and the proposer.**

*Mea Culpa*

**Toshihiro Shimizu of Kawasaki, Japan** should have been credited with having solved problems $5367, 5368, 5369, 5370, 5371,$ and $5372$. His name was inadvertently omitted from the listing. Also omitted from the list of having solved problems were**Charles McCracken, of Dayton, OH** for 5367 , **Paolo Perfetti of the Mathematics Department of Tor Vergata University in Rome, Italy** for 5372, **The Prishtina Math Gymnasium Problem Solving Group of the Republic of Kosova** for 5368 and 5370, and **Albert Stadler, Herrliberg, Switzerland for 5368. Bruno Salgueiro Fanego of Viveiro, Spain** noted that problem 5386 appeared in this column previously as problem 5304. For the above errors, duplications and omissions, mea culpa. Editor.

*Editor′s addendum*: Albert's proof to 5368 was very different from the others that were received. The problem (posed by Ed Gray of Highland Beach FL) was to find a four digit number *abcd* in base 10 such that the last four digits of the square of the number *abcd* was again, *abcd*. Most solvers considered various cases for the digits *abcd*, starting with the digit $d \in \{1, 5, 6\}$, and then, employing the conditions of the problem, eliminated various values. Following is Albert's solution to 5368.

**5368: Solution by Albert Stadler, Herrliberg, Switzerland**

Let $x$ be the four digit number is base 10. By assumption, $x^2 \equiv x \mod (10^4)$, which implies that $x(x-1)$ is divisible by $10^4$. $x$ and $x-1$ are relatively prime. So either $2^4$ divides $x$ and $5^4$ divides $x-1$ or $5^4$ divides $x$ and $2^4$ divides $x-1$.

We now invoke the Chinese remainder theorem. The first alternative implies $x = 9376$, while the second implies 625. However, 625 is not a four digit number, so 9376 is the only solution.